On a Symbolic Proof of Fourier's Theorem

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pupils, and they may reasonably be asked to examine the facility with which these assumptions were made, and to consider whether the evidence can in any way be strengthened. This being done, the ideas of consistence and redundancy can be developed, and some idea of the structure of a science imparted. Even then it may probably be wise to lay little stress on analysis of the geometrical postulates; if the ideas are realised in connection with mechanics, we may well leave the seed to mature in minds to which it is congenial.

In the view of mathematics here taken, its various branches are regarded as structures with many possible entrances, and the discussion has been concerned with the choice of entrance and the route to be taken through the edifice. We cannot hope that our pupils will ever know more than the outline of each structure. Even we who are the guides cannot know each detail of any one; the labyrinth is too vast. But the best guide to a structure is he who knows its main outlines most completely, and a teacher who has clear ideas of mathematical principles can do much, in leading his pupils through such avenues of the structure as they can attain, to give them a view of the whole. Of the import and beauty of this view more need not be said.

G. ST. L. CARSON.

ON A SYMBOLIC PROOF OF FOURIER'S THEOREM.

1. Some time ago, when attempting to generalise certain methods of solving differential equations by operators, the writer came across the following mode of deducing Fourier's Theorem symbolically.

The method is so simple that it seems improbable that it should be unknown. The nearest approach to it known to the author, however, occurs in Oliver Heaviside's *Electromagnetic Theory*, Vol. II. §§ 275 *et seq.*, but the investigation there proceeds on somewhat different lines, and the interpretation of the operators used requires to be carefully defined.

Under these circumstances the following deduction may not be without interest—although, naturally, like most symbolic proofs, it makes no pretence to rigour.

2. We know that if $\phi(D)y=f(x)$ (1)

be a differential equation, $\phi(D)$ being a polynomial in D with constant coefficients and D denoting the differential operator $\frac{d}{dx}$, then, the most

general solution of equation (1) being denoted by $\frac{1}{\phi(D)}f(x)$, it is legitimate to develop $\frac{1}{\phi(D)}$ in partial fractions by the ordinary methods. The result, if given its most general interpretation, will yield the most general solution of equation (1); and therefore, if we operate upon it with $\phi(D)$, we shall get back the function $f(x)$.

The above result can be rigorously demonstrated when $\phi(D)$ is a polynomial. Let us now assume it to be true when $\phi(D)$ is a transcendental function, such that $\frac{1}{\phi(D)}$ can be expanded in partial fractions. (See Whittaker's *Modern Analysis*, Art. 74.)

In particular, consider $\phi(D)=\sinh aD$.

We know that

$$\operatorname{cosech} aD = \frac{1}{a} \left[\frac{1}{D} + i \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ \frac{1}{n\pi/a - iD} - \frac{1}{n\pi/a + iD} \right\} \right],$$

i being $\sqrt{-1}$.

Now

$$\begin{aligned}\frac{1}{n\pi/a - iD} f(x) &= \frac{1}{n\pi/a - iD} e^{-\frac{in\pi x}{a}} \left\{ e^{\frac{in\pi x}{a}} f(x) \right\} \\ &= e^{-\frac{in\pi x}{a}} \frac{1}{(-iD)} \cdot e^{\frac{in\pi x}{a}} f(x) \\ &= +ie^{-\frac{in\pi x}{a}} \int_{\alpha_n}^x e^{\frac{in\pi x}{a}} f(x) dx,\end{aligned}$$

α_n being an arbitrary lower limit.

Similarly

$$\frac{1}{n\pi/a + iD} f(x) = -ie^{-\frac{in\pi x}{a}} \int_{\beta_n}^x e^{-\frac{in\pi x}{a}} f(x) dx,$$

β_n being likewise arbitrary.

Hence

$$\begin{aligned}\operatorname{cosech} aD \cdot f(x) &= \frac{1}{a} \left[\int_{\alpha_0}^x f(x) dx - \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ e^{\frac{in\pi x}{a}} \int_{\beta_n}^x e^{-\frac{in\pi x}{a}} f(x) dx \right. \right. \\ &\quad \left. \left. + e^{-\frac{in\pi x}{a}} \int_{\alpha_n}^x e^{\frac{in\pi x}{a}} f(x) dx \right\} \right]. \dots (2)\end{aligned}$$

Now apply the operator $\sinh aD$ to both sides, and assume this operator may be distributed between the terms of the infinite series on the right-hand side of (2).

$$\begin{aligned}f(x) &= \frac{\sinh aD}{a} \left[\int_{\alpha_0}^x f(x) dx - \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ e^{\frac{in\pi x}{a}} \int_{\beta_n}^x e^{-\frac{in\pi x}{a}} f(x) dx \right. \right. \\ &\quad \left. \left. + e^{-\frac{in\pi x}{a}} \int_{\alpha_n}^x e^{\frac{in\pi x}{a}} f(x) dx \right\} \right]. \dots (3)\end{aligned}$$

Consider now the effect of operating with $\sinh aD$ upon any function of x , say $\phi(x)$.

$$\begin{aligned}\sinh aD \cdot \phi(x) &= \frac{e^{aD} - e^{-aD}}{2} \cdot \phi(x) \\ &= \frac{\phi(x+a) - \phi(x-a)}{2},\end{aligned}$$

by the symbolic form of Taylor's Theorem.

$$\begin{aligned}\therefore \frac{\sinh aD}{a} \int_{\alpha_0}^x f(x) dx &= \frac{\sinh aD}{a} \int_{\alpha_0}^x f(\xi) d\xi \\ &= \frac{\int_{\alpha_0}^{x+a} f(\xi) d\xi - \int_{\alpha_0}^{x-a} f(\xi) d\xi}{2a} = \frac{1}{2a} \int_{x-a}^{x+a} f(\xi) d\xi.\end{aligned}$$

Again,

$$\begin{aligned}&\frac{\sinh aD}{a} \left\{ e^{\frac{in\pi x}{a}} \int_{\beta_n}^x e^{-\frac{in\pi \xi}{a}} f(\xi) d\xi \right\} \\ &= \frac{1}{2a} \left\{ e^{\frac{in\pi x}{a} + in\pi} \int_{\beta_n}^{x+a} e^{-\frac{in\pi \xi}{a}} f(\xi) d\xi - e^{\frac{in\pi x}{a} - in\pi} \int_{\beta_n}^{x-a} e^{-\frac{in\pi \xi}{a}} f(\xi) d\xi \right\} \\ &= \frac{(-1)^n}{2a} e^{\frac{in\pi x}{a}} \int_{x-a}^{x+a} e^{-\frac{in\pi \xi}{a}} f(\xi) d\xi.\end{aligned}$$

Similarly

$$\frac{\sinh aD}{a} \left\{ e^{-\frac{in\pi x}{a}} \int_{\alpha_n}^x e^{\frac{in\pi \xi}{a}} f(\xi) d\xi \right\} = \frac{(-1)^n}{2a} e^{-\frac{in\pi x}{a}} \int_{x-a}^{x+a} e^{\frac{in\pi \xi}{a}} f(\xi) d\xi.$$

Equation (3) therefore becomes

$$\begin{aligned} f(x) &= \frac{1}{2a} \left[\int_{x-a}^{x+a} f(\xi) \left\{ 1 + \Sigma \left(e^{\frac{in\pi}{a}(x-\xi)} + e^{-\frac{in\pi}{a}(x-\xi)} \right) \right\} d\xi \right] \\ &= \frac{1}{2a} \int_{x-a}^{x+a} f(\xi) \left\{ 1 + 2\Sigma \cos \frac{n\pi}{a}(x-\xi) \right\} d\xi. \dots\dots\dots(4) \end{aligned}$$

This equation holds for all values of x .

Now let $f(x)$ be a function either naturally periodic of period $2a$, or else made artificially periodic by taking the natural values of the function from $-a$ to $+a$, and repeating those indefinitely.

By a well known theorem, the integral of any periodic function taken over any one period is equal to the integral taken over any other period. Now, under the restrictive hypothesis just made, $f(\xi)$ and $\cos \frac{n\pi}{a}(x-\xi)$ are both periodic in ξ of period $2a$. Hence $f(\xi) \left\{ 1 + 2\Sigma \cos \frac{n\pi}{a}(x-\xi) \right\}$ is periodic in ξ of period $2a$.

Hence
$$\int_{x-a}^{x+a} = \int_{-a}^a,$$

and we have
$$f(x) = \frac{1}{2a} \int_{-a}^a f(\xi) \left\{ 1 + 2\Sigma \cos \frac{n\pi}{a}(x-\xi) \right\} d\xi, \dots\dots\dots(5)$$

which is Fourier's Theorem in its usual form.

3. The same method can be applied to find the development of $f(x)$ from $x=0$ to $x=a$ in a series of sines (or cosines) of odd multiples of $\pi x/2a$.

This is done by considering the identity

$$f(x) = \cosh aD \{ \operatorname{sech} aD \cdot f(x) \},$$

where in the last bracket $\operatorname{sech} aD$ is developed in partial fractions.

It seems probable that this method might be successfully applied to discover other developments in series.

L. N. G. FILON.

ON SURFACES TRACED OUT BY THE MOTION OF AN INVARIABLE CURVE.

§ 1. Suppose that a curve Γ of constant shape moves in space so as to trace out a surface Σ . The problem has suggested itself to various authors whether it is possible for Γ to be always a line of curvature, geodesic, asymptotic line, etc., on Σ . The following elementary method of solving such problems may be of interest.

Let (x, y, z) be the coordinates of a point P on Γ referred to fixed rectangular axes of reference, and let (ξ, η, ζ) be the coordinates of P referred to moving rectangular axes a, b, c rigidly attached to Γ , and meeting in the moving point O .

Then if $(l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3)$ are the direction-cosines of a, b, c , and (p_1, p_2, p_3) are the coordinates of O , referred to the fixed reference-axes,

$$x = l_1\xi + m_1\eta + n_1\zeta + p_1, \quad y = l_2\xi + m_2\eta + n_2\zeta + p_2, \quad z = l_3\xi + m_3\eta + n_3\zeta + p_3.$$

We may suppose ξ, η, ζ functions of a parameter u , and the l 's, m 's, n 's, p 's functions of a parameter v , which may be considered as the time taken by a, b, c to arrive at their present positions from standard positions.

We denote differentiation with respect to u and v by dashes and dots respectively.