THE MACLAURIN SUM-FORMULA

By E. W. BARNES.

[Received and Read, February 9th, 1905.]

1. The formula

$$\Sigma\phi(x) = C + \int^x \phi(x)\,dx - \tfrac{1}{2}\phi(x) + \frac{B_1}{2!}\frac{d\phi(x)}{dx} + \dots \tag{1}$$

was originally obtained in 1732 by Euler,* and independently rediscovered before 1742 by Maclaurin.† A full discussion of the question of priority appears in Cantor, *Geschichte der Mathematik*, Bd. III. (1898), p. 663. Then Plana,‡ in 1820, and Abel,§ in 1823, obtained the formula from which it can be developed:

$$\Sigma\phi(x) = \int^x \phi(x)\,dx - \tfrac{1}{2}\phi(x) + 2\int_0^\infty \frac{\phi(x+it)-\phi(x-it)}{2\iota}\,\frac{dt}{e^{2\pi t}-1}. \tag{2}$$

A generalisation of this formula, which it is interesting to compare with my own generalisation with which this paper concludes, was given by Abel‖ in 1825. In 1889 Kronecker¶ discussed the formula (2) by Cauchy's theory of residues, and another discussion appeared in 1898 in a text-book of Petersen.** More recently Lindelöf †† has given certain applications, and I have myself ‡‡ generalised the analysis and considered the question from the point of view of the theory of asymptotic series in 1903.

I now propose to take up the question again, and, without the intervention of such a theory, to obtain a form for the remainder different from that obtained by other investigators, and to give a fresh demonstration of the conditions under which my extensions of the formula (1) are valid.§§

---

* Euler, *Commentarii Acad. Sci. Imp. Petropolitanæ*, T. VI. (1732 and 1733), pp. 68–97.

† Maclaurin, *Treatise on Fluxions*, p. 672, Edinburgh, 1742.

‡ Plana, *Mem. Accad. Torino*, T. XXV.

§ Abel, *Œuvres Complètes* (1881), T. I., pp. 21–25; T. II., p. 77.

‖ Abel, *ibid.*, T. I., pp. 34–39.

¶ Kronecker, *Crelle*, T. CV., pp. 157–159 and pp. 345–354.

** Petersen, *Vorlesungen über Functionentheorie*, §§ 78–80.

†† Lindelöf, *Acta Soc. Sci. Fennicæ*, T. XXXI.; *Acta Mathematica*, T. XXVII., pp. 305–311.

‡‡ Barnes, *Quarterly Journal of Mathematics*, Vol. XXXV., pp. 175–188.

§§ References to other investigations will be found in Boole, *Finite Differences*, Third Edition, p. 153.

2. The discussion of the question is based upon the properties of the generalised Riemann ζ function, which is defined as follows:—Let s, a, and ω be any complex quantities, and let $S_n(a\,|\,\omega)$, or more briefly $S_n(a)$, be the n-th Bernoullian function defined by the expansion, valid when $|x| < |2\pi\iota/\omega|$,

$$\frac{xe^{-ax}}{1-e^{-\omega x}} = \sum_{n=0}^{\infty} \frac{S_n'(a)}{n!}(-x)^n,$$

the accent denoting differentiation with regard to x, coupled with the condition $S_n(0) = 0$.

Let $B_n(\omega)$ be the n-th Bernoullian number of parameter ω given by

$$B_n(\omega) = S_n'(0\,|\,\omega)/n.$$

Construct the function

$$S_{s,l}(a) = \sum_{m=0}^{l+1} \frac{S_m'(0\,|\,\omega)}{m!}\frac{d^m}{da^m}\frac{a^{1+s}}{1+s}$$

where the many-valued functions with s as index have their principal values with respect to the axis of $-\omega$, *i.e.*, with respect to a line drawn from the origin to $-\omega$ and produced to infinity. Thus $a^s = \exp[s\log a]$, where $\log a$ has a cross-cut along the axis of $-\omega$, and is real when a is real and positive.

[We assume for convenience that ω is not real and negative: in this case further definition is necessary.]

Then, if $R(s) > -(l+1)$, where l is a finite positive integer, the series

$$-S_{-s,l}(a)+\sum_{n=0}^{\infty}\left[(a+n\omega)^{-s}-S_{-s,l}\{a+(n+1)\omega\}+S_{-s,l}(a+n\omega)\right]$$

is absolutely convergent except when

$$s = 1 \quad \text{or} \quad a = -n\omega \quad (n = 0, 1, 2, \ldots, \infty),$$

and is denoted by $\zeta(s, a)$ or by $\zeta(s, a\,|\,\omega)$.

When $R(s) > 1$,

$$\zeta(s, a) = \sum_{n=0}^{\infty}(a+n\omega)^{-s},$$

and, when $a = \omega = 1$, we have Riemann's ζ function.*

When $R(a/\omega)$ is positive, it can be deduced from the previous definition that, for all values of s and ω,

$$\zeta(s, a) = \frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-az}}{1-e^{-\omega z}}(-z)^{s-1}\,dz,$$

the integral being taken along a contour embracing the axis of $1/\omega$,

* Riemann, *Gesammelte Werke*, pp. 136–144.

starting from $+\infty/\omega$, enclosing the origin but no other singularity of the subject of integration, and returning on the other side of the axis of $1/\omega$ to $+\infty/\omega$. The expression $(-z)^{s-1} = \exp\{(s-1)\log(-z)\}$ has its principal value with respect to the axis of $1/\omega$, which is a cross-cut for the logarithm.

The definition of the function $\zeta(s, a)$ is a development of that due to Mellin*: it is an application of the well-known theory of Mittag-Leffler.† For a proof of the convergency of the series I may refer to my "Theory of the Double Gamma Function" ‡ (p. 341), and for the properties of $\zeta(s, a)$ to my "Theory of the Gamma Function" § (Part III.).

3. The general result obtained in the present paper is as follows:— The Maclaurin sum-formula for a single parameter may, with the greatest generality, be written

$$\sum_{n=0}^{m-1} \phi(a+m\omega) = Y(a)+\sum_{n=0}^{l+1} \frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\int^x \phi(x)\,dx\right]_{x=m\omega}+J_l, \qquad \text{(A)}$$

where $Y(a)$ is a definite function of a and of the coefficients in the expansion of $\phi(x)$, and where $\phi(x)$ is to be expanded by Laurent's series and the lower limit so chosen in the integral that the corresponding term vanishes at this limit.

When $\phi(x)$ is a (possibly non-uniform) function which has no singularities outside a circle centre the origin and finite radius outside which a, $a+\omega$, ..., $a+m\omega$, and $m\omega$ all lie, J_l is a quantity which, when m is large, has its modulus at most of order $1/m^{l+1}$.‖

When $\phi(x)$ is an integral function of order ¶ less than unity, J_l tends to zero as l tends to infinity for all values of m, however large, and the Maclaurin series is absolutely convergent. In general, when $\phi(x)$ is an integral function of order $\geqslant 1$, the Maclaurin sum-formula has no meaning unless we apply the theory of asymptotic series to evaluate both the series for $Y(a)$ and the series which succeeds it in the enunciation.

The above formula can be generalised for any number of parameters, and corresponding propositions hold good.

---

* Mellin, *Acta Soc. Sci. Fennicæ*, T. XXIV., No. 10 (1899).

† Mittag-Leffler, *Acta Mathematica*, T. IV., pp. 1–79.

‡ Barnes, *Phil. Trans. Roy. Soc.* (A), Vol. CXCVI., pp. 265–387.

§ Barnes, *Messenger of Mathematics*, Vol. XXIX., pp. 64–128. This paper will be referred to for convenience under the initials *G.F.*, and the previous one as *D.G.F.*

‖ [*Note added April 5th*, 1905.]—By this statement we mean that $|J_l m^{l+1-\epsilon}|$, where ϵ is an arbitrarily small positive quantity, can be made as small as we please by taking m sufficiently large.

¶ As originally defined by Borel, *Acta Mathematica*, T. XX., p. 360.

4.* THEOREM I.—*If* $s = u + \iota v$ *where* u *is finite, then* $e^{-(\pi-\epsilon')|v|}|\omega^s\zeta(s, a)|$ *tends to zero as* $|v|$ *tends to infinity, if* $0 \leqslant \epsilon' < \epsilon \leqslant \pi$ *and*

$$|\arg(a/\omega)| = \pi - \epsilon.$$

We have, if $R(s) > -(l+1)$,

$$\zeta(s, a) = -S_{-s,l}(a) + \sum_{n=0}^{\infty}\left\{\frac{1}{(a+n\omega)^s} - S_{-s,l}\{a+(n+1)\omega\} + S_{-s,l}(a+n\omega)\right\},$$

where
$$S_{-s,l}(a) = \sum_{m=0}^{l+1}\frac{S'_m(0)}{m!}\frac{d^m}{da^m}\frac{a^{1-s}}{1-s}.$$

Now [*G.F.*, p. 80] $S_n\left(\frac{a}{\omega}\middle| 1\right) = \frac{1}{\omega^n}S_n(a\,|\,\omega).$

Also $(a+n\omega)^s$, when the many-valued function has its principal value with respect to the axis of $-\omega$, is equal to $(a/\omega+n)^s\omega^s$, where each function has its principal value defined as usual with respect to the axis of -1.

Hence
$$\omega^s S_{-s,l}(a) = \sum_{m=0}^{l+1}\frac{S'_m(0\,|\,1)}{m!}\left[\frac{d^m}{dx^m}\frac{x^{1-s}}{1-s}\right]_{x=a/\omega}$$

and

$$\omega^s\zeta(s, a) = -\sum_{m=0}^{l+1}\frac{S'_m(0\,|\,1)}{m!}\left[\frac{d^m}{dx^m}\frac{x^{1-s}}{1-s}\right]_{x=a/\omega}$$
$$+\sum_{n=0}^{\infty}\left\{\frac{1}{(a/\omega+n)^s} - \sum_{m=0}^{l+1}\frac{S'_m(0\,|\,1)}{m!}\left[\frac{d^m}{dx^m}\frac{x^{1-s}}{1-s}\right]_{x=a/\omega+n+1}\right.$$
$$\left. + \sum_{m=0}^{l+1}\frac{S'_m(0\,|\,1)}{m!}\left[\frac{d^m}{dx^m}\frac{x^{1-s}}{1-s}\right]_{x=a/\omega+n}\right\},$$

principal values of the many-valued functions with respect to the axis of -1 being taken.

We can take a finite number N, such that, if $n > N$,

$$R(a/\omega)+n > R(a/\omega)+N = \eta > 1,$$

and, if N be sufficiently large, the modulus of $\arg(a/\omega+n) = \theta_n$ may be made as small as we please. Also, if $n > N$, $|a/\omega+n| > R(a/\omega+n) > \eta$.

The expansion
$$\frac{1}{(x+1)^s} = \sum_{t=0}^{\infty}(-)^t\frac{\Gamma(s+t)}{\Gamma(s)\,\Gamma(t+1)\,x^{s+t}}$$

is valid provided $|x| > 1$, for all finite values of $|s|$ however large.

Now it has been seen that,† if $|x/\omega| > 1$ and we expand

$$\{S_{-s,l}(x+\omega) - S_{-s,l}(x) - x^{-s}\}\,\omega^s$$

* *April 5th*, 1905.—This paragraph has been modified since the paper was communicated. The reader may compare Mellin, *Acta Soc. Sci. Fennicæ*, T. xxix., No. 4, pp. 47–48.

† *D.G.F.*, p. 341.

in ascending powers of $1/x$, the initial $(l+1)$ terms vanish, and we have

$$\sum_{r=2}^{\infty} \frac{P_{l+r}(s)}{x^{s+l+r}}\,\omega^{s+l+r},$$

where $$P_{l+r}(s) = (-)^{r+l}\frac{\Gamma(s+l+r)}{\Gamma(s)\,\Gamma(r+1)}\sum_{m=0}^{l+1}\frac{S'_m(0\,|\,1)\,r!}{m!\,(l+r+1-m)!}.$$

Thus $$|P_{l+r}(s)| < \kappa\,\frac{\Gamma(|s|+l+r)}{\Gamma(|s|)\,\Gamma(r+1)},$$

where κ is a definite finite positive quantity independent of r, N, n, and v. Now we have

$$|\omega^s\zeta(s,a)| \leqslant \left| -S_{-s,l}(a)\,\omega^s - \sum_{n=0}^{N-1}\omega^s\left[S_{-s,l}(x+\omega)-S_{-s,l}(x)-x^{-s}\right]_{x=a+n\omega}\right|$$
$$+\sum_{n=N}^{\infty}\sum_{r=2}^{\infty}\frac{|P_{l+r}(s)|}{|(a/\omega+n)^{s+l+r}|}$$

The last series is less than

$$\kappa\sum_{n=N}^{\infty}\frac{1}{|(a/\omega+n)^{s+l+2}|}\sum_{r=0}^{\infty}\frac{\Gamma(|s|+l+r+2)}{\Gamma(r+3)\,\Gamma(|s|)\,\eta^r}$$
$$<\kappa\sum_{n=0}^{\infty}\frac{e^{|\theta_N v|}}{(\eta+n)^{R(s)+l+2}}\,\frac{\Gamma(|s|+l)}{\Gamma(|s|)}\,\frac{\eta^2}{(1-1/\eta)^{|s|+l}}.$$

Suppose now that $|v|$ becomes very large. Choose N so that $N : |v| : \eta$ tends to a ratio of equality. Then $(1-1/\eta)^{|s|+l}$ tends to a finite limit. The series $\sum_{n=0}^{\infty}\frac{1}{(\eta+n)^{R(s)+l+2}}$ tends to zero. The product $\kappa e^{|\theta_N v|}\frac{\Gamma(|s|+l)}{\Gamma(|s|)}\eta^2$ tends to zero when multiplied by $e^{-(\pi-\epsilon)|v|}$. Hence the double series tends to zero when multiplied by this expression.

We have now to consider the first series

$$\left|-S_{-s,l}(a)\,\omega^s - \sum_{n=0}^{N-1}\left[S_{-s,l}(a/\omega+n+1\,|\,1)-S_{-s,l}(a/\omega+n\,|\,1)-(a/\omega+n)^{-s}\right]\right|$$

The number of terms in the series ultimately bears a ratio of equality to $|s|$. Therefore, if each tends to zero when multiplied by $e^{-(\pi-\epsilon)|v|}$, the same will be true of the sum.

Now, if we put $a/\omega+n = r_n e^{\iota\theta_n}$, we have $|\theta_n| \leqslant \pi-\epsilon$ and $|(a/\omega+n)^{s+r}|$, where r is an integer, $= r_n^{u+r}\exp\{-\theta_n v\}$.

Hence $\exp\{-(\pi-\epsilon')|v|\}/|(a/\omega+n)^{s+r}|$ tends to zero as $|v|$ tends to infinity, at least like $\exp\{(\epsilon'-\epsilon)|v|\}$. This is true for all finite values of r. And the same thing will be true if $(a/\omega+n)^{s+r}$ be multiplied by any algebraical polynomial or quotient of such polynomials in s.

Hence, finally, $e^{-(\pi-\epsilon')|v|}\,|\,\omega^s\zeta(s,a)\,|$ tends to zero as $|\,v\,|$ tends to infinity.

5. THEOREM II.—*If k be any complex quantity of finite modulus, if $|\arg(a/\omega)| = \pi-\epsilon$, where $0<\epsilon\leqslant\pi$, and if $|\arg(t/\omega)| = \epsilon'$, where $0\leqslant\epsilon'<\epsilon$, the integral*

$$\int\zeta(s,a)\,t^{s-k}\,\Gamma(s)\,\Gamma(k-s)\,ds$$

vanishes when taken along any part of the great circle at infinity for which u is finite and $R(s)>-(l+1)$, t^{s-k} having its principal value with respect to the axis of $-\omega$.

By the asymptotic formula for $\Gamma(z)$, where $|z|$ is large and not in the vicinity of the negative half of the real axis, we know that $\Gamma(s)\,\Gamma(k-s)$ behaves like

$$2\pi\cdot\exp\left[(s-\tfrac{1}{2})\log s-s+(k-s-\tfrac{1}{2})\log(k-s)-k+s\right]$$

$$= 2\pi\exp\left\{-(u+\iota v)\left[\log(k-s)-\log s\right]-\tfrac{1}{2}\log s+(k-\tfrac{1}{2})\log(k-s)-k\right\}$$

the principal values of the logarithms being taken

$$= 2\pi\exp\left[-\pi\,|v|+\text{ terms of lower order}\right],$$

for, when v is positive and large, $\log(k-s)-\log s = -\pi\iota$ approximately; and, when v is negative and large, $\log(k-s)-\log s = \pi\iota$ approximately.

The modulus of the subject of integration behaves approximately like

$$\left|\left(\frac{t}{\omega}\right)^{s-k}\right|\;|\,\omega^s\zeta(s,a)|\;\;|\,\Gamma(s)\,\Gamma(k-s)\,|,$$

where $(t/\omega)^{s-k}$ and ω^s have their principal values with respect to the axis of -1. And this expression by Theorem I. behaves at most like

$$\exp\{(\epsilon'-\epsilon)|v|\}.$$

Thus the integral along the part of the great circle specified vanishes.

COR.—The same integral is finite when taken along any line, drawn in the finite part of the plane parallel to the imaginary axis, which does not pass through the finite singularities of the subject of integration.

6. THEOREM III.—*The previous integral vanishes when taken along that part of the great circle at infinity for which u is large and positive (v having any real value), provided*

(1) $|\arg(a/\omega)| = \pi-\epsilon,$ *where* $0<\epsilon\leqslant\pi$;

(2) $|\arg(t/\omega)| = \epsilon',$ *where* $0\leqslant\epsilon'<\epsilon$;

(3) $|t| < |a+n\omega|,$ $n = 0, 1, 2, \ldots, \infty$;

(4) *the great circle passes between the points* $k+n$.

For the values of s considered $t^s\zeta(s, a) = \sum_{n=0}^{\infty} t^s/(a+n\omega)^s$.

Let $\log\dfrac{t/\omega}{a/\omega+n} = p_n+\iota q_n,$ and let $s = Re^{\iota\phi}$; then

$$|t^s\zeta(s, a)| < \sum_{n=0}^{\infty}\exp\{R(p_n\cos\phi-q_n\sin\phi)\}.$$

Now, when $|s|$ is large,

$$\Gamma(s)\Gamma(k-s) = \frac{\pi\Gamma(s)}{\sin\pi(k-s)\Gamma(1-k+s)}$$

behaves like
$$\frac{\pi}{\sin\pi(k-s)}\exp\{(k-1)\log s+1-k\}.$$

Hence $|\zeta(s, a)t^s\Gamma(s)\Gamma(k-s)|$, under the restriction (4), behaves at most like $\sum_{n=0}^{\infty}\exp\{R[p_n\cos\phi-q_n\sin\phi-\pi|\sin\phi|]\}$, and this will tend to zero like $\exp\{-\eta R\}$, where $\eta>0$, provided p_n is negative and $|q_n|<\pi$.

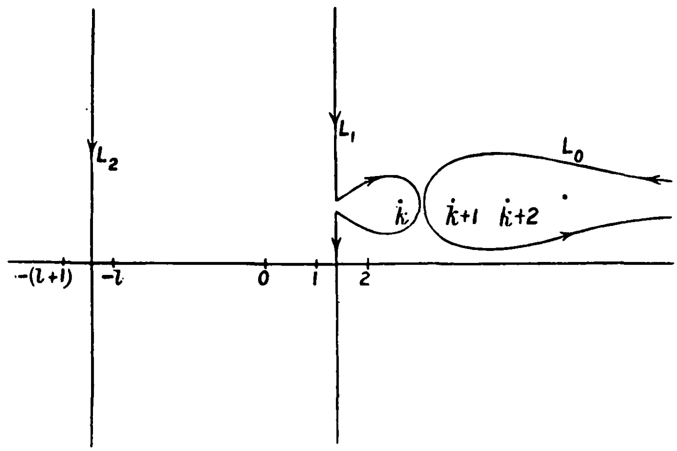
If p_n be negative, we have the condition (3). If $|q_n|<\pi$, we have $\arg(t/\omega)-\arg(a/\omega+n)\,|<\pi$, and therefore we have the conditions (1) and (2).

We therefore have the theorem stated.

7. THEOREM IV.—*If k, a, and t be any complex quantities of finite moduli subject to the conditions* (1), (2), (3), *and* (4) of § 6, *we have*

$$\zeta(k, a)-\zeta(k, a+t) = \frac{1}{2\pi\iota}\int_{L_0}\zeta(s, a)\,t^{s-k}\,\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}\,ds,$$

where t^{s-k} has its principal value with respect to the axis of $-\omega$, and *where the contour L_0 encloses the points $k+1$, $k+2$, ... as in the figure, but no other singularities of the subject of integration.*



We have, by Taylor's theorem,

$$\zeta(k, a+t) = \zeta(k, a) + \sum_{r=1}^{\infty} \frac{t^r}{r!} \zeta^{(r)}(k, a)$$

$$= \zeta(k, a) + \sum_{r=1}^{\infty} \frac{(-t)^r}{r!} \frac{\Gamma(r+k)}{\Gamma(k)} \zeta(r+k, a),$$

provided $|t| < |a+n\omega|$ for $n = 0, 1, 2, \ldots, \infty$. But

$$\frac{(-t)^r}{r!} \frac{\Gamma(r+k)}{\Gamma(k)} \zeta(r+k, a)$$

is the residue of the function

$$-\zeta(s, a)\, t^{s-k} \frac{\Gamma(s)\,\Gamma(k-s)}{\Gamma(k)}$$

at its pole $s = r+k$, since, if $s = r+k+\epsilon$,

$$\Gamma(k-s) = \frac{\Gamma(-\epsilon)}{(-\epsilon-1)(-\epsilon-2)\ldots(-\epsilon-r)} = \frac{(-)^{r-1}}{r!\,\epsilon} + \ldots.$$

Hence, under the conditions (1), (2), (3), and (4),

$$\zeta(k, a) - \zeta(k, a+t) = \frac{1}{2\pi\iota} \int_{L_0} \zeta(k, a)\, t^{s-k} \frac{\Gamma(s)\,\Gamma(k-s)}{\Gamma(k)}\, ds.$$

8. THEOREM V.—*Let L_1 denote a contour, as in the figure, parallel to the imaginary axis, cutting the real axis between $s = 1$ and $s = 2$, and with a loop to ensure that k is to the left and $k+1$ to the right of the contour; then, under the conditions* (1) *and* (2) *solely,*

$$\zeta(k, a) - \zeta(k, a+t) = \frac{1}{2\pi\iota} \int_{L_1} \zeta(s, a)\, t^{s-k} \frac{\Gamma(s)\,\Gamma(k-s)}{\Gamma(k)}\, ds.$$

The integral vanishes by Theorem III. when taken round an infinite contour to the right of the imaginary axis. The subject of integration is one-valued with the prescription assigned to t^{s-k}. When we deform the contour L_0 into the contour L_1, we pass over no poles of the subject of integration. Hence, by Cauchy's theorem, the required equality is valid under the four conditions of § 6. But each side of the equality is, by Theorem II., a uniform analytic continuous function of t even though the conditions (3) and (4) no longer hold. We may therefore eliminate these conditions and obtain the given theorem.

9. THEOREM VI.—*If* $R(k) > -(l+1)$, *and* $|\arg(a/\omega)| < \pi$,

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \zeta(k, a) + \sum_{n=0}^{l+1} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n}\int_{\infty,\,0}^{x} \frac{dx}{x^k}\right]_{x=m\omega}$$
$$+\frac{1}{2\pi\iota}\int_{L_2} \zeta(s, a)\,(m\omega)^{s-k}\,\frac{\Gamma(s)\,\Gamma(k-s)}{\Gamma(k)}\,ds,$$

where the many-valued functions have their principal values with respect to the axis of $-\omega$, *the lower limit of the integral is* ∞ *if* $R(k) > 1$ *and* 0 *if* $R(k) < 1$, *and where* L_2 *is a contour, as in the figure, parallel to the imaginary axis, cutting the real axis between* $s = -l$ *and* $s = -(l+1)$, *and such that the point* k *is on its positive side.*

We have [*G.F.*, p. 89]

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \zeta(k, a) - \zeta(k, a+m\omega).$$

Hence, putting $t = m\omega$ in the previous theorem, we have, if $|\arg(a/\omega)| < \pi$,

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \frac{1}{2\pi\iota}\int_{L_1} \zeta(s, a)\,(m\omega)^{s-k}\,\frac{\Gamma(s)\,\Gamma(k-s)}{\Gamma(k)}\,ds.$$

By Theorem II. we may apply Cauchy's theorem and modify the contour of the integral till it assumes the position of the line L_2. We must take account of the poles of the subject of integration which we pass over in the deformation. These poles are at the points

$$s = 1,\ 0,\ -1,\ \ldots,\ -l \qquad \text{and} \qquad s = k.$$

The residue at $s = k$ is $-\zeta(k, a)$. The residue at $s = 1$ is [*G.F.*, p. 95]

$$\frac{(m\omega)^{-k+1}}{\omega}\,\frac{\Gamma(k-1)}{\Gamma(k)}.$$

The residue at $s = -n$ $(n = 0, 1, \ldots, l)$ is [*G.F.*, p. 97]

$$\frac{(-)^{n-1} S_{n+1}'(a)}{n+1}\,\frac{(m\omega)^{-k-n}}{n!}\,\frac{\Gamma(k+n)}{\Gamma(k)} = -\frac{S_{n+1}'(a)}{(n+1)!}\left[\frac{d^n}{dx^n}\,x^{-k}\right]_{x=m\omega}.$$

Hence

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \zeta(k, a) + S_0'(a) \frac{(m\omega)^{1-k}}{1-k} + \sum_{n=0}^{l} \frac{S_{n+1}'(a)}{(n+1)!} \left[\frac{d^n}{dx^n} x^{-k}\right]_{x=m\omega}$$
$$+ \frac{1}{2\pi\iota} \int_{L_2} \zeta(s, a) (m\omega)^{s-k} \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} ds.$$

If, now, $\int_{\infty, 0}^{x} \frac{dx}{x^k}$ has a lower limit ∞ if $R(k) > 1$, 0 if $R(k) < 1$, the last series may be written*

$$\sum_{n=0}^{l+1} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n} \int_{\infty, 0}^{x} \frac{dx}{x^k}\right]_{x=m\omega}.$$

We thus have the theorem stated.

10. The form of the previous theorem obviously requires modification when k has any value which makes some of the sets of points

$$\left.\begin{matrix} k, \ k+1, \ \ldots \\ 1, \ 0, \ -1, \ \ldots, \ -l \end{matrix}\right\}$$

coincide. In these cases we may appeal to the principle of continuity to establish the final result. For example, when $k = 1$, the points $s = k$ and $s = 1$ coincide. In this case the residue of the function

$$\zeta(s, a) (m\omega)^{s-1} \Gamma(s) \Gamma(1-s)$$

at the point $s = 1$ is the coefficient of $1/\epsilon$ in the expansion of

$$-\frac{\pi}{\sin \epsilon\pi} \left[\frac{S_0'(a)}{\epsilon} - \frac{d}{da} \log \Gamma_1(a) \ldots\right] [1 + \epsilon \log m\omega + \ldots],$$

and is therefore $-S_0'(a) \log m\omega + \psi_1^{(1)}(a)$ where $\psi_1^{(1)}(a) = \frac{d}{da} \log \Gamma_1(a)$. This expression is evidently [*G.F.*, p. 95] the limit when $k = 1$ of

$$-\left[\zeta(k, a) + S_0'(a) \frac{x^{1-k}}{1-k}\right]_{x=m\omega}.$$

The previous theorem is thus true in general if limiting values be taken when infinite terms arise.

11. THEOREM VII.—*If* $R(k) > -(l+1)$ *and* $|\arg(a/\omega)| < \pi$,

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \zeta(k, a) + \sum_{n=0}^{l+1} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n} \int^x \frac{dx}{x^k}\right]_{x=m\omega} + J_l,$$

where, when m is very large, $|J_l|$ is of lower order than $m^{-l-R(k)}$.

* When $R(k) = 1$ and k is not real, such lower limit must be chosen as makes the integral vanish there.

We have $$J_l = \frac{1}{2\pi\iota}\int_{L_2} \zeta(k, a)\, m^{s-k}\, \omega^{s-k}\, \frac{\Gamma(s)\,\Gamma(k-s)}{\Gamma(k)}\, ds.$$

Hence $$|J_l| < \frac{1}{2\pi}\int_{L_2} \left|\frac{m^{s-k}}{\omega^k\,\Gamma(k)}\right| |\zeta(s, a)\, \omega^s\, \Gamma(s)\, \Gamma(k-s)\, ds|$$

$$< \frac{|m^{-l-k}|}{2\pi\,|\omega^k\,\Gamma(k)|}\int_{L_2} |\zeta(s, a)\, \omega^s\, \Gamma(s)\, \Gamma(k-s)|\,|ds|$$

$$< \frac{C}{m^{l+R(k)}},$$

where C is a finite quantity.

We have thus, provided $|\arg(a/\omega)| < \pi$, established the asymptotic expansion

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \zeta(k, a) + \sum_{n=0}^{\infty} \frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\,\frac{x^{1-k}}{1-k}\right]_{x=m\omega}$$

for all finite values of $|k|$ except $k = 1$. We have shown that the expansion is truly asymptotic in that, when m is large, the error committed by stopping at any term of the series has a modulus less than that of the last term retained. This expansion, the use of which was justified by the theory of divergent series, was made fundamental in my "Theory of the Gamma Function."

In the exceptional case $k = 1$, we have

$$\sum_{n=0}^{m-1} \frac{1}{a+n\omega} = -\psi_1^{(1)}(a) + \sum_{n=0}^{\infty} \frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\log x\right]_{x=m\omega}$$

12. THEOREM VIII.—*If $\phi(x)$ is a (possibly non-uniform) function which admits outside a circle of finite radius ρ the expansion $\sum_{r=2}^{\infty} c_r/x^r$, and if the points a, $a+\omega$, ..., $a+m\omega$, and $m\omega$ all lie outside this circle, we have, when $|\arg(a/\omega)| < \pi$, the Maclaurin sum-formula*

$$\sum_{n=0}^{m-1} \phi(a+n\omega) = \sum_{r=2}^{\infty} c_r\,\zeta(r, a) + \sum_{n=0}^{l+1} \frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\int_{\infty}^{x} \phi(x)\, dx\right]_{x=m\omega} + J_l,$$

where $|J_l|$ is, when m is large, at most of order $1/m^{l+2}$.

By Theorem VI. we have

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^r} = \zeta(r, a) + \sum_{n=0}^{l+1} \frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\int \frac{dx}{x^r}\right]_{x=m\omega}$$

$$+ \frac{1}{2\pi\iota}\int_{L_2} \zeta(s, a)\,(m\omega)^{s-r}\, \frac{\Gamma(s)\,\Gamma(r-s)}{\Gamma(r)}\, ds.$$

Hence

$$\sum_{n=0}^{m-1}\phi(a+n\omega)=\sum_{n=0}^{m-1}\sum_{r=2}^{l+2}\frac{c_r}{(a+n\omega)^r}+\sum_{n=0}^{m-1}\sum_{r=l+3}^{\infty}\frac{c_r}{(a+n\omega)^r}$$

$$=\sum_{r=2}^{l+2}c_r\zeta(r,a)+\sum_{n=0}^{l+1}\frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\sum_{r=2}^{l+2}\int_\infty^x\frac{c_r}{x^r}dx\right]_{x=m\omega}+\sum_{n=0}^{m-1}\sum_{r=l+3}^{\infty}\frac{c_r}{(a+n\omega)^r}$$

$$+\frac{1}{2\pi\iota}\int_{L_2}\zeta(s,a)(m\omega)^s\Gamma(s)\sum_{r=2}^{l+2}\frac{\Gamma(r-s)}{\Gamma(r)}\frac{c_r}{(m\omega)^r}ds$$

$$=\sum_{r=2}^{\infty}c_r\zeta(r,a)+\sum_{n=0}^{l+1}\frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\int_\infty^x\phi(x)\,dx\right]_{x=m\omega}$$

$$+\sum_{r=l+3}^{\infty}c_r\left\{\sum_{n=0}^{m-1}\frac{1}{(a+n\omega)^r}-\zeta(r,a)\right\}-\sum_{n=0}^{l+1}\frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\sum_{r=l+3}^{\infty}\int_\infty^x\frac{c_r\,dx}{x^r}\right]_{x=m\omega}$$

$$+\frac{1}{2\pi\iota}\int_{L_2}\zeta(s,a)(m\omega)^s\Gamma(s)\sum_{r=2}^{l+2}\frac{\Gamma(r-s)\,c_r}{\Gamma(r)(m\omega)^r}ds.$$

By hypothesis, when r is large, $|c_r|=\rho^r$ approximately. Again, if $|x|<|a+n\omega|$ $(n=0,1,2,\ldots,\infty)$, $\log\Gamma_1(a+x)$ admits the expansion

$$\sum_{r=0}^{\infty}\frac{x^r}{r!}\frac{d^r}{da^r}\log\Gamma_1(a)=\sum_{r=0}^{\infty}\zeta(r,a)\frac{(-x)^r}{r}\quad[G.F., \text{p. } 95].$$

Hence $|r^{-1}\zeta(r,a)|$, when $r>R$, is less than $(1+\epsilon)/\mu^r$, where μ is the minimum value of $|a+n\omega|$ and $\epsilon>0$. This minimum value is not zero, since $|\arg(a/\omega)|<\pi$.

By taking R sufficiently large, we may make ϵ as small as we please. Hence $|c_r\zeta(r,a)|$ is less than $\{\rho(1+\epsilon)/\mu\}^r$, and therefore $\sum_{r=2}^{\infty}c_r\zeta(r,a)$ is convergent if $a+n\omega$, $n=0,1,\ldots,\infty$, lies outside the circle of convergence of $\phi(x)$.

Again,

$$\sum_{r=l+3}^{\infty}c_r\left\{\sum_{n=0}^{m-1}\frac{1}{(a+n\omega)^r}-\zeta(r,a)\right\}=\sum_{r=l+3}^{\infty}c_r\sum_{n=m}^{\infty}\frac{1}{(a+n\omega)^r}=\sum_{r=l+3}^{\infty}c_r\zeta(r,a+m\omega).$$

This series is a finite quantity which, when m is large, is at most of order $1/m^{l+3}$.

In the next place

$$\sum_{n=0}^{l+1}\frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\sum_{r=l+3}^{\infty}\int_\infty^x\frac{c_r}{x^r}dx\right]_{x=m\omega}$$

$$=\sum_{n=0}^{l+1}S_n'(a)\int_\infty^x\sum_{r=l+3}^{\infty}\left[(-)^{n+1}\frac{c_r\,r(r+1)\ldots(r+n)}{n!\,x^{r+n+1}}dx\right]_{x=m\omega}.$$

Now the series $\sum_{r=l+3}^{\infty} \frac{r(r+1)\dots(r+n)}{n!} \frac{c_r}{x^{r+n+1}}$ is obviously convergent when $|x| > \rho$, and when m is large is at most of order $1/m^{l+n+4}$. Hence, when $|m\omega| > \rho$ this group of series is finite, and when m is very large at most of order $1/m^{l+3}$. Again, when $r = 2, 3, \dots, l+2$, the integrals

$$\frac{1}{2\pi\iota}\int_{L_2} \zeta(s, a)(m\omega)^s\,\Gamma(s)\,\frac{\Gamma(r-s)}{\Gamma(r)}\,\frac{c_r}{(m\omega)^r}\,ds$$

are each finite, and each, when m is large, is of order less than $1/m^{l+2}$. Thus we see that, under the conditions enunciated, we have

$$\sum_{n=0}^{m-1} \phi(a+m\omega) = \sum_{r=2}^{\infty} c_r\,\zeta(r, a) + \sum_{n=0}^{l+1} \frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\int_{\infty}^{x} \phi(x)\,dx\right]_{x=m\omega} + J_l,$$

where $|J_l|$, when m is large, is at most of order $1/m^{l+2}$.

We have excluded the first term c_1/x from the expansion of $\phi(x)$, since, when $k = 1$, it has been seen that a slight modification of the fundamental formula is necessary. The general theory remains unimpaired. When $\phi(x) = \sum_{r=1}^{\infty} c_r/x^r$, $|J_l|$ is at most of order $1/m^{l+1}$.

Further slight generalisations as to the nature of $\phi(x)$ may be made. We may state the general theorem—*The Maclaurin sum-formula* [§ 3 (A)] *is, when* $|\arg(a/\omega)| < \pi$, *valid for any (not necessarily uniform) function which has no singularities outside a finite circle outside which* a, $a+\omega$, $\dots$, $a+m\omega$, *and* $m\omega$ *all lie.*

13. THEOREM IX.—*If* $\phi(x)$ *is an integral function of order less than unity and* $|\arg a/\omega| < \pi$, *the Maclaurin sum-series is absolutely convergent, and we have the equality*

$$\sum_{n=0}^{m-1} \phi(a+n\omega) = \sum_{r=0}^{\infty} c_r\,\zeta(-r, a) + \sum_{n=0}^{\infty} \frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\int_0^x \phi(x)\,dx\right]_{x=m\omega},$$

where

$$\phi(x) = \sum_{r=0}^{\infty} c_r x^r.$$

If we put $k = -r$ in Theorem VII., we have

$$\sum_{n=0}^{m-1} (a+n\omega)^r = \zeta(-r, a) + \sum_{n=0}^{r+1} \frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\,\frac{x^{r+1}}{r+1}\right]_{x=m\omega},$$

for, when $n = r+2, r+3, \dots, l+1$, the terms of the series vanish, and, when $k = -r$, $1/\Gamma(k) = 0$, and therefore the integral J_l vanishes.

If, now, $\phi_k(x) = \sum_{r=0}^{k} c_r x^r$, we have

$$\sum_{n=0}^{m-1} \phi_k(a+n\omega) = \sum_{r=0}^{k} c_r \zeta(-r, a) + \sum_{r=0}^{k} \sum_{n=0}^{r+1} c_r \frac{S'_n(a)}{n!} \left[\frac{d^n}{dx^n} \frac{x^{r+1}}{r+1}\right]_{x=m\omega}$$

$$= \sum_{r=0}^{k} c_r \zeta(-r, a) + \sum_{n=0}^{k+1} \frac{S'_n(a)}{n!} \left[\frac{d^n}{dx^n} \sum_{r=n-1}^{k} \frac{c_r x^{r+1}}{r+1}\right]_{x=m\omega},$$

wherein, when $n = 0$ in the second series, r ranges from 0 to k,

$$= \sum_{r=0}^{k} c_r \zeta(-r, a) + \sum_{n=0}^{k+1} \frac{S'_n(a)}{n!} \left[\frac{d^n}{dx^n} \int_0^x \phi_k(x)\, dx\right]_{x=m\omega} \qquad (1)$$

Now [*G.F.*, p. 97] $\zeta(-r, a) = -S'_{r+1}(a)/(r+1)$.

Again, $S'_{r+1}(a)/(r+1)!$ is the coefficient of $(-z)^{r+1}$ in the expansion of $\frac{z e^{-az}}{1-e^{-\omega z}}$ in ascending powers of z, and this expansion has a radius of convergence equal to $2\pi/|\omega|$.

Hence, when r is large, $|c_r \zeta(-r, a)|$ behaves like $|c_r|\, r!\, \{|\omega|/2\pi\}^r$. Therefore, when k tends to infinity, $\sum_{r=0}^{k} c_r \zeta(-r, a)$ is convergent if $|c_r|$ behaves, when r is large, like $\{r!\}^{-1-\epsilon}$, where $\epsilon > 0$; that is to say, if the order of $\phi(x)$ is less than unity.

When $\phi(x)$ is of order greater than unity, $\sum_{r=0}^{\infty} c_r \zeta(-r, a)$ is divergent: it is only convergent in particular cases when $\phi(x)$ is of order unity.

In the next place, when k tends to infinity, the second series in the equality (1) becomes

$$\sum_{n=0}^{\infty} \frac{S'_n(a)}{n!} \left[\frac{d^n}{dx^n} \int_0^x \phi(x)\, dx\right]_{x=m\omega}.$$

This series is absolutely convergent if $\phi(x)$ is of order less than unity. For its general term is $\frac{S'_n(a)}{n!} \phi^{(n-1)}(m\omega)$. Now $\frac{\phi^{(n-1)}(m\omega)}{(n-1)!}$ is the coefficient of a^{n-1} in the expansion of $\phi(a+m\omega)$ in powers of a, and therefore $\left|\frac{\phi^{(n-1)}(m\omega)}{(n-1)!}\right|$ behaves, when n is large, like $\frac{1}{\{(n-1)!\}^{1+\epsilon}}$, where $\epsilon > 0$. Hence $\left|\frac{S'_n(a)}{n!} \phi^{(n-1)}(m\omega)\right|$ behaves like $\left\{\frac{|\omega|}{2\pi}\right\}^n \frac{1}{\{(n-1)!\}^{\epsilon}}$, and therefore the series is absolutely convergent.

We thus have the theorem stated.

14. We have now established the results stated in § 3 for the case of a single parameter. We see that it is hopeless to expect, when $\phi(x)$ is an integral function of order > 1, to apply the Maclaurin sum-formula, for in such cases $\sum_{r=0}^{\infty} c_r \zeta(-r, a)$ becomes divergent. I have, however, shewn in my previous paper on this subject that in such cases the theory of divergent series will often enable us to interpret such a formula. This mode of interpretation is, however, foreign to the range of ideas of the present investigation.

15. I will now indicate briefly the generalisation of the previous theory to the case of any number of parameters.*

Let ${}_rS_n(a \mid \omega_1, \ldots, \omega_r)$, or briefly ${}_rS_n(a)$, be the n-th r-ple Bernoullian function defined by the expression [*M.G.F.*, § 3]

$$\frac{(-)^r z e^{-az}}{\prod_{k=1}^{r} (1-e^{-\omega_k z})} = \sum_{s=1}^{r} \frac{(-)^s {}_rS_1^{(s+1)}(a)}{z^{s-1}} + \sum_{n=1}^{\infty} \frac{(-)^{n-1} {}_rS_n'(a) z^n}{n!},$$

coupled with the condition ${}_rS_n(0) = 0$. The expansion is valid when $|z|$ is less than the least of the quantities $|2\pi\iota/\omega_k|$ $(k = 1, 2, \ldots, r)$.

Take

$${}_rS_{-s,\,l}(x) = \sum_{m=0}^{l+r} \frac{{}_rS_m^{(r)}(0)}{m!} \frac{d^m}{dx^m} \left\{ \frac{x^{r-s}}{(r-s) \ldots (1-s)} \right\}.$$

As in my previous theory [*M.G.F.*, §§ 12 and 17], it is necessary to introduce a symbolic notation to simplify the cumbrous expressions to which the algebra otherwise gives rise.

Let F_r be a symbolic operator which is such that

$$F_r[\phi(x)]_{x=m\omega} = \phi(m_1\omega_1 + \ldots + m_r\omega_r) - \sum_{*=1}^{r} \phi(m_1\omega_1 + \ldots + {}^* + \ldots + m_r\omega_r)$$
$$+ \sum_{*=1}^{r} \sum_{*=1}^{r} \phi(m_1\omega_1 + \ldots {}^* + \ldots {}^* + \ldots + m_r\omega_r) - \ldots + (-)^{r-1} \sum_{k=1}^{r} \phi(m_k\omega_k).$$

In the first summation the star denotes that one of the ω's is to be omitted: in the second summation every two different pairs of ω's must be successively omitted, and so on.

We assume that in the Argand diagram the points $\omega_1, \ldots, \omega_r$ all lie on the same side of some straight line P through the origin. Let $1/L$ denote a line perpendicular to P drawn from the origin into the region in which the

* This theory is based on my researches in the domain of multiple gamma, Bernoullian, and Riemann ζ functions. An account will be found in "The Theory of the Multiple Gamma Function," *Cambridge Phil. Trans.*, Vol. XIX., pp. 374-425. This paper will be referred to as *M.G.F.*

ω's lie, and let L be the line conjugate to this line with respect to the real axis.

Further, let $(a+\Omega)^{-s} = \exp[-s \log(a+\Omega)]$, where

$$\Omega = n_1\omega_1+\ldots+n_r\omega_r,$$

the logarithm being rendered one-valued by a cross-cut along the axis of $-1/L$ (*i.e.*, the negative direction of the axis of $1/L$) and $\log(a+\Omega)$ being such that it is real when $a+\Omega$ is a positive quantity.

Then, if $R(s) > -(l+1)$, where l is a positive integer, the r-ple Riemann ζ function $\zeta_r(s, a\,|\,\omega_1, \ldots, \omega_r)$, or more briefly $\zeta_r(s, a)$, is defined by the equality [*M.G.F.*, § 20]

$$\zeta_r(s, a)$$
$$= -\sum_{n_1=0}^{\infty}\ldots\sum_{n_r=0}^{\infty}\left\{F_r\left[{}_rS_{-s,l}(a+\Omega+x)\right]_{x=\omega}+(-)^r\,{}_rS_{-s,l}(a+\Omega)-\frac{1}{(a+\Omega)^s}\right\}$$
$$+(-)^r\,{}_rS_{-s,l}(a).$$

When $R(s) > r$, we obtain

$$\zeta_r(s, a) = \sum_{n_1=0}^{\infty}\ldots\sum_{n_r=0}^{\infty}\frac{1}{(a+n_1\omega_1+\ldots+n_r\omega_r)^s}.$$

Let L denote a contour, embracing the axis L, similar to the contour defined in § 2. Further, let $(-z)^{s-1} = \exp\{(s-1)\log(-z)\}$ where the logarithm is rendered one-valued by a cross-cut along the axis L, and where $\log(-z)$ is such that it is real when z is real and negative. Then. provided a lies on the same side of the line P as the ω's, or, as we shall say, provided a is positive with respect to the ω's, we have

$$\zeta_r(s, a) = \frac{\iota\Gamma(1-s)}{2\pi}\int_L \frac{e^{-az}(-z)^{s-1}\,dz}{\prod_{k=1}^{r}(1-e^{-\omega_k z})},$$

where the contour L encloses no poles of the subject of integration except the origin.

16. We have [*M.G.F.*, p. 401], if k be any complex quantity,

$$\zeta_r(k, a)-\zeta_r(k, a+m_1\omega_1) = \sum_{n_1=0}^{m_1-1}\zeta_{r-1}(k, a+n_1\omega_1\,|\,\omega_2, \ldots, \omega_r).$$

Hence

$$\sum_{n_1=0}^{m_1-1}\ldots\sum_{n_r=0}^{m_r-1}\frac{1}{(a+n_1\omega_1+\ldots+n_r\omega_r)^k} = \zeta\,(k, a)+(-)^r F_r\left[\zeta_r(k, a+x)\right]_{x=m\omega},$$

or, since [*M.G.F.*, § 22] $F_r[1]_{x=m\omega} = (-)^{r-1}$,

$$\sum_{n_1=0}^{m_1-1} \dots \sum_{n_r=1}^{m_r-1} \frac{(-1)^r}{(a+n_1\omega_1+\dots+n_r\omega_r)^k} = F_r[\zeta_r(k, a+x)-\zeta_r(k, a)]_{x=m\omega}.$$

Again, it has been shewn [*M.G.F.*, § 56] that, if $R(s) > -(l+1)$ and $s = u+\iota v$, the quantity

$$|\zeta_r(s, a)x^s| e^{-\pi|v|},$$

where x^s has its principal value with respect to the axis of $-1/L$, tends exponentially to zero as $|s|$ tends to infinity, provided

$$\frac{x}{a+\Omega} = r_n e^{\iota\psi_n}, \qquad \Omega = n_1\omega_1+\dots+n_r\omega_r,$$

and $r_n < 1$, $|\psi_n| < \pi$.

We now define the contours L_0, ${}_rL_1$, and L_2 to be the same as those introduced in the diagram in § 7, except that ${}_rL_1$ cuts the real axis between r and $(r+1)$.

In the same way as formerly, we may now prove that

$$\frac{1}{2\pi\iota}\int \zeta_r(s, a)x^{s-k}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}\,ds$$

taken along the contours L_0, ${}_rL_1$, and L_2 is finite provided $r_n < 1$ and $|\psi_n| < \pi$, and that the integrals along the last two contours are finite provided $|\psi_n| < \pi$ for all values of $n_1, \dots, n_r$, whatever be the value of r_n.

Further, the integral along the contour L_0 is equal to that along the contour ${}_rL_1$.

Now the integral along the contour L_0

$$= \text{the sum of the residues of the subject of integration at } k+1, k+2, \dots$$

$$= \sum_{p=1}^{\infty} \zeta_r(p+k, a)\,x^p\,\frac{\Gamma(p+k)}{\Gamma(k)}\,\frac{(-)^{p-1}}{\Gamma(p+1)}$$

$$= -\sum_{p=1}^{\infty}\frac{x^p}{p!}\frac{\partial^p}{\partial a^p}\zeta_r(k, a)$$

$$\left[\text{since } \frac{\partial^p}{\partial a^p}\zeta_r(k, a) = \frac{(-)^p\,\Gamma(k+p)}{\Gamma(k)}\,\zeta_r(p+k, a)\right]$$

$$= \zeta_r(k, a)-\zeta_r(k, a+x).$$

Hence, under the sole set of conditions $|\psi_n| < \pi$, we have

$$\zeta_r(k, a+x)-\zeta_r(k, a) = -\frac{1}{2\pi\iota}\int_{{}_rL_1}\zeta_r(s, a)\,x^{s-k}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}\,ds.$$

Hence $\sum_{n_1=0}^{m_1-1} \dots \sum_{n_r=0}^{m_r-1} \frac{1}{(a+n_1\omega_1+\dots+n_r\omega_r)^k}$

$$= (-)^r F_r[\zeta_r(k, a+x) - \zeta_r(k, a)]_{x=m\omega}$$

$$= \frac{(-)^{r-1}}{2\pi\iota} \int_{rL_1} \zeta_r(s, a) F_r[x^{s-k}]_{x=m\omega} \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} ds,$$

k being any complex quantity, and the many-valued functions having their principal values with respect to the axis of $-1/L$.

17. We may now show that, *provided a lie on the same side of P as the ω's,*

$$\sum_{n_1=0}^{m_1-1} \dots \sum_{n_r=0}^{m_r-1} \frac{1}{(a+n_1\omega_1+\dots+n_r\omega_r)^k}$$

$$= \sum_{n=0}^{l+r} \frac{{}_rS_n^{(r)}(a)}{n!} F_r\left[\frac{d^n}{dx^n}\int^x \dots \int^x \frac{dx^r}{x^k}\right]_{x=m\omega}$$

$$+\zeta_r(k, a) + \frac{(-)^{r-1}}{2\pi\iota} \int_{L_2} \zeta_r(s, a) F_r[x^{s-k}]_{x=m\omega} \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} ds,$$

where the integration in $\int^x \dots \int^x$ is r times repeated, and such lower limits are taken that successive integrals vanish at them.

For, by Cauchy's theorem,

$$\frac{(-)^{r-1}}{2\pi\iota} \int_{rL_1} \zeta_r(s, a) F_r[x^{s-k}]_{x=m\omega} \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} ds$$

$= \frac{(-)^{r-1}}{2\pi\iota} \int_{L_2} +$ the sum of the residues of

$$(-)^r \zeta_r(s, a) F_r[x^{s-k}]_{x=m\omega} \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}$$

at the poles $s = k$ and $s = r, r-1, \dots, 2, 1, 0, -1, \dots, -l$.

The equality is limited by the condition

$$\left|\arg\left(\frac{m_1\omega_1+\dots+m_r\omega_r}{a+n_1\omega_1+\dots+n_r\omega_r}\right)\right| < \pi,$$

which must hold for all positive integral values of the n's, and when any but not all of the m's are zero. The inequality is equivalent to saying that a must not be such that

$$a+n_1\omega_1+\dots+n_r\omega_r = -\theta(m_1\omega_1+\dots+m_r\omega_r),$$

where θ is a real positive quantity. It is satisfied provided a lie on the same side of the line P as the ω's. In particular we may modify the formula so as to take $a=0$.

Now the residue of the function to be considered at $s=r-n$, where $n=0, 1, 2, \ldots, r-1$, is [*M.G.F.*, § 31]

$$(-)^{r-n}\,{}_rS_1^{(r-n+1)}(a)\,F_r[x^{r-n-k}]_{x=m\omega}\,\frac{\Gamma(k-r+n)}{\Gamma(k)}$$
$$=\frac{{}_rS_n^{(r)}(a)}{n!}\,F_r\left[\frac{d^n}{dx^n}\int^x\ldots\int^x\frac{dx^r}{x^k}\right]_{x=m\omega},$$

the integration being r times repeated and the lower limit each time being so chosen that the corresponding integral vanishes at it. The residue at $s=-n$, where $n=0, 1, \ldots, l$, is [*M.G.F.*, § 31]

$$\frac{{}_rS'_{n+1}(a)}{n+1}\,F_r\left[x^{-k-n}\right]_{x=m\omega}\frac{\Gamma(n+k)}{\Gamma(k)}\,\frac{(-)^n}{\Gamma(n+1)}$$
$$=\frac{{}_rS_{n+r}^{(r)}(a)}{(n+r)!}\,F_r\left[\frac{d^{n+r}}{dx^{n+r}}\int^x\ldots\int^x\frac{dx^r}{x^k}\right]_{x=m\omega}\quad[M.G.F., \S\ 6].$$

The residue at $s=k$ is

$$-(-)^r\,\zeta_r(k, a)\,F_r\,[1]_{x=m\omega}=\zeta_r(k, a).$$

We thus obtain the theorem stated.

It is evident that exceptional cases which must be treated by the calculus of limits arise when $k=1, 2, \ldots, r$. [Cf. *M.G.F.*, §§ 24 and 28.]

18. Substituting $p_1 m$ for $m_1, \ldots, p_r m$ for m_r, where the p's are finite positive integers, we now have the important asymptotic equality

$$\sum_{n_1=0}^{p_1m-1}\ldots\sum_{n_r=0}^{p_rm-1}(a+n_1\omega_1+\ldots+n_r\omega_r)^{-k}$$
$$=\zeta_r(k, a)+\sum_{n=0}^{l+r}\frac{{}_rS_n^{(r)}(a)}{n!}\,F_r\left[\frac{d^n}{dx^n}\,\frac{x^{-k+r}}{(1-k)(2-k)\ldots(r-k)}\right]_{x=pm\omega}+J_l,$$

where $|J_l|$ is, when m is very large, at most of order less than $\dfrac{1}{m^{l+R(k)}}$. The modulus of the last term of the series is of order $m^{-l-R(k)}$, and hence $|J_l|$ is of order less than this last term. The expansion is therefore truly asymptotic. It is the expansion obtained previously [*M.G.F.*, § 18], where its use was justified by the theory of divergent series.

19. Suppose now that a is positive with respect to the ω's, and that $\phi(x)$ admits outside a circle of finite radius ρ the expansion $\sum\limits_{k=r}^{\infty}\dfrac{c_k}{x^k}$; then, if the points $a+\Omega$ all lie outside this circle, we have, when m is large,

the equality

$$\sum_{n_1=0}^{p_1 m-1} \dots \sum_{n_r=0}^{p_r m-1} \phi(a+\Omega)$$

$$= \sum_{k=r}^{\infty} c_k \zeta_r(k, a) + \sum_{n=0}^{l+r} \frac{{}_rS_n^{(r)}(a)}{n!} F_r\left[\frac{d^n}{dx^n}\int_\infty^x \dots \int_\infty^x \phi(x)\,dx^r\right]_{x=pm\omega} + J_l,$$

where $|J_l|$ is at most of order m^{-l-r}.

The first r terms of the series for $\phi(x)$ have been omitted because of the modification of the fundamental formula thereby introduced. The general theory remains unimpaired: when $\phi(x) = \sum_{k=1}^{\infty} c_k/x^k$, J_l is at most of order m^{-l-1}.

The proof proceeds as for the case of a single parameter, and the general theory stated for a single parameter holds good.

20. If a is positive with respect to the ω's and $\phi(x)$ is an integral function of order less than unity, which admits the expansion $\sum_{k=0}^{\infty} c_k x^k$, we have the absolute equality

$$\sum_{n_1=0}^{p_1 m-1} \dots \sum_{n_r=0}^{p_r m-1} \phi(a+\Omega)$$

$$= \sum_{k=0}^{\infty} c_k \zeta_r(-k, a) + \sum_{n=0}^{\infty} \frac{{}_rS_n^{(r)}(a)}{n!} F_r\left[\frac{d^n}{dx^n}\int_0^x \dots \int_0^x \phi(x)\,dx^r\right]_{x=pm\omega}.$$

In other cases the series $\sum_{k=0}^{\infty} c_k \zeta_r(-k, a)$ is, in general, divergent.