$n_1 + n_2 = -1$, connecting the positive and negative degrees of two associated spherical harmonics, corresponds the relation

$$3n'_1 n'_2 - 2(n'_1 + n'_2) + 1 = 0,$$

connecting the degrees of two associated solutions of the dual equation. Such two associated solutions differ by a factor which is a power of $\left(\frac{du'}{dx'}\right)^2 + \left(\frac{du'}{dy'}\right)^2 + \left(\frac{du'}{dz'}\right)^2$, where u' is either of the two.

On certain Quartic Curves of the Fourth Class and the Porism of the Inscribed and Circumscribed Polygon. By R. A. ROBERTS.

Received March 8th, 1892. Read March 10th, 1892.

In a former communication to the *Proceedings* of the Society, I showed (see Vol. xvi., p. 53) that for certain plane curves of the fourth degree there existed an infinite number of polygons simultaneously inscribed in and circumscribed about the curve; and I gave the means of determining the parameter in the equation of the curve in the same way as Professor Cayley has done for the similar problem of two conics, namely, by equating to zero determinants formed by the coefficients in the expansion of a radical expression. It may be worth while, however, making the investigation directly, with the use of the elliptic integrals by means of which the results are arrived at.

In the place referred to, I showed that the curves are unicursal and possessed of two cusps and a node. Slightly altering the notation, I write the curve now

$$(xy + xz - yz)^2 - m^2 z^2 xy = 0,$$

where xz , yz are the cusps, and xy is the node. This equation, it is easy to verify, is satisfied by taking

$$x = 1 - \theta^2 - m\theta, \quad y = \theta^2 (1 - \theta^2 - m\theta), \quad z = \theta^2,$$

where θ is a parameter. Hence, if a line

$$\lambda x + \mu y + \nu z = 0$$

meet the curve, we have

$$\lambda(1-\theta^2-m\theta)+\mu\theta^3(1-\theta^2-m\theta)+\nu\theta^2=0,$$

from which we find that the four parameters of the points of intersection are connected by the relations

$$\Sigma\theta=-m, \quad \Sigma\frac{1}{\theta}=m.$$

Hence, if the tangent to the curve at the point θ meet the curve again at the points θ_1, θ_2 , we have

$$\theta_1+\theta_2+2\theta=-m, \quad \frac{1}{\theta_1}+\frac{1}{\theta_2}+\frac{2}{\theta}=m;$$

whence, eliminating θ , we obtain

$$(\theta_1+\theta_2)^2+m(\theta_1+\theta_2)(1-\theta_1\theta_2)-(m^2+4)\theta_1\theta_2=0\ldots\ldots(1),$$

which is consequently the relation connecting the parameters of two points, such that the line joining them touches the curve. Now this is of the well known form which occurs in the case of two conics; in fact, if we consider the conic $x=1, y=\theta^2, z=2\theta$, i.e., $z^2-4xy=0$, the tangents at the points θ_1, θ_2 intersect in

$$x=1, \quad y=\theta_1\theta_2, \quad z=\theta_1+\theta_2,$$

so that (1) expresses that two tangents to $z^2-4xy=0$ intersect on the conic

$$z^2+mz(x-y)-(m^2+4)xy=0\ldots\ldots(2).$$

The problem is thus reduced to finding the conditions that it should be possible to inscribe an infinite number of polygons in (2), so that they should be circumscribed about $z^2-4xy=0$.

It will now be very convenient to make use of a result given by Mr. L. J. Rogers (*Proceedings*, Vol. xvi., p. 306).

$$\text{Let} \quad S=z^2+mz(x-y)-(m^2+4)xy,$$

$$S'=z^2-4xy;$$

then let us form first the discriminant of $\xi S'+S$, when we find

$$\{4\xi+m^2+4\}\{4\xi^2+(m^2+8)\xi+4\}.$$

Now, in order to render all the factors rational, we put

$$m^2=\frac{4(t-1)^2}{t},$$

when the discriminant becomes proportional to

$$\left(\xi + t + \frac{1}{t} - 1\right)(\xi + t) \left(\xi + \frac{1}{t}\right).$$

But Mr. L. J. Rogers has shown that, if the discriminant is proportional to

$$(\xi - \xi_1) \left(\xi - \xi_1 \frac{\text{cn}^2 v}{\text{dn}^2 v}\right) (\xi - \xi_1 \text{cn}^2 v),$$

the condition for a polygon of n sides is

$$v = \frac{2K}{n}.$$

In this case we have

$$\xi_1 = -t,$$

and

$$\text{cn}^2 v = \frac{1}{t^2}, \quad \frac{\text{cn}^2 v}{\text{dn}^2 v} = \frac{1 + t^2 - t}{t^2};$$

whence

$$k^2 = \frac{1 - \text{dn}^2 v}{1 - \text{cn}^2 v} = \frac{t^2}{1 + t^2}.$$

We thus have

$$\text{sn } v = \frac{\sqrt{(t^2 - 1)}}{t}, \quad \text{cn } v = \frac{1}{t},$$

$$\text{dn } v = \frac{1}{\sqrt{(1 + t^2 - t)}} \dots \dots \dots (3).$$

To find $\text{sn } 2v$, we have

$$\text{sn } 2v = \frac{2 \text{sn } v \text{ cn } v \text{ dn } v}{1 - k^2 \text{sn}^4 v} = \frac{2\sqrt{\{(t-1)(t^2+1)\}}}{t(2t-1)},$$

and

$$\text{cn } 2v = \frac{1 - 2 \text{sn}^2 v + k^2 \text{sn}^4 v}{1 - k^2 \text{sn}^4 v} = \frac{2-t}{t(2t-1)},$$

$$\text{dn } 2v = \frac{1 - 2k^2 \text{sn}^2 v + k^2 \text{sn}^4 v}{1 - k^2 \text{sn}^4 v} = \frac{1}{2t-1} \dots \dots \dots (4).$$

$$\text{Again, since} \quad \text{sn } v \text{ sn } 3v = \frac{\text{sn}^2 2v - \text{sn}^2 v}{1 - k^2 \text{sn}^2 v \text{sn}^2 2v},$$

we get

$$\text{sn } 3v = \frac{3\sqrt{(t^2-1)}}{5t-4} \dots \dots \dots (5);$$

$$\text{whence} \quad \text{cn } 3v = \frac{5-4t}{5t-4}, \quad \text{dn } 3v = \frac{4t^2-7t+4}{(5t-4)\sqrt{(t^2-t+1)}},$$

as can also be proved otherwise.

Again, we have

$$\begin{aligned} \operatorname{cn} 4v &= \frac{1 - 2 \operatorname{sn}^2 2v + k^2 \operatorname{sn}^4 2v}{1 - k^2 \operatorname{sn}^4 2v} \\ &= \frac{1 - \frac{8(t-1)(t^3+1)}{t^2(2t-1)^2} + \frac{16t^3}{1+t^3} \frac{(t-1)^2(t^3+1)^2}{t^4(2t-1)^4}}{1 - \frac{16t^3}{1+t^3} \frac{(t-1)^2(t^3+1)^2}{t^4(2t-1)^4}} \\ &= \frac{-16t^3 + 33t^2 - 24t + 8}{t(8t^3 - 24t^2 + 33t - 16)} \dots \dots \dots (6). \end{aligned}$$

We may now apply these expressions to finding the values of t , when n takes different values.

Let $n = 3$; then $3v = 2K$, and $\operatorname{sn} 3v = 0$, which gives, from (5), $t = \pm 1$. Now these values are manifestly irrelevant, as, from (3), they actually correspond to $v = 0$, and $v = 2K$.

Thus we see that there are no curves simultaneously inscribed in and circumscribed about an infinite system of triangles.

Next, let $n = 4$; then $2v = K$, and $\operatorname{cn} 2v = 0$; hence $t = 2$, and

$$m^3 = \frac{4(t-1)^2}{t} = 2;$$

therefore the curve $(xy + xz - yz)^3 = 2xyz^3$

is simultaneously inscribed in and circumscribed about an infinite series of quadrilaterals.

Again, let $n = 5$; then $5v = 2K$ or $4K$, so that

$$\operatorname{cn} 2v \pm \operatorname{cn} 3v = 0.$$

Hence, from (4) and (5),

$$(2-t)(5t-4) \pm t(2t-1)(5-4t) = 0;$$

which, after dividing by $t-1$ and $t+1$, gives

$$8t^3 - 17t + 8 = 0, \quad 8t^3 - 11t + 8 = 0.$$

Each of these equations gives only one value of m , for

$$m^3 = 4 \left(t + \frac{1}{t} - 2 \right);$$

thus

$$m^3 = \frac{1}{2} \quad \text{or} \quad -\frac{3}{2}.$$

Hence, for the pentagon, the curves may be written

$$2(xy + zx - yz)^2 - xyz^2 = 0,$$

$$2(xy + zx + yz)^2 - 5xyz^2 = 0,$$

where, in the second case, we have changed the sign of y .

Let $n = 6$; then $6v = 2K$, $3v = K$ and $\text{cn } 3v = 0$, so that (5) gives $t = \frac{5}{4}$, and $m^2 = \frac{1}{5}$; thus the curve is

$$5(xy + zx - yz)^2 - xyz^2 = 0.$$

Let $n = 7$; then $7v = 2K$ or $4K$, and $\text{cn } 3v \pm \text{cn } 4v = 0$.

Hence, from (5) and (6), we get

$$\frac{5-4t}{4-5t} \mp \frac{16t^4-33t^2+24t-8}{t(8t^3-24t^2+33t-16)} = 0;$$

which, after dividing by $t-1$ and $t+1$, gives

$$32t^4 - 37t^3 + 14t^2 - 37t + 32 = 0,$$

$$32t^4 - 184t^3 + 297t^2 - 184t + 32 = 0;$$

or, putting

$$t + \frac{1}{t} = \lambda,$$

$$32\lambda^4 - 37\lambda - 50 = 0, \quad 32\lambda^4 - 184\lambda + 233 = 0.$$

Hence, since from these equations there are four real values of

$$\lambda = 2 + \frac{1}{2}m^2,$$

we infer that there are four curves corresponding to the figure of seven sides.

Let $n = 8$; then $8v = 2K$, $4v = K$, and $\text{cn } 4v = 0$; whence, from (6),

$$16t^4 - 33t^2 + 24t - 8 = 0.$$

This equation has only one real root, as its Hessian quadratic $7t^2 - 40t + 22$ has both its factors real. There is thus but one real curve in the case of the figure of eight sides.

Let $n = 9$; then $9v = 2K$. Let $3\alpha = 2K$, and $\alpha = 3v$. Now it is easy to show that, if $3\alpha = 2K$ or $4K$, then

$$\text{cn } \alpha = (1 \pm \text{cn } \alpha) \text{dn } \alpha.$$

Hence, from (5), we have

$$(5-4t)(5t-4)\sqrt{(1+t^2-t)} = 9(t-1)(4t^2-7t+4),$$

or $(5-4t)(5t-4)\sqrt{(1+t^2-t)} = (t+1)(4t^2-7t+4),$

which, if we put $t + \frac{1}{t} = \lambda$,

become $(20\lambda - 41)^2 (\lambda - 1) - 81 (\lambda - 2)(4\lambda - 7)^2 = 0$,

$$(20\lambda - 41)^2 (\lambda - 1) - (\lambda + 2)(4\lambda - 7)^2 = 0,$$

so that there are six curves for $n = 9$.

Let $n = 10$; then $10v = 2K$, $5v = K$, $3v = K - 2v$, and

$$\operatorname{sn} 3v = \operatorname{sn} (K - 2v) = \frac{\operatorname{cn} 2v}{\operatorname{dn} 2v};$$

therefore $9t^2 (t^2 - 1) = (2 - t)^2 (5t - 4)^2$,

which gives four curves for $n = 10$.

A few additional remarks on the curve corresponding to the quadrilateral may be of interest. Putting $m^2 = 2$, and $\theta\sqrt{2}$ for θ , we may represent the curve thus

$$x = 2\theta^2 (2\theta^2 + 2\theta - 1), \quad y = 2\theta^2 + 2\theta - 1, \quad z = 2\theta^2,$$

in which case, if the tangent at θ meet the curve again at θ_1, θ_2 , we have

$$\theta_1 + \theta_2 + 2\theta = -1, \quad \frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{2}{\theta} = 2,$$

whence $\theta_1^2 + \theta_2^2 + \theta_1 + \theta_2 - 2\theta_1\theta_2(\theta_1 + \theta_2 + 2) = 0$.

Now this relation may be written

$$\phi(\theta_1) + \phi(\theta_2) = 0,$$

where $\phi(\theta) = \frac{\theta(1+\theta)}{2\theta-1}$;

and the form of the equation proves the existence of an infinite number of quadrilaterals; for the four equations

$$\begin{aligned} \phi(\theta_1) + \phi(\theta_2) &= 0, & \phi(\theta_2) + \phi(\theta_3) &= 0, & \phi(\theta_3) + \phi(\theta_4) &= 0, \\ \phi(\theta_4) + \phi(\theta_1) &= 0, \end{aligned}$$

are manifestly equivalent to only three independent equations.

Let a, b, c, d be the parameters of the vertices; then, if λ is a parameter, we may evidently write

$$\left. \begin{aligned} a^2 + a &= \lambda(2a - 1), & b^2 + b &= -\lambda(2b - 1) \\ c^2 + c &= \lambda(2c - 1), & d^2 + d &= -\lambda(2d - 1) \end{aligned} \right\} \dots\dots\dots (a),$$

from the first and third of which we get

$$a + c + 1 = 2\lambda, \quad \text{and} \quad b + d + 1 = -2\lambda;$$

similarly from the second and last. Hence

$$a + b + c + d = -2.$$

Now, if θ is the point of contact of ab ,

$$2\theta = -(1 + a + b), \quad \text{and} \quad 2\theta' = -(1 + c + d),$$

where θ' is the point of contact of cd . Therefore $\theta + \theta' = 0$, which, it is easy to see, shows that θ, θ' are collinear with the node. Thus we see that the line joining the points of contact of opposite sides passes through the node; or we may put it: the node is the intersection of the diagonals of the quadrangle formed by the points of contact.

Again, if the curve be reciprocated, it will be transformed into a similar curve, and the property of the quadrilateral remains unaltered. Hence, from the last result, we deduce that the tangents at opposite vertices of the quadrilateral intersect on the bitangent of the curve.

We now prove a property of this curve. The points where the nodal tangents meet the curve again are found by altering the sign of θ in the quadratic giving the parameters of the node, that is

$$2\theta^2 + 2\theta - 1 = 0,$$

so that the nodal tangents cut the curve again in

$$2\theta^2 - 2\theta - 1 = 0;$$

but these are the points of inflexion; for the latter points, we have

$$3\theta + \theta' = -1, \quad \frac{3}{\theta} + \frac{1}{\theta'} = 2,$$

whence, eliminating θ' , we get

$$2\theta^2 - 2\theta - 1 = 0.$$

This property is the reciprocal of the result noticed by Professor Cayley in my former paper, namely, that the cuspidal tangents intersect the curve again in the points of contact of the bitangent. From the first and third of the equations (a), we easily get $\lambda = ac$, and from the second and fourth $\lambda = -bd$, from which we further deduce

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 4, \quad ac + bd = 0.$$

And from these we propose to prove a result concerning a certain conic circumscribing the quadrilateral. If any conic meet the curve, we can easily show that

$$\sum_8^1 \theta = -2, \quad \sum_8^1 \frac{1}{\theta} = 4,$$

for the eight points of intersection. Hence, if the conic pass through the node whose parameters are given by

$$2\theta^2 + 2\theta - 1 = 0,$$

we have for the remaining six points of intersection

$$\sum_6^1 \theta = -1, \quad \sum_6^1 \frac{1}{\theta} = 2.$$

Hence, if the conic passing through a, b, c, d meet the curve again in θ_1, θ_2 , we have from the above

$$\theta_1 + \theta_2 - 2 = -1;$$

or
$$\theta_1 + \theta_2 = 1, \quad \frac{1}{\theta_1} + \frac{1}{\theta_2} + 4 = 2,$$

or
$$\frac{1}{\theta_1} + \frac{1}{\theta_2} = -2;$$

but these equations, as we have shown already, determine the points of inflexion. Thus we have proved that we can describe a conic through the vertices of any one of the quadrilaterals, so as to pass through the node and the two points of inflexion.

Furthermore, we can show that this conic passes through another fixed point; for the nodal tangents being

$$x^2 + y^2 - 4xy = 0,$$

and the line of inflexion $4z + y - x = 0,$

a conic circumscribing that triangle may be written

$$x^2 + y^2 - 4xy - (lx + my)(4z + y - x) = 0,$$

which intersects the curve in points given by

$$(2\theta^2 + 2\theta - 1)^2 (2\theta^2 - 2\theta - 1) - (2l\theta^2 + m)(2\theta^2 + 2\theta - 1)(2\theta^2 - 2\theta - 1)(1 - 4\theta - 2\theta^2) = 0;$$

or, dividing out by factors,

$$(2\theta^2 + 2\theta - 1)^2 - (2l\theta^2 + m)(1 - 4\theta - 2\theta^2) = 0,$$

which becomes

$$4(l+1)(\theta^4 + 2\theta^2) + 2(m-l)\theta^2 + (m-1)(4\theta-1) = 0,$$

of which a, b, c, d are to be the roots.

$$\text{Now we have} \quad ac + bd = 0,$$

$$\text{which, in the case of } \theta^4 - p\theta^3 + q\theta^2 - r\theta + s = 0,$$

$$\text{gives} \quad p^2 + \frac{r^2}{s} = 4q,$$

and, applied to the biquadratic just found, gives, since $p = -2$,

$$\text{and} \quad \frac{r}{s} = 4, \quad 1 + r = q.$$

$$\text{Thus} \quad \frac{m-l}{2(l+1)} = 1 + \frac{1-m}{l+1},$$

$$\text{or} \quad m-l = 4,$$

which seems to show that the conic passes through the additional point

$$x+y=0, \quad 8z-5y=0.$$

Reciprocating this result, we deduce that a conic can be described to touch the four sides of the quadrilateral, the double tangent, the two cuspidal tangents and another fixed line.

Interesting particular cases might be found by projecting the curve. Thus, if the points of inflexion became the circular points at infinity, the vertices of the quadrilaterals would lie on a system of coaxial circles. Also, if the points of contact of the bitangent were the circular points, the circles circumscribing the four triangles formed by the sides of the quadrilateral would all pass through a fixed point.

We might investigate the relation connecting points whose tangents intersect on the curve as follows. Let the tangent at θ meet the curve again in α, β , then

$$2\theta + \alpha + 1 = -\beta, \quad \frac{2}{\theta} + \frac{1}{\alpha} - 2 = -\frac{1}{\beta};$$

therefore, eliminating β , we get

$$2\theta^2(1-2\alpha) + \theta(1+2\alpha-2\alpha^2) + 2\alpha(\alpha+1) = 0.$$

Now the roots of this equation in θ must be the points of contact of

the tangents which intersect in a . Hence we get

$$\theta_1 + \theta_2 = \frac{2a^2 - 2a - 1}{2(1 - 2a)}, \quad \theta_1 \theta_2 = \frac{a(a+1)}{1 - 2a},$$

so that, eliminating a , we obtain, as the relation between θ_1, θ_2 ,

$$1 + 4\theta_1^2 \theta_2^2 - 4(\theta_1^2 + \theta_2^2) = 0.$$

Now, from the curve, we have

$$y_1/x_1 = 2\theta_1^2, \quad y_2/x_2 = 2\theta_2^2.$$

Hence $x_1 x_2 + y_1 y_2 - 2(x_1 y_2 + y_1 x_2) = 0$,

which shows that the two points are conjugate with regard to

$$x^2 + y^2 - 4xy = 0,$$

i.e., the nodal tangents. Reciprocating this result, we find that tangents at the vertices of a quadrilateral divide the bitangent harmonically. Hence if the curve be projected so that the points of contact of the bitangent become the circular points, the tangents at the extremities of a chord which touches the curve cut at right angles; and the tangents at the vertices of the simultaneously inscribed and circumscribed quadrilateral form a rectangle.

Thursday, April 14th, 1892.

Prof. GREENHILL, F.R.S., President, in the Chair.

Mr. Arthur Lee Dixon, B.A., Fellow of Merton College, Oxford, was elected a member.

The meeting then balloted for the following six mathematicians—Messrs. Poincaré, Hertz, Mittag-Leffler, Schwarz, Beltrami, and Willard Gibbs, whose names had been submitted by the Council for the compliment of foreign membership. All were unanimously elected.

The following communications were made:—

Second Note on a Quaternary Group of 51840 Linear Substitutions: Dr. G. G. Morrice.

Note on the Skew Surfaces applicable upon a given Skew Surface: Prof. Cayley.

Mr. Kempe made an impromptu communication on "Regular Graphs."

The following presents were received:—

"Royal Society—Proceedings" Vol. L., Nos. 305, 306.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xvi., St. 3; Leipzig, 1892.

"Gesells. der Wissenschaften zu Leipzig—Berichte," Pt. iv., 1892.

"Gesells. der Wissenschaften zu Leipzig—Abhandlungen," Band xviii., Nos. 3, 4.

"Bulletin of New York Mathematical Society," Vol. i., No. 1, October, 1891, and No. 6, March, 1892. From Prof. W. W. Johnson.

"Kjobenhavn Tidsskrift for Mathematik," A. Tredje Aargang, Nos. 1 and 2.

"Kjobenhavn Tidsskrift for Mathematik," B. Tredje Aargang, No. 1.

"Bulletin de la Société Mathématique de France," Tome xx., No. 1.

"Bulletin des Sciences Mathématiques," Tome xvi., Fév., 1892.

"Roma—Atti della Reale Accademia dei Lincei," Vol. i., Sem. 1, Fasc. 3, 4, 5.

"Educational Times" for April.

"Toulouse—Annales de la Faculté des Sciences," Tome vi., Fasc. 1; Paris, 1892.

"Milano—Annali di Matematica," Tome xix., Fasc. 4; February, 1892.

"Journal für die Reine und Angewandte Mathematik (Crelle)," Bd. cix., Heft 3; Berlin.

"Mathematical Questions, with their Solutions, from 'Educational Times,'" Vol. lvi.; London, 1892.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zurich," Zurich, 1891.

"Prace Matematyczno Fizyczne," Tome iii.; Warszawa, 1892.

"Washington Observations, 1887. Observations made during the year 1887, at the U. S. Naval Observatory," with 3 Appendices and 10 Plates; 1892.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Nos. 149, 150; Firenze, 1892.

"J. Timtchenko—Fondements de la Théorie des Fonctions Analytiques" Mémoires de la Section Mathématique de la Société des Naturalistes de la Nouvelle-Russie," Tome xii.; Odessa, 1892).