These values show that the conductivity of sea-water for all frequencies as used in wireless telegraphy is very nearly equal to the value of the same for steady currents to within less than half a percent., the small differences obtained most probably being due to a small capacity effect of parts of the shunt on other parts.

A calculation of the true direct current conductivity from the resistance and the dimensions of the tube yielded the value

$$\cdot 0377 \Omega^{-1} \text{ per centimetre cube}$$

at $12^{\circ}5$ C. corresponding to

$$\sigma = 3 \cdot 77 \cdot 10^{-11} \text{ electromagnetic unit.}$$

As the conductivity varies very much with the temperature and the origin of the sample, a value of σ between 1 and $5 \cdot 10^{-11}$ is therefore appropriate to form the numerical basis for the theory of propagation of electromagnetic waves over the surface of the sea.

In conclusion I wish to thank Sir J. J. Thomson for his valuable advice and kindness in putting at my disposal the instruments of the laboratory.

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VII. *General Relativity without the Equivalence Hypothesis.*

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1. *Purpose and scope of the present inquiry.*

THE generalized theory of relativity as proposed by Einstein in 1912, and since that time repeatedly modified by himself and by his followers, has one very strong point, the requirement of general covariance of all physical laws, and one weak point, to wit, the originally so-called "equivalence hypothesis" which places gravitation on an entirely exceptional and privileged footing, bringing it into intimate connexion with the fundamental tensor which appears in the line-element of the world. I propose to retain the strong point and to reject the weak one, and thus to develop the implications of the general principle of relativity without the equivalence hypothesis, in fact, without privileging gravitation at all. This is the purpose of the

* Communicated by the Author.

present paper. And its scope will be a limited one, viz. to treat only some chief aspects of the physical implications of the Principle and to illustrate them to a certain extent, but by no means to try to embrace in a few generally covariant formulæ all the marvels of Nature. The strength of the "strong" point is indisputable and does not call for lengthy remarks; it amounts, in its ultimate analysis, to claiming that real, phenomenal, contents should be expressed, or expressible at least, in a way showing their independence of the particular language or scaffolding adopted. The weakness of the "weak" point, however, does require some explanations. First of all, then, independently of agreement or disagreement with experimental facts, the equivalence hypothesis is a vulnerable point because of its very special nature and of the great number of assumptions which it tacitly implies. In the next place, however, serious doubts with regard to its acceptability arise, according to my opinion, from the obstinately negative results quite recently obtained by St. John at the Mount Wilson Observatory*. The mean displacement (which according to Einstein should be about $\frac{1}{100}$ Å.U. towards the red) is at the sun's centre for 25 lines -0.001 , and for 18 lines $+0.0014$ Å.U., with a mean of zero for the 43 lines in the band spectrum of nitrogen (cyanogen); again, the mean displacement at the limb is 0.000 for 17 lines, and $+0.0063$ for 18 lines, with a mean of $+0.0018$ for the 35 lines. The final conclusion is that "within the limits of error there is no evidence of a displacement to longer wave-length, either at the centre or at the limb of the sun, of the order 0.008 Å." (*loc. cit.* p. 265). This negative result certainly outweighs the much more dubious and less numerous figures quoted in 1914 by E. Freundlich (*Phys. Ztschr.* xv. p. 370) in favour of Einstein's prediction. As a matter of fact, Einstein himself, while mentioning Freundlich's star-spectra testimony, is of the opinion that "a final verification is still [1916] outstanding" (*Ann. d. Physik*, xlix. p. 820). Notice also that the masses of Freundlich's stars can only be guessed in a rough manner while our sun's mass is sufficiently well known ($M/c^2 \doteq 1.5$ km.) to be substituted into Einstein's shift formula. Whence the obvious superiority of St. John's results. It is well-known that the predicted shift was one of the most immediate consequences of the equivalence hypothesis, even in its original form of 1911; in Einstein's recent theory

* Charles E. St. John, "The Principle of general Relativity and the Displacement of Fraunhofer Lines, etc.," *Astrophys. Journ.* xlvii., Nov. 1917, p. 249.

that shift or the decrease of vibration frequency is directly embodied in the tensor component g_{44} , the coefficient of $c^2 dt^2$, produced by the gravitating centre, viz., in a first approximation,

$$g_{44} = 1 - \frac{2M}{c^2 r}.$$

Another consequence of the theory, the bending of light rays by a gravitating mass (represented by the above in conjunction with other tensor components) still awaits its verification. Hitherto there is not the slightest evidence for the reality of such a phenomenon. It constitutes one of the chief points of the programme for the 1919 Solar Eclipse Expedition which seems particularly favourable for the proposed observations, as was pointed out by Sir Frank Dyson. Thus far the only positive and, one must confess, very conspicuous and fascinating success of Einstein's theory is the formula it gives without difficulty for the perihelion motion, amounting for instance in the case of Mercury to 43" per century, the famous excess which has occupied the attention of astronomers since the times of Le Verrier. But it so happens that this remarkable result relating to the secular motion of the perihelion is most vitally conditioned by the same tensor component, g_{44} *, which—to everybody's true regret—has thus discredited itself at the Mount Wilson Observatory.

Such being the state of things, one is justified, if not in condemning the equivalence hypothesis, at least in doubting its validity and in not attributing to it anything like the importance one cannot help ascribing to the general principle of relativity. Whence the natural desire of the writer to draw a sharp line between the two utterly heterogeneous elements and, rejecting the former, to investigate some general physical problems from the point of view of the latter alone. It may be well to notice that the importance and utility of the requirement of general covariance has been felt, and expressed with much force, by the mathematicians many years ago and in a much wider field, viz. that of "the geometry" of manifolds of any number of dimensions as represented by quadratic differential forms,—of which the physicist's "world" or space-time is but a particular example. Is not

* The perihelion motion, as a delicate feature of planetary motion, is, of course, given by g_{11} in co-operation with the remaining coefficients. To drop the variable part of g_{11} itself, after the others have been neglected, would make it impossible to get even the ordinary Keplerian planetary motion. In short, the second term of g_{11} is the chief term involved in that motion.

this need—of studying properties intrinsically connected with the manifold itself and of developing appropriate methods—clearly and with much emphasis expressed in such purely mathematical tracts as that* of Prof. Wright? But a strong tendency of that kind, together with great expectations for the future, manifests itself even in the old book of Lamé on curvilinear coordinates (1859); see the concluding paragraph, p. 367, of these memorable Lessons.

2. Space-time (world) of any constant curvature.

What is here called curvature is a certain invariant of the manifold and, as such, an intrinsic property of the manifold, as real as and possibly more real than the mass of a lump of matter. Whatever its value, nil, positive, or negative, it cannot be settled either by mere reasoning or by convention, but has to be found out by experiment or observation. Being ignorant as to its sign or amount, the best way is to leave it undetermined and to develop all formulæ with the corresponding degree of generality. Its evaluation is the task of the future physicist or, more likely, the astronomer. On the other hand, the reason why it is enough to limit oneself to *constant* curvature, *i. e.* the same through all times and everywhere, will be readily seen. Again, as concerns the mathematical technicalities, it is almost as easy to study a four-manifold of any constant curvature as a non-curved or homaloidal one (from *ὁμαλός* = even; an old name for flat or Euclidean space, of any number of dimensions). To deprive ourselves of generality would thus be a badly compensated sacrifice.

Let x_1, x_2, x_3, x_4 , the first three space-like, and the fourth time-like, be any coordinates fixing a world-point, and let the invariant line-element, determining the metric properties of the four-dimensional manifold, be given by the quadratic differential form

$$ds^2 = \Sigma \Sigma g_{ij} dx_i dx_j,$$

where $g_{ij} = g_{ji}$ are, in general, some functions of all the coordinates. If we pass to any other system x_i' , then the new g_{ij}' will be linear homogeneous functions of the g_{ij} , viz. in the usual abbreviated notation †,

$$g_{ij}' = \frac{\partial x_\kappa}{\partial x_i'} \frac{\partial x_\lambda}{\partial x_j'} g_{\kappa\lambda}.$$

* Cambridge Tracts, No. 9; cf. especially pp. 3–4.

† In which the sum signs are omitted, the tacit prescription being that the sum, from 1 to 4, is to be taken over each term in which a suffix occurs at least *twice*.

The g_{ij} thus constitute what is called a (covariant) *tensor* of rank two, viz. a symmetrical one, since $g_{ij}=g_{ji}$. Any array or matrix of 4×4 constituents a_{ij} which are transformed according to the same rule is a covariant tensor of the second rank. The differentials of the four coordinates themselves, which are transformed into

$$dx_i' = \frac{\partial x_i'}{\partial x_j} dx_j,$$

constitute a contravariant tensor of rank one or a four-vector. The reader is supposed to be acquainted with these and higher tensors and with their transformational properties*. Here, therefore, it will be enough to recall that the importance of all tensors consists in the linearity and homogeneity of their transformation formulæ; whence, if all the constituents of a tensor vanish in one system, they will vanish also in any other system of coordinates (provided, of course, that $\partial x_k/\partial x_i'$ are not infinite). Thus, if a physical law is written entirely in tensors, it will retain its form in passing from one system of reference to any other. Tensors, and tensors only, thus furnish the material for writing down such laws. (This does not imply, of course, that they necessarily will, but only that they may be obeyed by Nature.) The fundamental tensor, g_{ij} , will manifestly play a prominent part.

Now, to come to our subject. In Einstein's theory the tensor g_{ij} is intimately connected with gravitation so that the latter codetermines the metrical properties of the world or space-time. If there is no gravitation, or as we will say, far away from heavy masses and disregarding the feeble contribution due to electromagnetic and other energy, Einstein's world, at least that of 1916, is Euclidean or homaloidal, amounting to $ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2$, $x_4 = ct$, or to

$$g_{11}=g_{22}=g_{33}=-1, \quad g_{44}=1 \text{ (others zero)}. \quad . \quad . \quad (a)$$

In presence of gravitation this is changed. To make things plain by an illustration, suppose there is but one conspicuous body in the universe, say, our sun of mass M , the gravitational contribution of a testing particle or "planet" being negligible. Then, far away from the sun, and the farther

* Those readers who are not familiar with the subject can inform themselves in the easiest way by reading §§ 5-13 of Einstein's paper, *Ann. d. Physik*, xlix. (1916), and Chaps. I. and II. of Wright's 'Invariants,' Cambr. Tract No. 9 (1908).

the more exactly, the tensor is as in (a). On approaching the sun we have instead, sensibly,

$$g_{44}=1-\frac{2\alpha}{r}, \quad g_{11}=-1-\frac{2\alpha}{r}\frac{x_1^2}{r^2}, \quad g_{12}=-\frac{2\alpha}{r}\frac{x_1x_2}{r^2}, \text{ etc.}^*, \quad . \quad (b)$$

where $r^2=x_1^2+x_2^2+x_3^2$ and $\alpha=M/c^2$, which is about 1.5 km.; and this departure from the previous tensor cannot, of course, be transformed away; the change due to the sun is an essential one, a change of the metric properties all around that body. The eqs. of motion of a particle which in absence of the sun were given by the geodesic $\delta\int ds=0$ with the tensor (a), expressing uniform motion, are now again given by the geodesic $\delta\int ds=0$ with the modified tensor (b), however. It is this system of eqs. which yielded the remarkable result concerning the motion of the perihelion as a welcome accessory of the classical planetary motion. But what mainly interests us here is that according to Einstein's theory the tensor g_{ij} is changed not only within the sun but in all the circumjacent region of the world, the supplementary terms fading away with distance. And similarly in the presence of two or more lumps of "matter," which includes not only ordinary matter but also the electromagnetic field, for instance. The tensor components thus modified, as (b) for instance, are (approximate) solutions of Einstein's "field equations," certain generally covariant differential equations of the second order written down by him in terms of a tensor derived by contraction from the famous Riemann-Christoffel tensor of rank four. The particular form of his eqs. is here of no avail. It is enough to notice that, according to these eqs., within matter not only the several g_{ij} but also a certain differential invariant, the world-curvature, is changed in value, while outside of matter the modified g_{ij} are so distributed that the world-curvature remains nil as in absence of matter. To repeat it, however, even outside of matter the modification of g_{ij} is an essential one and cannot be transformed away.

To illustrate it by a bidimensional picture, imagine an ordinary surface populated by one-dimensional beings using one coordinate u for their space or extension, and another v for their time; their sun will be a line segment, Δu , and the world-tube of the sun a certain strip of the surface. Let our surface (and *their* "world"), in strict analogy to the above, be an ordinary plane in absence of the sun; then, in presence of

* In Einstein's formulæ (70), *loc. cit.* p. 819, α is a misprint for 2α , as the reader will readily convince himself.

that body, the Gaussian curvature within the strip will differ from zero' while outside it will remain nil, the neighbourhood of the solar strip being bent and possibly strained somehow but remaining developable upon a plane, as is a piece of a cylinder, say, or of a cone*. The geodesic lines of the surface, and therefore also the eqs. of motion in the vicinity of the strip, will then be changed correspondingly.

Now, what I propose is to *emancipate the fundamental tensor*, at least outside ordinary matter, *from the influence of gravitation* (as well as of any other agents), in spite of the well-known exceptional properties of gravitational fields. In other words, I propose to reject the gravitational "equivalence hypothesis," but to retain the postulate of general covariance of physical laws.

But here, at the very outset, a fundamental question presents itself. If the coefficients of the invariant line-element $ds^2 = g_{ij}dx_i dx_j$ are not manufactured or moulded by gravitating bodies, what does determine them physically? What determines the values of those tensor components, *if* in different cases they were to be essentially different, *i.e.* not reducible to one another by mere transformations of coordinates? A radical answer to this question easily suggests itself, and is already announced by having emphasized the "if." It is this:

Let the fundamental tensor g_{ij} be not different in different physical circumstances *but always, under all circumstances* (at least *in vacuo*) *essentially the same*. In other words, let us assume that ds^2 is, in vacuo, throughout the world essentially the same quadratic form, or that it is always possible to choose such coordinates $x_1, x_2, x_3, x_4 = ct$ in which ds^2 acquires a certain standard form, no matter whether suns or galaxies are near at hand or very remote. This amounts to postulating *homogeneity* of the four-manifold, which—in view of the principle of causality, in its heuristic aspect—seems to be a perfectly sound requirement.

Now, our world, as any multi-dimensional manifold, has a host of invariants, the differential invariants of various orders of its line-element. Thus, if the world is to be homogeneous (always *in vacuo*, at least), clearly all of its invariants must each have throughout one and the same numerical value; and since one of them (and even prominent amongst them) is the differential invariant of 2nd

* The idea of reducing physical phenomena to changes of curvature, especially in connexion with particles of matter, is not altogether new. It was suggested nearly fifty years ago by Clifford, with the only difference that Clifford had no opportunity of associating the time-coordinate with the remaining three. Cf. Clifford's 'Math. Papers,' p. 21, and his 'Common Sense of Exact Sciences,' 5th ed. p. 224 *et seq.*

order repeatedly called world-curvature, we shall claim for our world a *constant curvature*. This will henceforth be denoted by

$$k = 1/R^2.$$

Not pretending to know, or to be able to decide a priori what its sign or value might be, we shall leave them undetermined.

If k is positive, R is a real length, and if negative, then iR is a real length. If $|R| = \infty$, the world is homaloidal. Notice that, whatever the results of future observations, they can not lead to the conclusion that the world is strictly homaloidal but can give only a lower limit of $|R|$, say 10^9 astr. units or more; this under the assumption that the results of observation will be nil-effects, as in the case of Einstein's shift and of all the æther-drift observations and experiments. It may, however, happen that some observations will point to a lower and an upper limit of $|R|$ together with a definite sign of R^2 . Then, whatever the actual sign of R^2 , the result will be a very positive and an interesting one. It may run thus, for instance: $R^2 < 0$, and $10^8 < |R| < 10^{10}$, stating that the world has a negative curvature and fixing its amount between two, more or less narrow limits. I must warn the reader, however, that if he lives long enough to hear of such a result, he must not say that "the three-space" is negatively curved or hyperbolic of curvature $k = -10^{-18}$ astr. un.⁻², but only that the four-manifold or the world is so. In fact, if such be *the* world, he can choose in it a space* just of the curvature $k = -10^{-18}$ (but not below it), as well as a linear infinity of hyperbolic spaces, the homaloidal and all positively curved spaces without upper limit. This freedom of conventional or opportunist choice, limited only at the lower end by the invariant k , is based upon a remarkable and very general theorem on manifolds of any number of dimensions proved 33 years ago by Killing† and in part before him by Beltrami, which may shortly be rendered thus:—

Every n -dimensional space of constant curvature contains in itself spherical space forms (Kugelgebilde) of less dimensions (ν) whose Riemannian curvatures form a continuous manifold having no maximum but a minimum, viz. equal to the curvature of the n -space, this minimum curvature belonging to the ν -dimensional plane.

For our case it is enough to put in this admirable theorem $n=4$ and $\nu=3$. After this lengthy but (in view of certain recent misunderstandings) not altogether needless digression, let us return to our subject.

Having assumed a homogeneous world we have *eo ipso* accepted one of *constant* curvature, $k = \frac{1}{R^2}$. (This being at any rate a necessary condition, it will be still incumbent to show that the line-element to be written down presently leads also to all other constant invariants,—which task may be postponed to another opportunity.) Now, to obtain the

* And a homogeneous one, or hypersphere of three dimensions.

† W. Killing, *Die Nicht-Euklidischen Raumformen*, Leipzig, 1885, pp. 79–83. This excellent old book will be helpful to every student of general relativity.

corresponding quadratic form for the line-element let us take, with Beltrami-Killing's theorem as guide, for our particular three-space just the extreme appearing in that theorem, viz. a space of constant curvature equal to that of the world. The line-element $d\lambda$ of such a space, no matter what the sign of R^2 , can be written, as is well-known, in polar coordinates r, ϕ, θ , for instance,

$$d\lambda^2 = dr^2 + R^2 \sin^2 \frac{r}{R} (d\phi^2 + \sin^2 \phi d\theta^2).$$

Such being the space part of the line-element, let us use a system in which $g_{14} = g_{24} = g_{34} = 0$, which is always possible, and let us tentatively take $g_{44} = 1$. Thus, with $x_4 = ct$, the required expression for the line-element will be $dx_4^2 - d\lambda^2$, i. e.

$$ds^2 = c^2 dt^2 - dr^2 - R^2 \sin^2 \frac{r}{R} (d\phi^2 + \sin^2 \phi d\theta^2). \quad . \quad . \quad (1)$$

Attaching (mentally) the suffixes 1, 2, 3, 4 to the radial, the meridional, the latitudinal, and the time-direction, respectively, the equivalent fundamental tensor will conveniently be written, with $g_{ii} = g_i$,

$$g_1 = -1, \quad g_2 = -R^2 \sin^2 \frac{r}{R}, \quad g_3 = g_2 \cdot \sin^2 \phi, \quad g_4 = 1, \quad . \quad (2)$$

all other components being zero. That the form (1) or the corresponding tensor (2) do actually express (in a particular, convenient reference system) the said four-manifold, will be seen hereafter in more than one way.

Our original assumption is now reduced to the assumption that, outside of matter, it is always possible to choose such a system of coordinates in which the line-element takes the form (1). We shall refer to such variables by the short name of *natural* coordinates.

It will be well understood, however, that we do not postulate the invariance of the particular form (1) or of the corresponding tensor (2) which, of course, could be preserved only with respect to certain very particular transformations, whereas we require all physical laws to be generally covariant. Thus, in any not "natural" system of coordinates, which we will generally denote by u_1, u_2, u_3, u_4 , the line-element (1) will assume the form

$$ds^2 = g_{ij} du_i du_j, \quad . \quad . \quad . \quad . \quad (3)$$

where g_{ij} will be some linear homogeneous functions of the

natural components $g_1, \dots, g_4$ given in (2), viz., by the general transformation rule already quoted,

$$g_{ij} = \frac{\partial x_\kappa}{\partial u_i} \frac{\partial x_\kappa}{\partial u_j} (g_\kappa), \quad . \quad . \quad . \quad . \quad (4)$$

the office of () being to remind us that the values (2) are meant, if x_1, x_2, x_3, x_4 stand for r, ϕ, θ, ct . Similarly we shall have for the contravariant tensor $g^{ij} = \gamma_{ij}$ in any u -system, remembering that $(g^{ii}) = 1/(g_i)$,

$$\gamma_{ij} = \frac{\partial u_i}{\partial x_\kappa} \frac{\partial u_j}{\partial x_\kappa} \frac{1}{(g_\kappa)}, \quad . \quad . \quad . \quad . \quad (4a)$$

to be summed over $\kappa = 1$ to 4, as before. Thus, whenever required, it will be easy to pass from the above to any other system of reference.

Even taking for granted that (1) does represent a world of constant curvature and is thus equivalent to a generally covariant way of defining that manifold, yet the reader may feel formally unsatisfied by seeing the fundamental tensor g_{ij} thus to assume a variety of forms in different reference systems. It will be well, therefore, to give here already certain properties of that tensor which do preserve even their outward form in all systems of coordinates. In fact, let, in the very old notation, $(\mu\kappa\lambda)$ be the four-index symbols of Riemann belonging to the general quadratic form (3), certain differential expressions in g_{ij} to be quoted later on. These "symbols" are themselves the constituents of a tensor of rank four; in the case of n dimensions there are in the most general case only $\frac{n^2}{12}(n^2-1)$ linearly independent Riemann symbols*, which makes 20 for the four-dimensional world. Now, by a most remarkable, although half forgotten, theorem of general geometry†, the necessary and sufficient condition for a manifold to be developable upon a "sphere," *i. e.* to have constant Riemannian curvature, is that all the Riemann symbols $(\mu\kappa\lambda)$ should bear a constant ratio to the expressions $g_{i\kappa}g_{\mu\lambda} - g_{i\lambda}g_{\mu\kappa}$; that ratio being precisely what we have called the curvature of the manifold in question.

Thus, in our case, (3) being only (1) transformed, we have in *any* reference system, natural or not,

$$(\mu\kappa\lambda) = \frac{1}{R^2} (g_{i\kappa}g_{\mu\lambda} - g_{i\lambda}g_{\mu\kappa}). \quad . \quad . \quad . \quad . \quad (5)$$

* Cf. Wright, *l. c.* pp. 11 & 23.

† Killing, *l. c.* p. 232.

Passing from a system u to any other, u' , we shall have $(i\mu\kappa\lambda)' = \frac{1}{R^2} (g_{i\kappa}' g_{\mu\lambda}' g_{i\lambda}' g_{\mu\kappa}')$. The left-hand members being properly developed, (5) are ultimately partial differential eqs. for the g_{ij} . It is still to be proved that our above tensor components (2) do actually satisfy all these differential equations. This will be shown in the next section.

To conclude the present one, notice that by (1) the velocity of light, given by $ds=0$, becomes, in natural coordinates,

$$v = \frac{d\lambda}{dt} = c,$$

that is, constant and independent of direction, throughout the natural space (vacuum) of curvature $\frac{1}{R^2}$, whatever the value or the sign of the latter. The "rays" of light will be straight, shortest, lines in that particular space. Remember, however, that it is only *a* space, among many others at your disposal. In view of the above property we can call it *visual* or *optical space*. If then, by convention, we desire to choose as our reference space that among an infinity of others which has the above property, then there is certainly no objection to calling it simply *space* as a short for "optical space." And since all more remote objects are explored by optical means, such a choice will manifestly be by far the most convenient one. If $R^2 < 0$, so that

$$R \sin \frac{r}{R} = |R| \sinh \left| \frac{r}{R} \right|,$$

then the optical space will be hyperbolic or Lobatchewskyan, *i. e.* infinite but showing a defect in the angle sum of a triangle proportional to its area, and so on; if, in spite of the negative value of the invariant R^2 , somebody would prefer to use Euclidean geometry, there would be nothing to prevent him doing so; only in that case his optics will not be so simple. Similarly, *mutatis mutandis*, for $k=0$ or $k>0$. To put it in a few words: The world *is* so or so (to be explored), while space—even with the requirement of homogeneity—is, in very wide limits, a matter of convention, much as was predicted years ago by Poincaré.

Positive constant world-curvature is a feature of Einstein's 1917-theory; of course, only in absence of gravitation, and with the unavoidable cooperation of a certain hypothetical "world-matter." An interesting modification of Einstein's newest theory due to de Sitter will be

found in Monthly Notices R. A. S., for Nov. 1917. The latter is especially interesting because it does without the "world-matter," but has on the other hand $g_{44} = \cos^2 \frac{r}{R}$, instead of 1, to suit the gravitational field equations slightly amplified by Einstein. At any rate both authors are under the strange impression that the world cannot be infinite.

Finally, notice that in the case of a homaloidal world the theorem expressed by (5) gives $(\iota\mu\kappa\lambda) = 0$, as it should be, this being the well-known necessary and sufficient condition for ds^2 to be reducible to a form with all constant coefficients.

3. Mathematical Supplement to the preceding section.

In order to obtain the promised support for (1) as the expression for the line-element of a four-manifold of constant curvature, take $g_{14} = g_{24} = g_{34} = 0$ and measure u_1, u_2, u_3 along the principal axes of the three-dimensional linear vector operator $g_{\iota\kappa}$ (1, 2, 3). This operator (which itself is no relativistic entity, of course), being self-conjugated, has always such orthogonal axes, and three corresponding principal values, say, g_1, g_2, g_3 . Thus, u_1, u_2, u_3 being in general *curvilinear* coordinates, the expression for the line-element will become

$$ds^2 = g_1 du_1^2 + g_2 du_2^2 + g_3 du_3^2 + g_4 du_4^2, \quad \dots \quad (6)$$

and $\det. g_{ij} = g_1 g_2 g_3 g_4$, so that the components of the contravariant tensor will be, simply,

$$\gamma_{ii} = \frac{1}{g_i}, \text{ and nil for } i \neq j.$$

As space-coordinates of this kind can be employed conveniently the polar coordinates r, ϕ, θ or any other orthogonal curvilinear coordinates known since the times of Lamé. Now, what has been repeatedly called the curvature of that world which is given by the above differential form is itself an invariant (one of many) of the differential form, viz. proportional to

$$\mathcal{E} = \gamma^{\dot{i}\dot{j}} B_{\dot{i}\dot{j}},$$

that is, in our case, to

$$\mathcal{E} = \sum \frac{1}{g_i} B_{ii}, \quad \dots \dots \dots (7)$$

where

$$B_{ii} = \begin{Bmatrix} i\alpha \\ \beta \end{Bmatrix} \begin{Bmatrix} i\beta \\ \alpha \end{Bmatrix} - \begin{Bmatrix} ii \\ \alpha \end{Bmatrix} \begin{Bmatrix} \alpha\beta \\ \beta \end{Bmatrix} + \frac{\partial}{\partial u_i} \begin{Bmatrix} i\alpha \\ \alpha \end{Bmatrix} - \frac{\partial}{\partial u_\alpha} \begin{Bmatrix} ii \\ \alpha \end{Bmatrix}. \quad (7a)$$

These are the constituents of a covariant tensor of rank two, derived by Einstein from the Riemann-Christoffel tensor by "contraction" (equating two indices to one another and summing over them).

In (7a), the general definition of the three-index symbols of Christoffel is

$$\left\{ \begin{smallmatrix} ij \\ k \end{smallmatrix} \right\} = g^{\kappa\alpha} \left[\begin{smallmatrix} ij \\ \alpha \end{smallmatrix} \right] = \gamma_{\kappa\alpha} \left[\begin{smallmatrix} ij \\ z \end{smallmatrix} \right], \quad . \quad . \quad . \quad (8)$$

where

$$\left[\begin{smallmatrix} \mu\nu \\ \sigma \end{smallmatrix} \right] = \frac{1}{2} \left(\frac{\partial g_{\mu\sigma}}{\partial u_\nu} + \frac{\partial g_{\nu\sigma}}{\partial u_\mu} - \frac{\partial g_{\mu\nu}}{\partial u_\sigma} \right), \quad . \quad . \quad . \quad (9)$$

and, therefore, for the form (6), and with $u_i = x_i$, say, $x_1, x_2, x_3, x_4 = r, \phi, \theta, ct$, as in the preceding section,—

$$\left[\begin{smallmatrix} ii \\ i \end{smallmatrix} \right] = \frac{1}{2} \frac{\partial g_i}{\partial x_i}; \quad \left[\begin{smallmatrix} ii \\ \kappa \end{smallmatrix} \right] = -\frac{1}{2} \frac{\partial g_i}{\partial x_\kappa} (i \neq \kappa); \quad \left[\begin{smallmatrix} i\kappa \\ i \end{smallmatrix} \right] = \frac{1}{2} \frac{\partial g_i}{\partial x_\kappa},$$

and $\left[\begin{smallmatrix} ij \\ \kappa \end{smallmatrix} \right] = 0$ when i, j, κ are all different. In our case (8)

becomes

$$\left\{ \begin{smallmatrix} ij \\ \kappa \end{smallmatrix} \right\} = \Sigma \frac{1}{g_\kappa} \left[\begin{smallmatrix} ij \\ \kappa \end{smallmatrix} \right],$$

so that finally,

$$\left\{ \begin{smallmatrix} ij \\ i \end{smallmatrix} \right\} = \frac{1}{2g_i} \frac{\partial g_i}{\partial x_j}, \quad \left\{ \begin{smallmatrix} ii \\ j \end{smallmatrix} \right\} = -\frac{1}{2g_j} \frac{\partial g_i}{\partial x_j} (i \neq j), \quad . \quad (10)$$

and $\left\{ \begin{smallmatrix} i\kappa \\ \lambda \end{smallmatrix} \right\} = 0$ when all indices are different. Thus far g_1

etc. were any functions of x_1 , etc. or r, ϕ, θ, ct . Henceforth it will be enough to develop the sub-case in which

$$g_1 = g_1(r), \quad g_2 = g_2(r), \quad g_3 = g_2 \sin^2 \phi, \quad g_4 = g_4(r), \quad . \quad . \quad (11)$$

where, however, g_1, g_2, g_4 continue to be any functions of $r = x_1$. Then, by (7a) and (10), with dashes used for derivatives, and introducing the abbreviations

$$\left. \begin{aligned} h_1 &= \log(-g_1), \quad h_2 = \log(-g_2), \quad h_4 = \log g_4, \quad h = \log \frac{g_2^2 g_4}{-g_1}, \\ B_{11} &= h_2'' + \frac{1}{2} h_4'' + \frac{1}{2} (h_2' - h_1') h_2' + \frac{1}{4} h_4' (h_4' - h_1') \\ B_{22} &= \frac{1}{\sin^2 \phi} B_{33} = \frac{g_2}{2g_1} (h_2'' + \frac{1}{2} h_2' h') - 1 \\ B_{44} &= \frac{g_4}{2g_1} (h_4'' + \frac{1}{2} h_4' h'); \quad B_{i\kappa} = 0 (i \neq \kappa), \end{aligned} \right\} \quad . \quad (12)$$

and the corresponding expression for the curvature will be, by (7),

$$\mathcal{E} = \frac{1}{g_1} B_{11} + \frac{2}{g_2} B_{22} + \frac{1}{g_4} B_{44}, \quad . \quad . \quad . \quad (13)$$

This is valid for any fundamental tensor of the form (11); our case, given by (1) or (2), is but a sub-case of (11). Taking, as in (2), $g_1 = -1$, we have

$$\mathcal{E} = - \left\{ 2h_2'' + h_4'' + \frac{3}{2}h_2'^2 + \frac{1}{4}h_4'^2 + h_2'h_4' + \frac{2}{g_2} \right\}, \quad . \quad (13a)$$

where $h_2 = \log(-g_2)$, $h_4 = \log g_4$ are still any functions of $x_1 = r$. These more or less general formulæ have been given here since they may be useful in some other connexion. For the present, however, our purpose is only to show that the element (1) actually belongs to a world of constant curvature.

Now, putting, as in (1) or (2), $g_2 = -R^2 \sin^2 \frac{r}{R}$, we have

$$h_2' = \frac{2}{R} \cot \frac{r}{R}, \quad h_2'' = -\frac{2}{R^2} \operatorname{cosec}^2 \frac{r}{R},$$

and, equating $\mathcal{E}$ to a constant, the differential equation for $h_4 = \log g_4$ becomes

$$\frac{6}{R^2} - h_4'' - h_4'h_2' - \frac{1}{2}h_4'^2 = \mathcal{E} = \text{const.} \quad . \quad . \quad (13b)$$

This equation can be satisfied by $g_4 = \cos^2 ar$, $a = \text{const.}$, which would give

$$2a^2 + \frac{4a}{R} \tan ar \cot \frac{r}{R} = \mathcal{E} - \frac{6}{R^2},$$

and this can be satisfied either by $a = 1/R$, *i. e.* $g_4 = \cos^2 \frac{r}{R}^*$, with $\mathcal{E} = 12/R^2$, or more simply by

$$a = 0, \text{ and } \mathcal{E} = \frac{6}{R^2},$$

* This is mentioned here because Prof. de Sitter's element, in absence of gravitation (M. N., Nov. 1917) has precisely $g_{44} = \cos^2 \frac{r}{R}$. And this, with $g_{22} = -R^2 \sin^2 \frac{r}{R}$, was de Sitter's *only possibility*, since he has had to satisfy not only $\mathcal{E} = \text{const.}$ but also the four (amplified) "field equations" of Einstein, *i. e.* without "world-matter,"

$$B_{ii} = \frac{1}{2} \mathcal{E} g_{ii},$$

and these cannot be satisfied by $a = 0$ or $g_{44} = 1$. (With appropriate "world matter," as in Einstein's case, quoted also by de Sitter, the equations are so modified as to admit $g_{44} = 1$.) In the theory we are proposing, it will be remembered, there are no "field equations" to satisfy; the fundamental tensor has here nothing to do with gravitation.

and, therefore, $g_4=1$, which is precisely the case of the element (1). Thus, that line-element characterizes a space-time of constant curvature, the value of $k=1/R^2$ being one-sixth of $\mathcal{E}$, the above differential invariant. Q. E. D.

Such being the case, we know beforehand that the theorem (5) will hold. Yet it may be well to verify it by calculating directly some at least of the Riemann symbols corresponding to the tensor (2), in part also to make the reader more familiar with the handling of these remarkable symbols. Now, their general definition may be put into the form

$$(ijkh) = \frac{\partial}{\partial x_k} \left\{ \frac{ij}{h} \right\} - \frac{\partial}{\partial x_j} \left\{ \frac{ik}{h} \right\} + \left\{ \frac{ij}{\alpha} \right\} \left\{ \frac{\alpha k}{h} \right\} - \left\{ \frac{ik}{\alpha} \right\} \left\{ \frac{\alpha j}{h} \right\}, \quad (14)$$

to be summed as usual. Thus, with the normal form (6), and therefore, with the values (10), we find, for instance, remembering that $g_1 = \text{const.} = -1$,

$$(2112) = -\frac{1}{2} \frac{\partial^2 g_2}{\partial x_1^2} + \frac{1}{4g_2} \left(\frac{\partial g_2}{\partial x_1} \right)^2,$$

which becomes in the case of (2), with $x_1=r$, $(2112) = -\sin^2 \frac{r}{R}$. Most symbols vanish, as for ex. all (iii) , and more generally all $(ijkk)$,—this in virtue of the general property $(ijkh) = -(ijkh)$; again, many symbols that in general would not vanish do so in our case owing to $\frac{\partial}{\partial x_3} = 0$, and so on. Finally we find, f. ex., another non-vanishing symbol,

$$(3113) = -\frac{1}{2} \frac{\partial^2 g_3}{\partial x_1^2} + \frac{1}{4g_3} \left(\frac{\partial g_3}{\partial x_1} \right)^2 = -\sin^2 \frac{r}{R} \cdot \sin^2 \phi,$$

and so on. Thus we have

$$(2112) = \frac{1}{R^2}(g_2); \quad (3113) = \frac{1}{R^2}(g_2),$$

as it should be; for, by (5), $(2112) = \frac{1}{R^2}(g_{12}^2 - g_{22}g_{11})$, and

in our system $g_{12}=0$, $g_{11}=-1$; and so on. Having thus verified (5) in a pair of examples, we can safely apply that theorem to the tensor g_{ij} transformed from (2) to any system u_i . If we use, for instance, normal coordinates, i. e. such that $ds^2 = \Sigma g_{ii} du_i^2$, then all Riemann symbols vanish with

the exception of those of the type $(ikik)$, and these become

$$(ikik) = \frac{1}{R^2} g_{ii} g_{kk} = - (kiik), \quad . \quad . \quad . \quad (15)$$

for $i \neq k$. The most useful, of course, is the formula (5) itself, since it enables us to write at once all the constituents of the Riemann-Christoffel tensor for the assumed world in any system of reference.

4. Natural Systems of Reference.

In Section 2 the coordinates r, ϕ, θ, ct in which ds^2 assumes the simple form (1), and in which, therefore, light is propagated uniformly and isotropically, were called *natural* coordinates. Now, it is interesting to inquire whether, in a world with any fixed curvature, there is but one or a whole class of natural systems,—apart from such, of course, as can be derived from the original one by mere three-space transformations. From the older Relativity we know that for $R = \infty$, when ds^2 becomes

$$c^2 dt^2 - dr^2 - r^2 (d\phi^2 + \sin^2 \phi d\theta^2) = c^2 dt^2 - dx^2 - dy^2 - dz^2,$$

there is an infinity of natural systems all derivable from x, y, z, ct by the Lorentz transformation. It can be expected that something analogous will hold for any finite R , real or imaginary. Let us, therefore, try to find such natural systems. More definitely, starting from the form (1), let us ask for such transformations $r = r(r', \phi', \theta', ct')$, etc. which turn (1) into

$$c^2 dt'^2 - dr'^2 - R^2 \sin^2 \frac{r'}{R} (d\phi'^2 + \sin^2 \phi' d\theta'^2).$$

Then, at least all these systems (if no others) will share with the original one the “natural” property of simple optical behaviour and other properties therewith connected. In short, let us find the generalization of the Lorentz transformation for a space-time of any constant curvature.

It will be formally convenient to introduce $|R|$ as the unit of length*, and similarly $\frac{|R|}{c}$ as the unit of time, no matter how long these units might turn out to be when compared with those usually employed by the physicist or the astronomer.

In these units, and therefore with the light velocity $c = 1$, further with

$$l = it = t\sqrt{-1},$$

* With the exception, of course, of the case of $|R| = \infty$.

and with Sin written for sin or sinh according as $R^2 > 0$ or $R^2 < 0$, the original line-element becomes

$$-ds^2 = dr^2 + \text{Sin}^2 r (d\phi^2 + \sin^2 \phi d\theta^2) + dl^2, \quad . \quad . \quad (16)$$

all four variables being now pure, dimensionless, numbers ; and the required *natural systems* will be defined by

$$dr^2 + \text{Sin}^2 r (d\phi^2 + \sin^2 \phi d\theta^2) + dl^2 = dr'^2 + \text{Sin}^2 r' (d\phi'^2 + \sin^2 \phi' d\theta'^2) + dl'^2.$$

In order to find them it will be most convenient to use Weierstrass coordinates, viz. to introduce a fifth, auxiliary, coordinate x_5 , such that (with $x_4 = l = it$)

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = R^2 = \pm 1; \quad . \quad . \quad . \quad (17)$$

then our standard form will become

$$-ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2. \quad . \quad . \quad (18)$$

The upper sign in (17) will correspond to an elliptic, and the lower to a hyperbolic, world which thus appears, in $x_1, \dots, x_5, l$, as a four-dimensional *sphere* or *pseudosphere*, respectively *.

The well-known connexions between the x_i and r , etc. will be given presently. Whatever these are, if we require that, for the natural systems of reference,

$$dx_1^2 + \dots + dx_5^2 = dx_1'^2 + \dots + dx_5'^2, \quad . \quad . \quad (A)$$

and at the same time,

$$x_1^2 + \dots + x_5^2 = x_1'^2 + \dots + x_5'^2, \quad . \quad . \quad (B)$$

then if x_i' are retransformed into r' , etc., thus getting rid of the temporary or auxiliary fifth variable, the natural form (16) will reappear in dashed letters, as is required.

Now, (A) and (B) can be satisfied only by taking for x_i' linear functions of the x_i . Let us write, therefore,

$$x_1' = a_{10} + a_{11}x_1 + \dots + a_{15}x_5, \text{ etc.}$$

or, in the usual abbreviated notation

$$x_i' = a_{i0} + a_{i\kappa}x_\kappa, \quad i=1, 2, \dots, 5, \quad . \quad . \quad (19)$$

where $a_{i\kappa}$ are thirty constant coefficients. Such, however, being the case, we have also

$$dx_i' = a_{i\kappa}dx_\kappa,$$

so that all the equations yielded by (A) are already contained

* That is to say as the four-dimensional analogy of the ordinary two-dimensional sphere or pseudosphere. In the variables x_1 , etc. and t (real) equation (17) represents a one-sheeted hyperboloid for $R^2 > 0$, and a two-sheeted hyperboloid for $R^2 < 0$.

among those required by (B). These equations, common to (A) and (B), are

$$a_{1\kappa}^2 + a_{2\kappa}^2 + \dots + a_{5\kappa}^2 = 1, \text{ five eqs.,}$$

and

$$a_{1i}a_{1\kappa} + a_{2i}a_{2\kappa} + \dots + a_{5i}a_{5\kappa} = 0, \text{ ten eqs.,}$$

in all 15 equations. In addition to these (B) itself gives the condition

$$a_{10}^2 + a_{20}^2 + \dots + a_{50}^2 = 0,$$

and five more equations of the type

$$a_{10}a_{1\kappa} + a_{20}a_{2\kappa} + \dots + a_{50}a_{5\kappa} = 0. \quad (20)$$

The latter, however, being a system of homogeneous equations for the a_{i0} , we have either $\det. (20) = 0$, when (20) are reduced to four independent equations only, or $\det. (20) \neq 0$, and $a_{10} = a_{20} = \dots = a_{50} = 0$. In the former case we have in all $15 + 1 + 4 = 20$ equations for $5 \times 6 = 30$ coefficients, and in the latter case, the first 15 equations only for 25 coefficients. Thus, in either case the coefficients can be expressed by 10 free parameters, or the transformations in question are *ten-parametric*. Without sacrifice of generality we can take the second case, *i. e.* $a_{i0} = 0$, and therefore, the homogeneous transformations

$$x'_i = a_{i\kappa} x_\kappa, \quad i = 1, 2, \dots, 5, \quad (19a)$$

with 15 equations

$$\left. \begin{aligned} a_{1\kappa}^2 + a_{2\kappa}^2 + \dots + a_{5\kappa}^2 &= 1, \\ a_{1i}a_{1\kappa} + a_{2i}a_{2\kappa} + \dots + a_{5i}a_{5\kappa} &= 0, \end{aligned} \right\} \quad (21)$$

for the 25 coefficients $a_{11}, a_{12}, \dots, a_{55}$. And, as every pair of transformations (19a) can be replaced by a single transformation of the same kind, the said natural transformations constitute a *group*, *viz. a ten-parametric one*. The relations (21) are exactly of the same form as the six equations which are well-known in connexion with the ordinary transformation of Cartesian coordinates by a rotation of the system, or the 10 equations connected with the Lorentz transformation (with *fixed* origin of time and space). In fact, (21) taken by themselves would correspond to a typical orthogonal transformation in five variables. Since, however, our five coordinates are not independent but bound to one another by (17), our case is better expressed by saying that it is the four-dimensional analogy of the (rotation or) motion in

itself of an ordinary two-dimensional sphere; the difference, even with $R^2 > 0$, being that the coordinate $x_4 = l = it$ is imaginary. Keeping this well in mind one can characterize the required transformations by saying that any one of them is a *rotation of the world, sphere or pseudosphere, in itself*, similarly as the Lorentz transformations were described as rotations of the Minkowskian, homaloidal world. Thus, notwithstanding the world-curvature, the said group of transformations is characterized in much the same way as the Lorentz group (with one difference to be explained presently). The details of its analytical expression will, of course, be different for non-vanishing curvature.

The result can now be stated shortly by saying that all the *natural systems* of reference are derivable from one another by a *rotation of the world in itself*, whatever its curvature. To pass from a natural system of coordinates x_1 , etc., $x_4 = l$ to any non-natural, $u_1, \dots, u_4$, is to *distort* the world sphere or pseudosphere (without changing, however, its invariant curvature), while to pass from that system to any other natural system is to effect a *mere rotation* of the sphere or pseudosphere, according to the sign of R^2 . The corresponding group of transformations, deriving one natural system from another, could appropriately be called *the natural group*, of which then the Lorentz group would be a particular case corresponding to $R^2 = \infty$. It must be expressly stated that I do not propose to limit the theory to the natural group; on the contrary, I require every physical law to be covariant (or contravariant) with respect to *any* transformations of the coordinates. The "natural" ones are treated here at some length only because of their eminently simple properties, as a class of reference systems among an infinity of others.

Now, as to the difference in relation to the Lorentz group, hinted at a moment ago. It is well known that the so-called general Lorentz transformations, viz. including pure space rotations, constitute a *six-parametric group**, while our natural group is a *ten-parametric* one, since (21) are but fifteen equations for the twenty-five coefficients $a_{11}, a_{12}, \dots, a_{55}$. This, however, is only an apparent discrepancy. For the Lorentz group just mentioned relates to a *fixed origin* of x, y, z, l , or to put it shortly, to a fixed world-origin O . When we add the four degrees of freedom to choose a world-point as O , the result will precisely be $6 + 4 = 10$. That is

* The narrower or three-parametric Lorentz transformations do *not* constitute a group, although they contain the subgroups for parallel velocities. (Cf. the author's 'Theory of Relativity,' Macmillan (1914), p. 170.

Four of the eqs. (21) will then become $a_{51}=a_{52}=a_{53}=a_{54}=0$, so that the group ultimately becomes

$$x'_\iota = a_{\iota\kappa} x_\kappa \quad (\iota, \kappa = 1, \dots, 4); \quad x'_5 = x_5. \quad \dots \quad (19b)$$

There being now 10 conditions for 16 coefficients, the group with fixed origin is six-parametric. Moreover, apart from $x_5 = x'_5$, the eqs. (19b) are now exactly of the form of those expressing the Lorentz transformation.

In short, the natural systems, for any constant world-curvature, are obtained by subjecting the *Weierstrass coordinates* of any one of them (with $x'_5 = x_5$) to a Lorentz transformation.

Of the six-parametric group the pure space-rotations are of no interest. It will thus be enough to take the case of

$$x'_2 = x_2, \quad x'_3 = x_3; \quad x'_5 = x_5.$$

The conditions (21) then become

$$a_{11}^2 + a_{41}^2 = a_{14}^2 + a_{44}^2 = 1; \quad a_{11}a_{14} + a_{41}a_{44} = 0,$$

and are satisfied by $a_{11}=a_{44}=\cos \tilde{\omega}$, $a_{14}=-a_{41}=\sin \tilde{\omega}$, with $\tilde{\omega}$ as the only parameter, so that the transformation ultimately becomes

$$\left. \begin{aligned} x'_1 - ix'_4 &= e^{i\tilde{\omega}}(x_1 - ix_4), \\ x'_2, x'_3, x'_5 &= x_2, x_3, x_5. \end{aligned} \right\} \dots \dots \dots (22)$$

The first line is familiar from the older relativity, the only difference being that now it holds for the *Weierstrass coordinates* of the world-point. To translate this result into our original coordinates, put

$$\left. \begin{aligned} x_1, x_2, x_3 &= R \sin \psi \cdot \sin \frac{r}{R} [\cos \phi, \sin \phi \cos \theta, \sin \phi \sin \theta], \\ x_4 = it &= R \cos \psi; \quad x_5 = R \sin \psi \cdot \cos \frac{r}{R}, \end{aligned} \right\} \dots (23)$$

so that $x_1^2 + x_2^2 + x_3^2 + x_5^2 = R^2 \sin^2 \psi$, and $\sum_1^5 x_i^2 = R^2$,

identically, as required. Then we get, as a translation of (22), keeping for the moment R , to avoid confusion*, the following equations between r, ϕ, θ, ψ and their dashed

* Since $|R|$ was taken as unit length, R will here stand for 1 or $\sqrt{-1}$ according as the curvature is positive or negative.

correspondents, with $\tilde{\omega}$ as parameter,

$$\left. \begin{aligned} \theta' = \theta; \tan \frac{r'}{R} \sin \phi' &= \tan \frac{r}{R} \sin \phi; \sin \psi' \cos \frac{r'}{R} = \sin \psi \cos \frac{r}{R} \\ \sin \psi' \cdot \sin \frac{r'}{R} \cos \phi' &= \cos \tilde{\omega} \cdot (\sin \psi \sin \frac{r}{R} \cos \phi) + \sin \tilde{\omega} \cdot \cos \psi \\ \cos \psi' &= \cos \tilde{\omega} \cdot \cos \psi - \sin \tilde{\omega} \cdot \left(\sin \psi \cdot \sin \frac{r}{R} \cos \phi \right) \end{aligned} \right\} \quad (24)$$

The first of these equations, expressing axial symmetry, needs no further remarks. The third may be put aside for the moment (one of the 5 eqs. being a consequence of the others), but will be useful hereafter. The second, fourth, and fifth can be written, remembering that $\cos \psi \equiv it/R$,

$$\left. \begin{aligned} \left(1 + \frac{t'^2}{R^2}\right)^{1/2} R \sin \frac{r'}{R} \cos \phi' &= \cos \tilde{\omega} \left(1 + \frac{t^2}{R^2}\right)^{1/2} R \sin \frac{r}{R} \cos \phi + it \sin \tilde{\omega}, \\ t' &= t \cdot \cos \tilde{\omega} + i \sin \tilde{\omega} \cdot \left(1 + \frac{t^2}{R^2}\right)^{1/2} R \sin \frac{r}{R} \cos \phi, \\ R \tan \frac{r'}{R} \cdot \sin \phi' &= R \tan \frac{r}{R} \cdot \sin \phi. \end{aligned} \right\} \quad (25)$$

These are the required transformations valid for any constant curvature $k = \frac{1}{R^2}$ of the world. If the world is elliptic, we have, with $|R|$ as unit length,

$$(1 + t'^2)^{1/2} \sin r' \cos \phi' = \cos \tilde{\omega} \cdot (1 + t^2) \sin r \cdot \cos \phi + it \sin \tilde{\omega}, \text{ etc.,}$$

and if hyperbolic, then

$$(1 - t'^2)^{1/2} \sinh r' \cos \phi' = \cos \tilde{\omega} \cdot (1 - t^2)^{1/2} \sinh r \cos \phi + it \sin \tilde{\omega}, \text{ etc.}$$

The detailed discussion of the several interesting terms may be left to the reader. If the world is homaloidal, *i. e.* Minkowskian, we have, as the simplest particular case of (25),

$$\left. \begin{aligned} r' \sin \phi' &= r \sin \phi; \quad r' \cos \phi' = r \cos \phi \cdot \cos \tilde{\omega} + it \sin \tilde{\omega}, \\ t' &= t \cos \tilde{\omega} + i \sin \tilde{\omega} \cdot r \cos \phi, \end{aligned} \right\} \quad (25^0)$$

which are the well-known formulæ of the older relativity theory*; the first of (25⁰) expresses conservation of lateral dimensions, and the two others, with $\tilde{\omega} = \arctan \left(\frac{iv}{c} \right)$ and

* See, for instance, my book, p. 127 *et seq.*

The differential equations of the *geodesics* or shortest lines of the four-dimensional world are immediately seen, by (18) and (17), to assume in Weierstrass coordinates the simple form

$$\frac{d^2 x_\kappa}{ds^2} = \frac{1}{R^2} x_\kappa, \quad \kappa=1, \dots, 5, \quad . \quad . \quad . \quad . \quad (28)$$

where s is measured along the geodesic itself. Needless to say that these lines are of prime importance, first, because—owing to their definition $\delta \int ds = 0$ —they are invariant, and then because they offer the first example of generally covariant laws, viz. the law of motion of a free particle. Notice in passing that, in any coordinates u the eqs. of a geodesic will be

$$\frac{d^2 u_i}{ds^2} + \left\{ \begin{matrix} \kappa \lambda \\ i \end{matrix} \right\} \frac{du_\kappa}{ds} \frac{du_\lambda}{ds} = 0,$$

the g_{ij} , entering through the Christoffel symbols, being always as in (4), since they are not moulded by gravitation or by any other agent. But let us return to the eqs. (28). Their general integrals are

$$x_\kappa = a_\kappa \cos\left(\frac{is}{R}\right) + ib_\kappa \sin\left(\frac{is}{R}\right), \quad . \quad . \quad . \quad (29)$$

where a_κ , b_κ are ten constants satisfying, by (17), the conditions

$$a_\kappa a_\kappa = -b_\kappa b_\kappa = R^2, \text{ and } a_\kappa b_\kappa = 0, \quad . \quad . \quad . \quad (30)$$

to be summed according to the usual rule. These are the equations of a geodesic of the world, for any constant curvature. In natural units,

$$x_\kappa = a_\kappa \cosh s - b_\kappa \sinh s, \text{ for } R^2 > 0,$$

and

$$x_\kappa = a_\kappa \cos s + ib_\kappa \sin s, \text{ for } R^2 < 0.$$

In order to find the *shortest distance* s between any two world-points* whose Weierstrass coordinates are x_κ and y_κ , write (29) for the former and for the latter, multiply them in pairs and add; then, in virtue of (30), the result will be

$$\cos \frac{is}{R} = \frac{1}{R^2} (x_\kappa y_\kappa), \quad . \quad . \quad . \quad . \quad (31)$$

a well-known formula of multidimensional non-euclidean

* Not exceeding certain obvious limits.

geometry *. According to the sign of the world's curvature we have, in natural units, $(x_{\kappa}y_{\kappa}) = \cosh s$ or $-\cos s$.

Without insisting any further on these formulæ let us only draw from the last one this simple but interesting consequence:—The shortest distance of two world-points being manifestly an invariant, so is also $x_{\kappa}y_{\kappa}$. That is to say, in passing from one to any other *natural* system (for in such only we have the above Weierstrass coordinates with all their simple properties), the sum of products of such co-ordinates of two world-points retains its value. This simple property, although arrived at by following upon the shortest path from x to y , is at any rate independent thereof, and belongs to that pair of world-points as such. This *restricted invariant* † $(x_{\kappa}y_{\kappa})$ which, by (27), can be written, with $x_4 = it_x$, $y_4 = it_y$,

$$x_{\kappa}y_{\kappa} = \left(1 + \frac{t^2}{R^2}\right)^{1/2} (\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 + \xi_5\eta_5) - t_x t_y, \quad (32)$$

must, of course, follow also from the group equations (26). So, in fact, it does. As an instructive verification of the above line of reasoning write the first five eqs. (26) for the point x , and then for y , add the products of corresponding pairs of coordinates and take account of the 15 conditions between the α_{κ} immediately derivable from those given in (26). Then the result will be

$$x_{\kappa}'y_{\kappa}' = x_{\kappa}y_{\kappa}, \quad . \quad . \quad . \quad . \quad . \quad (33)$$

which was the property to be proved. This is the non-homaloidal analogy of the invariance of the “scalar product” of two four-vectors well-known from the older relativistic vector algebra. The property (33) will follow even more immediately by considering x_{κ}, y_{κ} as five-vectors in a five-space, restricted by (17) to have the fixed “size” R , and by remembering that the transformations in question are rotations of the world-sphere or pseudosphere. With the aid of (26) we can at once develop the whole vector algebra for a non-homaloidal world, as an obvious generalization of the older one. It is needless to show in detail how this is to be done. We shall construct the entities of this kind together with the rules of operating upon them every time these

* The circumstance that the usual “ s ” is here replaced by is is due to the negative sign in (18), which I retain for the sake of uniformity with the present notation of most authors.

† *I. e.* invariant with respect to the 10-parametric natural group of transformations.

should be particularly required. It will, of course, be always kept in mind that these entities are applicable only when the employed reference system is a natural one. But then they will offer conspicuous analytical facilities.

Having thus sufficiently explained the properties of this particular, but important, class of systems of coordinates, let us now pass to physical laws endowed with covariance for any transformations of the four variables. We shall begin with the fundamental electromagnetic laws since these embrace a vast and ever growing domain of phenomena.

5. *Electromagnetic Vacuum-Equations.*

Even before the publication of Einstein's outlines of a generalized theory of relativity, Kottler*, although confining his investigation to the Minkowskian world, has made the capital discovery that Maxwell's amplified equations, now generally known as the "vacuum-equations" or the fundamental equations of the electron theory, were *generally covariant*, i. e. with respect to any coordinate transformations. More correctly, this property belongs not to the usual four equations $\partial \mathbf{E} / \partial t + \rho \mathbf{v} = \text{curl } \mathbf{M}$, etc. containing (beside $\mathbf{v}$) only the two vectors $\mathbf{E}$, $\mathbf{M}$, but to the broader system of equations, with u_4 as time coordinate,

$$\frac{\partial \mathfrak{M}}{\partial u_4} + \text{curl } \mathbf{E} = 0, \quad \text{div } \mathfrak{M} = 0; \quad . \quad . \quad . \quad . \quad (I)$$

$$\text{curl } \mathbf{M} - \frac{\partial \mathfrak{E}}{\partial u_4} = \rho \mathbf{v}, \quad \text{div } \mathfrak{E} = \rho, \quad . \quad . \quad . \quad (II)$$

containing four vectors which will be shortly referred to as electric and magnetic forces ($\mathbf{E}$, $\mathbf{M}$) and polarizations ($\mathfrak{E}$, $\mathfrak{M}$). The latter appear as certain linear vector functions of the former, the nature of the corresponding vector operators being dependent upon the choice of the system of the four coordinates. In the homaloidal world and in any "legitimate" or Lorentz system these operators degenerated into idemfactors, so that $\mathfrak{E} = \mathbf{E}$, $\mathfrak{M} = \mathbf{M}$, reducing (I) and (II) to their usual form. The said property is based upon the familiar assumption of the invariance of electric charge.

It will be enough to recall here Kottler's proof but briefly, giving however at the same time an explicit translation of the involved tensors into components of $\mathbf{E}$, etc. taken along

* F. Kottler, 'Raumzeitlinien der Minkowski'schen Welt,' Vienna *Sitzungsber.*, vol. cxxi. II a, Oct. 1912, pp. 1659-1759; see especially p. 1685 *et seq.*

any orthogonal curvilinear coordinates, such coordinates being indispensable for the treatment of a non-homaloidal world.

Let $F_{\iota\kappa}$ be a six-vector or a covariant antisymmetric tensor of rank two, so that $F_{\iota\iota}=0$, $F_{\iota\kappa}=-F_{\kappa\iota}$. Then, with any coordinates $u_1, \dots, u_4$, the four equations

$$\frac{\partial F_{\iota\kappa}}{\partial u_\lambda} + \frac{\partial F_{\kappa\lambda}}{\partial u_\iota} + \frac{\partial F_{\lambda\iota}}{\partial u_\kappa} = 0 \quad . \quad . \quad . \quad . \quad (I a)$$

are generally covariant, because their left-hand members are the (only four different) components of a tensor, to wit of an antisymmetric one of rank three. Using ordinary Cartesians, for instance, it will be seen at once that (I a), *i. e.*

$$\partial F_{23}/\partial u_4 + \partial F_{34}/\partial u_2 - \partial F_{24}/\partial u_3 = 0, \text{ etc.,}$$

$$\partial F_{23}/\partial u_1 + \partial F_{31}/\partial u_2 + \partial F_{12}/\partial u_3 = 0,$$

are exactly of the form of the group (I) of electromagnetic equations. It requires, however, some care to find the correct translation of F_{23} etc. into the electro-magnetic components along coordinates of a more general kind. Such translation will be given presently. Meanwhile, to cover the group (II) of equations, consider the contravariant of $F_{\iota\kappa}$, that is, with the usual prescription as to summations,

$$F^{\iota\kappa} = g^{\iota\alpha} g^{\kappa\beta} F_{\alpha\beta} = \gamma_{\iota\alpha} \gamma_{\kappa\beta} F_{\alpha\beta}, \quad . \quad . \quad . \quad . \quad (35)$$

where $\gamma_{\iota\kappa}$ is the contravariant fundamental tensor corresponding to the chosen system u_i . Further, let σ_κ be a contravariant four-vector, embodying in its space-part the convection current, and g the determinant of the g_{ij} . Then the four equations

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial u_\kappa} (\sqrt{-g} F^{\iota\kappa}) = \sigma_\iota, \quad . \quad . \quad . \quad (II a)$$

whose left hand members are the components of a contravariant four-vector, will again be generally contravariant*. That these equations are exactly of the form of the group (II) of electromagnetic equations is seen at a glance, at least for Cartesians, and with some attention, also for more general coordinates.

Thus, the eight equations contained in (I a), (II a), together with the six relations (35) express a set of electromagnetic

* Einstein usually employs a system with $g=-1$, so that (II a) are simplified to $\partial F^{\iota\kappa}/\partial u_\kappa = \sigma_\iota$. But to fix thus the value of the fundamental determinant would hamper us unnecessarily.

vacuum-laws which are covariant with respect to any transformations of the four coordinates. It will be remembered that in Einstein's theory the linear relations (35) depend, among other things, also upon the gravitational field. But, as we have rejected his "equivalence hypothesis," our g_{ij} , and therefore also the relations between the polarizations and the forces, will depend only upon the chosen system of reference and, of course, upon the assumed fixed properties of the world.

It remains to write explicitly the components of the six-vectors F_{κ} and F^{κ} in terms of the components of $\mathfrak{M}$, etc. along the curvilinear axes of the system u_i , and thus to find also the explicit relations between the polarizations and the forces. [Then the original form (I), (II) of the Maxwellian equations, with u_4 as time, may be readopted and conveniently applied to any electromagnetic problem concerning empty space.]

It will be enough to do this for *orthogonal* curvilinear coordinates u_1, u_2, u_3 , with any u_4 as time. The corresponding form of the line-element then becomes, as in (6),

$$ds^2 = g_{\kappa\kappa} du_{\kappa}^2 = g_{11} du_1^2 + g_{22} du_2^2 + \dots + g_{44} du_4^2, \quad (36)$$

and $g = g_{11} \dots g_{44}$. In order to compare (Ia) with (I), with our purpose in view, remember that, A_1, A_2, A_3 being the components of any three-vector $\mathbf{A}$ along the curvilinear coordinates in question, its divergence is

$$\text{div } \mathbf{A} = w_1 w_2 w_3 \left[\frac{\partial}{\partial u_1} \left(\frac{A_1}{w_2 w_3} \right) + \text{etc.} \right],$$

which covers the second of (I), and that the first of (I), with u_4 for t , splits into

$$\frac{\partial}{\partial u_4} \left(\frac{1}{w_2 w_3} \mathfrak{M}_1 \right) + \frac{\partial}{\partial u_2} \left(\frac{E_3}{w_3} \right) - \frac{\partial}{\partial u_3} \left(\frac{E_2}{w_2} \right) = 0;$$

and two similar equations, where w_1 , etc. are defined by $ds_{\kappa} = \frac{du_{\kappa}}{w_{\kappa}}$; ds_1, ds_2, ds_3 being the components of the (space) line-element. By (34), $ds_1^2 = -g_{11} du_1^2$, etc., so that $\frac{1}{w_1} = \sqrt{-g_{11}}$, etc. Keeping this in mind, a glance at (Ia) will suffice to see that these eqs. become identical with (I) if we put

$$F_{23} = \mathfrak{M}_1 \sqrt{g_{22} g_{33}}, \text{ etc.}; \quad F_{14} = E_1 \sqrt{-g_{11}}, \text{ etc.}$$

Again the fourth of (II a), for $\iota=4$, compared with the second of (II) gives

$$F^{14} = -\frac{1}{\sqrt{-g_{11}g_{44}}} \mathfrak{E}_1, \text{ etc.}$$

and at the same time $\sigma_4 = \frac{1}{\sqrt{g_{44}}} \rho$. Finally, the first of (II), compared with (II a) for $\iota=1, 2, 3$ gives, as the remainder of the required dictionary,

$$F^{23} = M_1 \sqrt{\frac{g_{11}}{g}}, \text{ etc.}; \quad \sigma_1 = \frac{\rho v_1}{\sqrt{-g_{11}g_{44}}}, \text{ etc.},$$

The relations between the forces and the polarizations, which are obviously of prime importance, follow at once. In fact, by (35) which for orthogonal coordinates becomes

$$F^{ik} = F_{ik} / g_{ii} g_{kk}, \quad, \quad (35 a)$$

we have $M_1 = \mathfrak{M}_1 \sqrt{g_{44}}$, $E_1 = \mathfrak{E}_1 \sqrt{g_{44}}$, and so on.

Thus, for *any* orthogonal system of curvilinear coordinates w , collecting the scattered formulæ,

$$\left. \begin{aligned} F_{23} &= \mathfrak{M}_1 \sqrt{g_{22}g_{33}}, \text{ etc.}; \quad F_{14} = E_1 \sqrt{-g_{11}}, \text{ etc.} \\ F^{23} &= M_1 \sqrt{\frac{g_{11}}{g}}, \text{ etc.}; \quad F^{14} = -\mathfrak{E}_1 \frac{1}{\sqrt{-g_{11}g_{44}}} \\ \sigma_1 &= \frac{\rho v_1}{\sqrt{-g_{11}g_{44}}}, \text{ etc.}, \quad \sigma_4 = \frac{\rho}{\sqrt{g_{44}}} \end{aligned} \right\} . \quad (37)$$

where v_1, v_2, v_3 are the components of $\mathbf{v}$ along the curvilinear "axes" u_1, u_2, u_3 and similarly for the remaining four vectors; and the relations between the forces and polarizations assume the remarkably simple form

$$\mathfrak{M} = \frac{1}{\sqrt{g_{44}}} \mathbf{M}, \quad \mathfrak{E} = \frac{1}{\sqrt{g_{44}}} \mathbf{E}. \quad . . . \quad (38)$$

These latter are the simple relations supplementing (I) and (II). If u_4 is used as time, $\frac{1}{\sqrt{g_{44}}}$ plays the part of magnetic "permeability," and at the same time of dielectric "permittivity." Of course, according to (36), $dt = du_4 \sqrt{g_{44}}$ would be the more appropriate time-element in the system under consideration. And if g_{44} is 1 or any constant, then

by a mere change of the time unit the equations (I), (II) will assume their usual form $\partial \mathbf{M} / \partial t + \text{curl } \mathbf{E} = 0$, etc. This is certainly the case, without further assumptions, in any natural system, for then we have, as in (2), $g_{44} = g_4 = 1$. The dictionary, (37), is then also considerably simplified, giving $\sigma_{1,2,3} = \rho \mathbf{v}$, $\sigma_4 = \rho$, and so on. Remember that if $x_1, \dots, x_4$ be natural coordinates, say : r, ϕ , etc., then for any other system the value of g_{44} to be substituted in (38) is, by (4) and (2),

$$g_{44} = \left(\frac{\partial x_\kappa}{\partial u_4} \right)^2 g_\kappa = \left(\frac{\partial t}{\partial u_4} \right)^2 - \left(\frac{\partial r}{\partial u_4} \right)^2 - R^2 \sin^2 \frac{r}{R} \left[\left(\frac{\partial \phi}{\partial u_4} \right)^2 + \sin^2 \phi \left(\frac{\partial \theta}{\partial u_4} \right)^2 \right]. \quad (39)$$

Thus, not only in all natural systems but also in an infinity of others, for which this expression has a constant value (say, r, ϕ, θ independent of u_4 , although arbitrary functions of u_1, u_2, u_3 , and $t = u_4$), the *polarizations become identical with the forces*, and the *Maxwellian equations retain their ordinary form*, no matter whether the world is homaloidal or curved. And for any system of orthogonal coordinates whatever the *broadier* form, (I), (II) of Maxwell's equations is retained, with the reciprocal square root of (39) as permeability and permittivity.

Maxwell's equations will thus occupy a unique position in General Relativity, hitherto at least unparalleled by any other physically satisfactory laws, since they not only are generally covariant (as, for instance, Einstein's gravitational equations), but have also, unlike Einstein's recent products, the remarkable property of not being hitherto contradicted by any experiments or observations.

Questions relating to ponderomotive forces, electromagnetic energy etc. must for the present be omitted.

Here it will be enough to add that (I) and (II) give, in any natural system (owing to $\mathfrak{E} = \mathbf{E}$, $\mathfrak{M} = \mathbf{M}$) for the velocity of propagation of electromagnetic waves the constant value c , in agreement with the definition $ds = 0$ of the velocity of light (cf. p. 104), as it should be,—since light consists in such waves. Needless to repeat, that this vacuum velocity after our rejection of “equivalence”—is not modified by gravitation. And until any such phenomenon is discovered we are fully justified in adhering to the proposed purified theory.

6. *Dynamics of a Particle. Example of a covariant law of motion.*

Return to the general expression $ds^2 = g_{ij} du_i du_j$, where $u_1, \dots, u_4$ are any coordinates. Since du_i is a contravariant four-vector and ds an invariant, du_i/ds is again a contravariant vector; similarly $d^2 u_i/ds^2$, and so on. Such expressions could, therefore, be employed for the construction of generally covariant or contravariant laws of motion of a particle, endowed, say, with some invariant "mass" or inertia-coefficient. It seems more convenient, however, to adopt another method.

As was already mentioned, the equations of motion of a free particle are contained in $\delta \int ds = 0$, the variational equation of the world-geodesics. And the idea easily suggests itself to derive possible laws of motion of a non-free particle (or one "acted on by external forces") from similar variational equations after an appropriate amplification of the integrand. The purpose of the present section is to give only a very simple *example* of a generally covariant law of motion obtainable by this method (but by no means to develop the general dynamics of a particle or of a system of particles).

Let Φ , a function of all the u_i , be a tensor of rank zero or what is called a *scalar*, and therefore a general invariant. Then Φds will again be invariant, and the laws of motion embodied in an equation of the form

$$\delta \int (1 - 2\Phi) ds = 0 \quad . \quad . \quad . \quad . \quad . \quad (40)$$

will obviously retain their form in any reference system whatever, or will be generally covariant. Understanding by u_i the space-time coordinates of the particle in question, and considering u_i as fixed at the limits of the integral, develop (40) by the usual methods. Then the result will be a system of four differential equations, one of which is a consequence of the others,

$$\frac{d}{ds} \left[(1 - 2\Phi) \frac{\partial \dot{s}}{\partial \dot{u}_\kappa} \right] + (2\Phi - 1) \frac{\partial \dot{s}}{\partial u_\kappa} = - \frac{\partial \Phi}{\partial u_\kappa}, \quad . \quad . \quad (41)$$

where $\dot{s}^2 = g_{ij} \dot{u}_i \dot{u}_j$, the dot denoting the derivative with respect to a parameter which is ultimately made to coincide with s itself. Thus, for instance, if r, ϕ, θ, t are used as

coordinates,

$$\frac{d}{ds} \left[(1-2\Phi) \frac{dr}{ds} \right] + \frac{1}{2}(1-2\Phi) \left[\frac{\partial g_2}{\partial r} \dot{\phi}^2 + \frac{\partial g_3}{\partial r} \dot{\theta}^2 \right] = \frac{\partial \Phi}{\partial r}, \text{ etc. } (41a)$$

where $\dot{\phi} = d\phi/ds$, etc. The eqs. (41) can also be written without trouble in Weierstrassian or any other coordinates.

It is seen at once that the (invariant and, say, constant) mass m of the particle is replaced in the "momentum" by $(1-2\Phi)m$, and that Φ plays the part of a scalar *potential* of the "force" solliciting the particle.

Without entering, for the present, into a detailed interpretation of these equations of motion let us concentrate our attention upon the potential Φ , and let us see whether it is possible to construct a single differential equation for Φ , preferably of the second order, which would be generally covariant.

In order to obtain such an equation, start from

$$f_{i\kappa} = \frac{\partial^2 \Phi}{\partial u_i \partial u_\kappa} - \left\{ \begin{matrix} i\kappa \\ \lambda \end{matrix} \right\} \frac{\partial \Phi}{\partial u_\lambda} \quad . \quad . \quad . \quad (42)$$

which (Φ being a scalar) is a covariant tensor of rank two, viz. a symmetrical one, $f_{i\kappa} = f_{\kappa i}$, since $\left\{ \begin{matrix} i\kappa \\ \lambda \end{matrix} \right\} = \left\{ \begin{matrix} \kappa i \\ \lambda \end{matrix} \right\}$. Equating the ten different constituents of this tensor to those of some other tensor (say, within matter) or to zero, in empty space, we should have at once a covariant system of equations for our potential. But these would be too many for our purpose, seeing that there is but one function to satisfy them. What we require is a single differential equation. In order to obtain it, the idea easily suggests itself to build up the mixed tensor of rank four $g^{a\beta} f_{i\kappa}$ and to derive from it by a twofold "contraction" a tensor of rank zero or a scalar, thus:—Put $\alpha = i$ and sum over α from 1 to 4, so that the result will be a mixed tensor of rank two

$$H_\kappa^\beta = \sum_i g^{i\beta} f_{i\kappa};$$

here put $\beta = \kappa$ and sum up over κ , obtaining the scalar

$$H_\kappa^\kappa = H = \sum_{\kappa i} g^{i\kappa} f_{i\kappa},$$

or in abbreviated notation

$$H = g^{i\kappa} f_{i\kappa} = \gamma_{i\kappa} f_{i\kappa},$$

$\gamma_{i\kappa}$ being, as always, the contravariant fundamental tensor.

If $f_{i\kappa}$ is as in (42), H is a genuine scalar or general invariant. It is in fact Beltrami's second differential parameter $\Delta_2\Phi$ whose definition can, obviously, be retained for a manifold of any number of dimensions.

Dropping the operand, Φ , we thus obtain the comparatively simple invariant differential operator

$$\Omega = \gamma_{i\kappa} \left[\frac{\partial^2}{\partial u_i \partial u_\kappa} - \left\{ \begin{matrix} i\kappa \\ \lambda \end{matrix} \right\} \frac{\partial}{\partial u_\lambda} \right]; \quad . \quad . \quad . \quad (43)$$

terms to be summed up as before. Operating with this upon a scalar, as is the above "potential," and equating the result to another given scalar T , say, we shall have in

$$\Omega\Phi = T \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

a differential equation of the second order for Φ , invariant with respect to any transformations of the coordinates. The scalar T can, for example, be zero outside of matter and, say, proportional to appropriately measured "density of mass" ρ within matter. I do not say that Φ is the gravitational potential; I am only constructing an example of an abstract generally covariant law of motion of a particle.

Having thus ascertained the general invariance of the operator Ω it is obviously interesting to see what its form is like in some simple reference system, more especially in a natural system. First of all, in any system of orthogonal coordinates we have, with $\gamma_{ii} = 1/g_{ii} = 1/g_i$,

$$\Omega = \frac{1}{g_i} \left(\frac{\partial^2}{\partial u_i^2} - \left\{ \begin{matrix} ii \\ \kappa \end{matrix} \right\} \frac{\partial}{\partial u_\kappa} \right), \quad . \quad . \quad . \quad . \quad (43a)$$

and, in the natural system r, ϕ , etc., for instance, developing the second term of (43a), with the tensor (2), and $c=1$,

$$\begin{aligned} \Omega = \frac{\partial^2}{\partial t^2} - \frac{1}{R^2 \sin^2 \frac{r}{R} \cdot \sin \phi} \left\{ \frac{\partial}{\partial r} \left(R^2 \sin^2 \frac{r}{R} \cdot \sin \phi \frac{\partial}{\partial r} \right) \right. \\ \left. + \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial}{\partial \phi} \right) + \frac{1}{\sin \phi} \frac{\partial^2}{\partial \theta^2} \right\}. \quad . \quad . \quad . \quad (43b) \end{aligned}$$

In the second part of this operational equation the reader will easily recognize the Laplacian, $\nabla^2 = \text{div } \nabla$, for a space of constant curvature $\frac{1}{R^2}$; thus, in any natural reference system the whole operator assumes the simple and familiar

form $\Omega = \partial^2/\partial t^2 - \nabla^2$, and the differential equation (44) becomes, with $T = 4\pi\rho$,

$$\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi = 4\pi\rho, \dots \dots \dots (45)$$

where the scalar ρ is to be considered as some given function of the four variables. The "potential" Φ is thus propagated, in any natural system, with light velocity c , here assumed as unit velocity. If $\rho \neq 0$ within a certain region, then apart from waves (satisfying the reduced equation), Φ can be represented as the retarded potential of that distribution, or it can be treated by the well-known four-dimensional method. This completes the eqs. of motion (41) or (41 *a*). In a first approximation we should have Newtonian planetary motion, with obvious complications in higher approximations. As has already been said, the above is intended merely as an example of generally covariant laws of motion of a particle. Yet, after all, (45) or, in general coordinates, (44), with (41) may turn out to be helpful in describing gravitation. It is true that in Einstein's "Entwurf" of 1913 (§ 7) the question about the possibility of reducing the gravitational field to a scalar is answered in the negative. Einstein's objections, however, are based upon various assumptions which are by no means unavoidable. Again, his chief objection (*loc. cit.* p. 22) is based upon the restricted (Lorentzian) covariance of that reduction to a scalar which he had in mind when writing that paragraph, while our set of equations is generally covariant.

At any rate the above example has seemed sufficiently interesting and instructive to be inserted here. Notice that if the "attracting body," *i. e.* the region of $\rho \neq 0$, with all of its distributional properties, is itself at rest in a natural system, then this can with advantage be taken as the reference system, converting the retarded potential into an ordinary one. In general, however, this will not be the case, and—if higher approximations are at all contemplated—the simple potential would have to be replaced by an appropriate solution of the general equation (44), with (43) as the differential operator.

March 25, 1918.

Note, added June 16th.—If the central point-mass M is at rest in a natural reference system, we have, by (43 *b*),

$$\Phi = \frac{M}{R} \cot \frac{r}{R} \dots \dots \dots (46)$$

128 *General Relativity without Equivalence Hypothesis.*

The equations of motion of the planet, which can easily be written down according to (41*a*), yield a plane orbit. If this plane is taken to be $\phi = \text{const.} = \pi/2$, then, with $c=1$, and $u = \frac{1}{R} \cot \frac{r}{R}$, the differential equation of the orbit is

$$\frac{d^2 u}{d\theta^2} + (1+\gamma)u = \frac{M}{h^2}, \quad (47)$$

where

$$h = R^2 \sin^2 \frac{r}{R} \cdot \frac{d\theta}{dt} = \text{const.},$$

$$\gamma = M^2/h^2.$$

The solution is

$$u = \frac{M}{h^2(1+\gamma)} \{1 + e \cos(\theta \sqrt{1+\gamma})\}, \quad (48)$$

where $e = \text{const.}$ With appropriate "initial" data this is an ellipse, with eccentricity e and with moving perihelion. The motion of the perihelion is, per period T of revolution,

$$\epsilon = -2\pi \left\{ 1 - \frac{1}{\sqrt{1+\gamma}} \right\},$$

i. e. rejecting γ^2 and higher powers, and replacing c ,

$$\epsilon = -\frac{\pi M^2}{h^2 c^2} = -\frac{4\pi^3 a^2}{c^2 T^2 (1-e^2)}, \quad (49)$$

where a is the major semiaxis. Thus, the proposed equations give a *negative* (or retrograde) *secular motion of the perihelion*, equal to *minus* one sixth of that yielded by Einstein's gravitation theory.

By (49) we should have, per century, the perihelion motion multiplied by the eccentricity,

$$e\delta\varpi = -1''.48 \quad \text{for Mercury,}$$

and $-0''.010$, $-0''.011$, $-0''.021$ for Venus, Earth, and Mars respectively. According to Newcomb ('Astronomical Constants') the secular excess for Mercury, not accountable for by the perturbation due to all the other planets of our system, is $+8''.48 \pm 0.43$. Thus, if the above equations of motion are accepted, the *true excess* for Mercury would be still greater, viz.

$$e\delta\varpi = 8''.48 + 1''.48 = 9''.96 \quad (\pm 0.43),$$

or $\delta\varpi = 48''.44$. I understand from a conversation with Mr. Harold Jeffreys, who has already found a satisfactory representation of Mercury's $8''.48$ and of the motion of the node of Venus by means of a modification of Seeliger's zodiacal-light matter (M.N., R.A.S., December 1916, p. 112), that the above, increased, excess of about $10''$ could equally well, and possibly "more easily," be accounted for by an appropriate distribution of the said disturbing matter.