As Schuster has especially emphasized, the question here is really one of the *group-velocity*. Approximately homogeneous light consists of a train of waves in which the amplitude and wave-length slowly vary. A *local* peculiarity of amplitude or wave-length travels in a dispersive medium with the *group* and not with the *wave-velocity*; and the relative retardation with which we are concerned is the relative retardation of the groups. From this point of view it is obvious that, what is to be made to vanish is not (1) in which μ is the ratio of wave-velocities V_0/V , but that derived from it by replacing μ by U_0/U , or by V_0/U , where U is the group-velocity in the dispersive medium. In vacuum the distinction between U_0 and V_0 disappears, but in the dispersive medium

$$U = \frac{d(kV)}{dk}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)*$$

k being the reciprocal of the wave-length in the *medium*. If we denote as usual the wave-length in *vacuo* by λ ,

$$k = \frac{2\pi\mu}{\lambda} = \frac{2\pi V_0}{\lambda V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Accordingly

$$\frac{V_0}{U} = \frac{V_0 dk}{d(kV)} = \frac{d(\mu/\lambda)}{d(1/\lambda)} = \mu - \lambda \frac{d\mu}{d\lambda}. \quad . \quad . \quad . \quad . \quad (8)$$

Substituting this for μ in (1), we see that the position of the most distinct band is given by

$$R + \left(\mu - 1 - \lambda \frac{d\mu}{d\lambda} \right) D = 0, \quad . \quad . \quad . \quad . \quad (9)$$

in agreement with (5).

XXXIV. *On Momentum in the Electric Field.* By J. J. THOMSON, M.A., F.R.S., Cavendish Professor of Physics, Cambridge †.

I HAVE for some years made considerable use of the principle proved in 'Recent Researches,' p. 9, that momentum as well as energy is distributed throughout the electromagnetic field, and that changes in the momenta of magnets, of circuits conveying electrical currents, and other material systems in that field are accompanied by equal and opposite changes in the momentum of the field itself. Thus

* 'Theory of Sound,' § 191, 1877.

† Communicated by the Author.

the system consisting of the field and the material systems in it, form a self-contained system the momentum of which remains constant in magnitude and direction. If we take the view that the effects in the electromagnetic field are to be explained on dynamical principles by the agency of a medium filling the field, the conception of momentum in this medium is as essential and as fundamental as that of energy. As the consideration of the momentum in the field throws a good deal of light on electromagnetic phenomena, and leads in a simple way to some of the most important laws of electromagnetic action; I have thought that a discussion of the application of the method to some simple cases might be useful, and also help students to get a better grasp of the principle and the methods of its application. A few such cases have already been given in my 'Electricity and Matter.' Some of these are repeated here for the sake of completeness.

The basis of the method is the result (Phil. Mag. xxxi. p. 149, 1891; 'Recent Researches,' p. 9) that in the electromagnetic field, there is at any point, an amount of momentum equal per unit volume to $\mu H D \sin \theta$, where H is the magnetic force at the point, D the electric polarization (Maxwell's Electric Displacement), μ the magnetic permeability of the medium, θ the angle between the directions of H and D : the direction of the momentum is at right angles to H and D , drawn according to the scheme on fig. 1.

Fig. 1.

Thus P , Q , R the components of the momentum parallel to the axes x , y , z are given by the equations

$$P = \mu(\beta h - \gamma g),$$

$$Q = \mu(\gamma f - \alpha h),$$

$$R = \mu(\alpha g - \beta f).$$

Case of a Pole and Charged Point.—If we have a pole of strength m at A and a charge e at B, the preceding rule shows that the momentum at P is at right angles to the plane APB and is per unit volume equal to

$$\frac{1}{4\pi} \frac{e}{BP^2} \frac{m}{AP^2} \sin \text{APB}.$$

This distribution of momentum can easily be shown to be equivalent to a moment of momentum em around AB; the direction of rotation being opposite to that for a right-handed screw when the direction of translation is from A to B.

The resultant momentum in any direction vanishes; as this is true for all positions of A and B it follows that however the point and pole may move about in the field the momentum in any direction gained by the pole must be equal and opposite to that gained by the point; *i. e.*, the forces on the pole and point must be equal and opposite.

To find these forces, let us suppose that in a time δt the point B comes to B', A remaining at rest; then the moment of momentum in the field has changed from em along AB to em along AB'; *i. e.*, the moment of momentum in the field has changed by $em \sin \phi$ at right angles to AB in the plane ABA'. The material system consisting of the pole and point must have gained the moment of momentum lost by the field; the moment of momentum $em \sin \phi$ in the assigned direction is equivalent to momentum $\frac{em \sin \phi}{AB}$ at A at right angles to the plane BAA' and an equal and opposite momentum at B. Thus we may suppose A to have gained and B lost this momentum in time δt , the momentum gained by A per unit time, *i. e.* the force acting upon A, is $\frac{em \sin \phi}{AB} / \delta t$. Now $\sin \phi = \frac{BB' \sin \theta}{AB}$, where θ is the angle between BB' the direction of motion of B and AB; thus the force on A is

$$\frac{em \sin \theta}{AB^2} \cdot \frac{BB'}{\delta t} = \frac{em v \sin \theta}{AB^2},$$

where v is the velocity of B.

Thus a moving charge is acted upon by a force at right angles to the direction of motion, at right angles also to the magnetic force, and equal to $Hev \sin \theta$: hence the principle of momentum gives us in a very simple way the force on a moving charged point.

Exactly the same result will apply if the pole moves instead

of the point ; a moving pole in an electric field will be acted on by a force $mFv \sin \theta$, where F is the electric force, v the velocity, and θ the angle between F and v ; the moving pole will exert a force upon the fixed charge.

If pole and point move with the same velocity, AB will move parallel to itself ; there will be no change in the moment of momentum and no forces on the pole or point.

If we had any number of poles of strengths $m_1, m_2, \dots m_n$ at the points $A_1, A_2, \dots A_n$, and a system of charges $e_1, e_2, e_3, \dots e_p$ at $B_1, B_2, \dots B_p$, the distribution of momentum in the field has for its moment of momentum the resultant of moments of momenta $\Sigma m_i e_s$ acting along $A_i B_s$.

The following results are often useful in calculating the moment of momentum due to distributions of electricity over the surfaces of spheres.

The moment of momentum, due to a pole at A and a charge at B , contained within a sphere of radius r with its centre at either A or B is $\frac{1}{3} me \cdot \frac{r^2}{AB^2}$, if r is less than AB ; while the moment of momentum outside a sphere of radius r , when r is greater than AB , is $\frac{2}{3} me \frac{AB}{r}$

Let us suppose that the pole is shielded from the electric force due to the charge by being placed at the centre of a metal sphere of radius a ; then inside this sphere there is no electric force. If we calculate the moment of momentum in this case, taking into account the charge induced on the sphere round the pole, we shall find that it is equal to

$$me \left(1 - \frac{a^2}{AB^2} \right),$$

AB being supposed greater than a ; the axis of the moment is along AB . In this case the moment of momentum involves the distance between the pole and the point ; so that if we make the point move towards the pole, the moment of momentum in the field will change, the moment of momentum lost or gained by the field is gained or lost by the sphere, so that the sphere moves as if acted on by a couple $\frac{d\Gamma}{dt}$, where Γ is the moment of momentum of the field. Using the value of Γ just found, we find that the couple on the sphere is

$$\frac{2meVa^2}{AB^3},$$

where V is the velocity of the point along AB . The existence

of this couple would, on the more usual way of regarding the phenomenon, be explained by noticing that when the charged point moves towards or away from the sphere the distribution of electricity over the surface of the sphere changes. There is thus a movement of electricity over the surface of the sphere constituting a system of electric currents, and the mechanical forces arising from the action on these currents of the magnetic field due to the pole at the centre gives rise to the couple whose moment we have just calculated.

If we consider the term em in the expression for Γ we get the same forces on the pole and the point as if the shielding sphere round the pole were absent, while the term $\frac{ema^2}{AB^2}$ may easily be seen to give rise to a couple acting on the system consisting of the pole and sphere whose components parallel to the axes of x, y, z are respectively

$$\begin{aligned} ema^2 \left\{ u \frac{d^2}{dx^2} \frac{1}{r} + v \frac{d^2}{dx dy} \frac{1}{r} + w \frac{d^2}{dx dz} \frac{1}{r} \right\}, \\ ema^2 \left\{ u \frac{d^2}{dx dy} \frac{1}{r} + v \frac{d^2}{dy^2} \frac{1}{r} + w \frac{d^2}{dy dz} \frac{1}{r} \right\}, \\ ema^2 \left\{ u \frac{d^2}{dx dz} \frac{1}{r} + v \frac{d^2}{dy dz} \frac{1}{r} + w \frac{d^2}{dz^2} \frac{1}{r} \right\}, \end{aligned}$$

where x, y, z are the coordinates of B, $r=AB$, u, v, w are the components of the velocity of B. We see from this expression that the axis of the couple on the sphere is parallel to the magnetic force at A due to a doublet at B magnetized parallel to the direction of motion of B. If the moment of this doublet is equal to the velocity of B, then if H is the force at A due to this doublet, the couple on the sphere has its axis parallel to H and its moment equal to $Hema^2$.

Let us suppose that instead of a charged point we have a uniformly electrified spherical surface. Let e be the charge on the surface, a the radius of the sphere, B its centre. Then if the pole is at B, the moment of momentum about AB of the field is

$$em \left(1 - \frac{1}{3} \frac{a^2}{AB^2} \right).$$

Here, as in the last case, we have, in addition to the forces which would act on a charge e placed at the centre, and on

the pole at A, a couple acting on the charged sphere whose components along the axes of x, y, z are respectively

$$\begin{aligned} \frac{1}{3} e m a^2 \left\{ u \frac{d^2}{dx^2} \frac{1}{r} + v \frac{d^2}{dx dy} \frac{1}{r} + w \frac{d^2}{dx dz} \frac{1}{r} \right\}, \\ \frac{1}{3} e m a^2 \left\{ u \frac{d^2}{dx dy} \frac{1}{r} + v \frac{d^2}{dy^2} \frac{1}{r} + w \frac{d^2}{dy dz} \frac{1}{r} \right\}, \\ \frac{1}{3} e m a^2 \left\{ u \frac{d^2}{dx dz} \frac{1}{r} + v \frac{d^2}{dy dz} \frac{1}{r} + w \frac{d^2}{dz^2} \frac{1}{r} \right\}, \end{aligned}$$

where u, v, w are the components of the velocity of the centre of the sphere relative to the pole.

The moment of momentum remains unaltered when the charged sphere is rotated about a diameter: thus rotation of the sphere leaves the momentum in the electromagnetic field unaltered. Hence the momentum of the material systems in the field must remain unchanged, or the forces between the point and the rotating sphere must be equal and opposite.

Momentum due to Moving Charges.—If a charge e is moving with uniform velocity w parallel to the axis of z , then, if α, β, γ are the components of the magnetic force, f, g, h the components of the polarization in the dielectric, then, when the field has become steady, if we neglect squares and higher powers of w/V , where V is the velocity of light, we have

$$\begin{aligned} f &= -\frac{e}{4\pi} \frac{d}{dx} \frac{1}{r}, & g &= -\frac{e}{4\pi} \frac{d}{dy} \frac{1}{r}, & h &= -\frac{e}{4\pi} \frac{d}{dz} \frac{1}{r}, \\ \alpha &= 4\pi w g, & \beta &= -4\pi w f, & \gamma &= 0; \end{aligned}$$

therefore

$$\alpha = -ew \frac{d}{dy} \frac{1}{r}, \quad \beta = ew \frac{d}{dx} \frac{1}{r}, \quad \gamma = 0.$$

Here x, y, z are the coordinates of the point at which the magnetic force is (α, β, γ) and the dielectric polarization f, g, h . Hence by the rule the components p, q, r of the momentum per unit volume at P are given by the equations

$$\begin{aligned} p &= \beta h - \gamma g = -\frac{e^2 w}{4\pi} \frac{xz}{r^6}, \\ q &= \gamma f - ah = \frac{e^2 w}{4\pi} \frac{yz}{r^6}, \\ r &= \alpha g - \beta f = \frac{e^2 w}{4\pi} \frac{(x^2 + y^2)}{r^6}. \end{aligned}$$

Hence the momentum contained in the space between the surface of the moving charged sphere (whose radius we represent by a) and a concentric sphere of radius R is equivalent to a momentum through the centre of the sphere parallel to z , the direction of motion of the sphere, and equal to $\frac{2}{3}e^2w\left(\frac{1}{a} - \frac{1}{R}\right)$. The whole momentum outside the sphere is $\frac{2}{3}e^2\frac{w}{a}$.

If the moving electrified body were ellipsoidal instead of spherical, then if the velocities parallel to the axes of the ellipsoid were u, v, w , the momenta parallel to the axes of x, y, z would be Au, Bv, Cw respectively, where A, B, C are not equal. Thus in this case the momentum would not be in the direction of the velocity unless the velocity were parallel to a principal axis of the ellipsoid: except in this case there would be a couple acting on the ellipsoid. For let O be the centre of the ellipsoid at any time, after a time δt let the centre come to O' ; let I be the momentum at t acting in the direction OL ; then at $t + \delta t$ the momentum is I in the direction $O'L'$, $O'L'$ being parallel to OL . Thus the momentum in the field at the time $t + \delta t$ is not the same as it was at the time t , the difference is equivalent to the moment of momentum due to $-I$ at O and I at O' . The moment of momentum lost by the field must be transferred to the moving body; hence this will be acted upon by a couple in the plane LOO' . The components of the moment of this couple parallel to the axes of x, y, z are respectively

$$(B-C)vw, \quad (C-A)uw, \quad (A-B)uv.$$

Charged Sphere rotating about an Axis.—If a uniformly charged sphere of radius a is rotating with angular velocity ω about an axis through its centre parallel to z , we can easily prove that the components of magnetic force at a point distant r ($r > a$) from the centre of the sphere are given by the equations

$$\alpha = \frac{e\omega a^2}{3} \frac{d^2}{dx dz} \frac{1}{r},$$

$$\beta = \frac{e\omega a^2}{3} \frac{d^2}{dy dz} \frac{1}{r},$$

$$\gamma = \frac{e\omega a^2}{3} \frac{d^2}{dz^2} \frac{1}{r};$$

e being the charge on the sphere.

Since the components f, g, h of the electric polarization are given by the equations

$$f = -\frac{e}{4\pi} \frac{d}{dx} \frac{1}{r}, \quad g = -\frac{e}{4\pi} \frac{d}{dy} \frac{1}{r}, \quad h = -\frac{e}{4\pi} \frac{d}{dz} \frac{1}{r},$$

we see that p, q, r the components of the density of the momentum are given by

$$p = \frac{a^2 e^2 \omega y}{3 \times 4\pi r^6}, \quad q = -\frac{a^2 e^2 \omega x}{3 \times 4\pi r^6}, \quad r = 0;$$

thus the momentum at any point is parallel to the direction in which that point would move if the point were rigidly connected with the sphere.

The moment of momentum in the field about the axis of either x or y vanishes; about the axis of z it is equal to

$$\frac{2}{9} \frac{a^2 e^2 \omega}{a}.$$

If the shell had a mass M , the moment of momentum of the shell would be $\frac{2}{3} a^2 \omega M$. Thus when the charge is rotating, the effective mass is $\frac{1}{3} \frac{e^2}{a}$. While when the motion is translatory the effective mass is $\frac{2}{3} e^2/a$, i.e. twice that for rotatory motion. In the case of an ellipsoid charged with electricity, the moment of momentum in the field corresponding to rotations $\omega_x, \omega_y, \omega_z$ about the axes of x, y, z , would be of the form $A'\omega_x, B'\omega_y, C'\omega_z$ where A', B', C' are different. Thus, unless the axis of rotation coincided with one of the principal axes, the axis of the moment of momentum would not coincide with the axis of rotation and the motion could not be steady.

Momentum in the Field before the Steady State is reached.—Hitherto we have considered the momentum when the field due to the moving charged particle has reached a steady state; we shall now consider the distribution of momentum before the steady state is reached. Let us suppose that a charged sphere is suddenly set in motion by an impulse parallel to the axis of z , then at a time t after the impulse has ceased to act, the distribution of magnetic force is as follows. Let O be the point from which the centre of the charged sphere starts. With centre O and radii $Vt, Vt + \delta$, where δ is small, describe two spherical surfaces; then outside the outer of these surfaces there is no magnetic force,

and therefore no momentum; in the region between the spheres the magnetic force at a point P has its components

Fig. 2.

α, β, γ given by the equations

$$\alpha = \frac{ewy}{\delta \cdot OP^2}, \quad \beta = -\frac{ewx}{\delta \cdot OP^2}, \quad \gamma = 0;$$

the components f, g, h of the dielectric polarization, which are required for calculating the momentum, may for this purpose, as we are neglecting terms involving w^2 , be taken as unaltered by the motion and given by the equations

$$f = \frac{e}{4\pi} \frac{x}{OP^3}, \quad g = \frac{e}{4\pi} \frac{y}{OP^3}, \quad h = \frac{e}{4\pi} \frac{z}{OP^3}.$$

Hence, calculating the momentum in the shell by the ordinary rule, we find that it is parallel to z and equal to

$$\frac{2}{3} \frac{e^2 w}{OB},$$

where OB is the radius of the inner surface of the shell. Inside the inner shell the steady field has been established, so that if O' is the position of O at the time t , the magnetic force at P' will have for components

$$\alpha = ew \frac{y}{O'P'^3}, \quad \beta = -ew \frac{x}{O'P'^3}, \quad \gamma = 0.$$

As we are neglecting squares of w and $OO' = wt$, we may in calculating the momentum suppose that O' coincides with O : hence by the result given on p. 337, the momentum inside the sphere is parallel to z and equal to

$$\frac{2}{3} e^2 w \left\{ \frac{1}{a} - \frac{1}{OB} \right\},$$

where a is the radius of the charged sphere. Thus the total momentum, viz., the sum of that inside the inner surface of the shell and the shell itself, is equal to

$$\frac{2}{3} e^2 w \left\{ \frac{1}{a} - \frac{1}{OB} \right\} + \frac{2}{3} e^2 w \frac{1}{OB} = \frac{2}{3} \frac{e^2 w}{a};$$

thus it is constant and equal to that in the field when the steady state is reached and the pulse has passed off to an infinite distance; the pulse in passing over any region leaving part of its momentum behind it.

When a particle which has been in uniform motion is suddenly stopped, then the state of things at a time t after the stoppage is as follows:—Let O be the centre of the sphere now at rest; describe two spheres centre O , radii Vt , $Vt + \delta$; then inside the inner sphere there is no magnetic force; in the space between the spheres the magnetic force is given by the equations

$$\alpha = -\frac{ewy}{\delta \cdot OP^2}, \quad \beta = \frac{ewx}{\delta \cdot OP^2}, \quad \gamma = 0,$$

while outside the outer sphere the magnetic force is the same as that before the stoppage of the particle and is given by

$$\alpha = \frac{ewy}{OP^3}, \quad \beta = -\frac{ewx}{OP^3}, \quad \gamma = 0,$$

We can easily see that the momentum in the shell is in this case equal and opposite to that in the space surrounding it, so that the total momentum in the field vanishes as soon as the charged point is reduced to rest.

Let us now consider the case of a charged particle suddenly set in motion in an external magnetic field. Suppose that this field is uniform and parallel to the axis of x , let it be due to two parallel plates of positive and negative magnetism, the plates being infinite and parallel to yz . Let us find the momentum due to the external magnetic field before the pulse started by the sudden movement of the sphere has reached the layers of magnetism. To calculate this momentum, we have to know the distribution of the dielectric polarization after the sphere has been started. Now in the pulse there is an additional polarization whose components f' , g' , h' are

$$f' = \frac{ew}{4\pi V\delta} \frac{xz}{r^3}, \quad g' = \frac{ew}{4\pi V\delta} \frac{yz}{r^3}, \quad h' = \frac{-ew}{4\pi V\delta} \frac{(x^2 + y^2)}{r^3},$$

where V is the velocity of light. The components of momentum due to a field H parallel to x and this polarization

are respectively :—

$$0, \quad \frac{Hew}{4\pi V\delta} \frac{(x^2+y^2)}{r^3}, \quad \frac{Hew}{4\pi V\delta} \frac{yz}{r^3}.$$

Integrating through the shell the components parallel to x and z vanish, and that parallel to y becomes

$$\frac{2}{3} \frac{Hew}{V} \cdot OA;$$

where OA is the inner radius of the sphere, and is equal to Vt . Hence the momentum in the shell parallel to y is equal to

$$\frac{2}{3} Hew \cdot t.$$

The electrostatic field inside the shell is at the time t that arising from a point O' which is not at the centre of the shell, but such that $OO' = wt$. Calculating the momentum due to this field and the external magnetic force H , we find that is $\frac{H}{3} HeOO' = \frac{1}{3} Hewt$; hence the total momentum on the shell and inside it is

$$Hewt.$$

Thus on the hypothesis that the pulse formed in the magnetic field is the same as that formed when the external magnetic field is absent, we conclude that the momentum in the field at right angles to the direction of the magnetic force, and also to the direction of motion of the charged particle, is increasing at the rate Hew . By the principle of the conservation of momentum, the material system, *i. e.* the charged body and the magnet, must be losing momentum at that rate, *i. e.* they must be acted on by a force parallel to y equal to $-Hew$. Now the charged particle is moving in a magnetic field, and so is acted on by a force parallel to y equal to $-Hew$; while, since the wave of magnetic force has not yet reached the magnets, the latter do not experience any mechanical force. Thus the whole force acting on the system is such as to give to the system the momentum lost by the field. Thus the view that the wave of magnetic and electric force produced by the sudden starting of the charged point is not affected by the magnetic field is consistent with the principle of the Conservation of Momentum. We notice that the momentum arising from the external field vanishes when $t=0$. Thus the momentum at that time is not affected

by the magnetic field, so that a given impulse will produce the same velocity in the particle whatever the magnitude of the field. The magnetic field affects the subsequent motion of the particle, but not its initial velocity or the pulses produced when it is started.

As another example of the principle of the Conservation of Momentum, let us consider the case where the electrified particle is suddenly started with a velocity w parallel to the axis of z in a uniform electric field parallel to x , the field being supposed to be due to uniformly electrified plates parallel to yz . If we calculate by the rule given above the part of the momentum arising from the electric field, we shall find that while the momentum in any direction vanishes the moment of momentum is finite.

If X is the electric force, K and μ the specific inductive capacity and magnetic permeability of the dielectric, then the moment of momentum about the axis of y in the region inside the pulse is equal to

$$\frac{\mu K X e w}{6} (OB^2 - a^2),$$

where OB is the inner radius of the pulse; it is equal to Vt , where V is the velocity of light and t the interval which has elapsed since the particle was started.

The moment of momentum in the pulse itself is equal to

$$\frac{\mu K \cdot X e w}{3} OB^2 ;$$

hence, the total moment of momentum about y is equal to

$$\begin{aligned} & \frac{\mu K \cdot X e w}{2} OB^2 - \frac{1}{6} \mu K \cdot X e w a^2 \\ &= \frac{1}{2} \mu K \cdot X e w V^2 t^2 - \frac{1}{6} \mu K \cdot X e w a^2. \end{aligned}$$

The rate of increase of this, since $\mu K = \frac{1}{V^2}$, is $X e w t$. Thus to keep the moment of momentum of the whole system constant, the material parts of the system, *i. e.* the charged point and the plates, must be acted on by a couple about the axis of y whose moment is $-X e w t$. We can see that this is the case, for suppose O is the position of the charged body before it is set in motion, O' the position after a time t , then, when the particle was at O the force on the plates was equal and opposite to that on the particle, it therefore passed through O and was equal to $X e$. When, however, the particle is set in

motion the force on the electrified plates will not alter until the pulse reaches them, and so will continue to pass through O. The force on the point will, however, pass through O'. Thus we get a couple whose moment is $XeOO'$ or $Xe\omega t$, *i. e.* the couple required to keep the moment of momentum of the system constant.

Momentum when a charged point at rest is placed in the field due to a moving charged particle.—If the external electrostatic field in the last example had not been uniform but that due, say, to a charged particle at P, then we should have found that the momentum as well as the moment of momentum in the field would change as the pulse spreads out. In this case the force acting on the moving particle when it is at O' is $ee'/O'P^2$, and acts along O'P, while the force acting on P is ee'/OP^2 along OP, O being the initial position of O'. In this case, since OP is not in general equal to O'P the forces on O' and P are not equal in magnitude, so that there is a resultant force acting on the system.

If e is the charge on the moving point O, u, v, w the components of its velocity, then α, β, γ the components of the magnetic force produced by it at a point P whose co-ordinates are x, y, z , are given by the equations

$$\begin{aligned}\alpha &= e \left(v \frac{d}{dz} \frac{1}{r} - w \frac{d}{dy} \frac{1}{r} \right) \\ \beta &= e \left(w \frac{d}{dx} \frac{1}{r} - u \frac{d}{dz} \frac{1}{r} \right) \\ \gamma &= e \left(u \frac{d}{dy} \frac{1}{r} - v \frac{d}{dx} \frac{1}{r} \right),\end{aligned}$$

where r is the distance of P from O. If Q is the position of a charge e' at rest, then f, g, h , the components of the electric polarization at P due to this charge, are given by the equations

$$f = -\frac{e'}{4\pi} \frac{d}{dx} \frac{1}{r'}, \quad g = -\frac{e'}{4\pi} \frac{d}{dy} \frac{1}{r'}, \quad h = -\frac{e'}{4\pi} \frac{d}{dz} \frac{1}{r'},$$

r' being equal to PQ.

U, the x component of the momentum in the field, is given by the equation

$$\begin{aligned}U &= \mu \iiint (\gamma g - \beta h) dx dy dz \\ &= \frac{\mu ee'}{4\pi} \iiint \left\{ \left(w \frac{d}{dx} \frac{1}{r} - u \frac{d}{dz} \frac{1}{r} \right) \frac{d}{dz} \frac{1}{r'} - \left(u \frac{d}{dy} \frac{1}{r} - v \frac{d}{dx} \frac{1}{r} \right) \frac{d}{dy} \frac{1}{r'} \right\} dx dy dz.\end{aligned}$$

We can easily show that

$$\iiint \frac{d}{dx} \frac{1}{r} \cdot \frac{d}{dz} \frac{1}{r'} dx dy dz = 2\pi \frac{d^2 R}{dx_2 dz_2}$$

$$\iiint \frac{d}{dz} \frac{1}{r} \frac{d}{dx} \frac{1}{r'} dx dy dz = 2\pi \frac{d^2 R}{dz_2^2},$$

where R is the distance OQ between the fixed and moving points, and x_2, y_2, z_2 are the coordinates of Q . Using these values for the integrals we find

$$\begin{aligned} U &= -\frac{\mu e e'}{2} \left\{ w \frac{d^2 R}{dx_2 dz_2} + v \frac{d^2 R}{dx_2 dy_2} - u \left(\frac{d^2 R}{dy_2^2} + \frac{d^2 R}{dz_2^2} \right) \right\} \\ &= -\frac{\mu e e'}{2} \left(w \frac{d^2 R}{dx_2 dz_2} + v \frac{d^2 R}{dx_2 dy_2} + u \frac{d^2 R}{dx_2^2} \right) + \mu e e' \frac{u}{R} \\ &= -\frac{\mu e e'}{2} \frac{d}{dx_2} \left(u \frac{dR}{dx_2} + v \frac{dR}{dy_2} + w \frac{dR}{dz_2} \right) + \mu e e' \frac{u}{R}. \end{aligned}$$

Similarly,

$$\begin{aligned} V &= -\frac{\mu e e'}{2} \frac{d}{dy_2} \left(u \frac{dR}{dx_2} + v \frac{dR}{dy_2} + w \frac{dR}{dz_2} \right) + \mu e e' \frac{v}{R} \\ W &= -\frac{\mu e e'}{2} \frac{d}{dz_2} \left(u \frac{dR}{dx_2} + v \frac{dR}{dy_2} + w \frac{dR}{dz_2} \right) + \mu e e' \frac{w}{R}. \end{aligned}$$

We notice that

$$\frac{dU}{dx_2} + \frac{dV}{dy_2} + \frac{dW}{dz_2} = 0.$$

We shall now show that the line of the resultant momentum passes through Q the fixed point. Take Q as origin, let x, y, z be the coordinates of any point in the field. Then the moment of momentum of the field about the axis of z is equal to

$$\begin{aligned} &\mu \iiint \{ (\gamma g - \beta h) y - (\alpha h - \gamma f) x \} dx dy dz \\ &= \mu \iiint \gamma (x f + y g + z h) dx dy dz \\ &\quad - \mu \iiint h (x \alpha + y \beta + z \gamma) dx dy dz; \end{aligned}$$

each of the integrals is equal to

$$\frac{1}{2} \frac{e e'}{R} (v x_1 - u y_1),$$

where R is the distance OQ between the fixed and moving

points and $x_1 y_1 z_1$ the coordinates of O when Q is taken as origin. Thus the moment of momentum about the axis of z vanishes. Similarly we may show that it vanishes about the axes of x and y , and thus the resultant momentum passes through Q the fixed point.

This is only a particular case of the much more general proposition that the resultant of the momentum due to a stationary electrified point and a magnetic field passes through the fixed point when the magnetic field is continuous and derivable from a vector potential. For, taking the fixed point as origin, Γ the moment of momentum about the axis of z is given by the equation

$$\Gamma = \mu \iiint \{ (\gamma g_1 - \beta h_1) y - (\alpha h_1 - \gamma f_1) x \} dx dy dz,$$

where f_1, g_1, h_1 are the components of the electric polarization due to the charge, and are respectively equal to

$$-\frac{e}{4\pi} \left\{ \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\} \frac{1}{r}.$$

Substituting these values we find

$$\Gamma = \frac{\mu e}{4\pi} \iiint \left[\left(\beta \frac{d}{dz} \frac{1}{r} - \gamma \frac{d}{dy} \frac{1}{r} \right) y - \left(\gamma \frac{d}{dx} \frac{1}{r} - \alpha \frac{d}{dz} \frac{1}{r} \right) x \right] dx dy dz.$$

Integrating by parts, we find if α, β, γ are continuous, and if at great distances from the origin they vary more rapidly than $1/r^2$,

$$\begin{aligned} \Gamma &= \frac{\mu e}{4\pi} \iiint \frac{1}{r} \left\{ \frac{d}{dy} (y\gamma) + \frac{d}{dx} (x\gamma) - \frac{d}{dz} (\beta y + \alpha x) \right\} dx dy dz \\ &= \frac{\mu e}{4\pi} \iiint \frac{1}{r} \left\{ y \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz} \right) + 2\gamma - x \left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx} \right) \right\} dx dy dz. \end{aligned}$$

Putting

$$\alpha = \frac{dH}{dy} - \frac{dG}{dz}, \quad \beta = \frac{dF}{dz} - \frac{dH}{dx}, \quad \gamma = \frac{dG}{dx} - \frac{dF}{dy},$$

we have

$$\Gamma = \frac{\mu e}{4\pi} \iiint \frac{1}{r} \left\{ x \nabla^2 G + 2 \frac{dG}{dx} - y \nabla^2 F - 2 \frac{dF}{dy} - \left(x \frac{dJ}{dy} - y \frac{dJ}{dx} \right) \right\} dx dy dz,$$

where

$$J = \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz}.$$

The terms in J cancel on integration, and we have

$$\begin{aligned}\Gamma &= \frac{\mu e}{4\pi} \iiint \frac{1}{r} \nabla^2 (xG - yF) dx dy dz \\ &= \mu e \text{ value of } (xG - yF) \text{ at the origin} \\ &= 0.\end{aligned}$$

Hence the moment of the momentum about any axis through the electrified point vanishes, so that the resultant momentum passes through the point.

The magnetic force due to a moving particle is the same as that due to an element of current at the particle. The direction of the current being parallel to the direction of motion of the particle, and the intensity i of the current being determined by the relation $ids = e$ (velocity of particle), ds is the length of the element of current. Hence we may apply the preceding results to find the momentum in the field due to the mutual action of any system of currents and a fixed electrified point; and hence it follows that the resultant momentum passes through the electrified point.

We have previously seen that when the magnetic field is due to a system of magnetic poles in a non-magnetic medium, the momentum in the field due to the action of the magnetic field on a fixed electrified point, has zero resultant but finite moment of momentum, and hence can be represented by a momentum passing through the point and an equal and opposite momentum through some point of the system producing the magnetic force. In the case when the magnetic field is due to currents the second momentum, *i. e.* that passing through the system producing the magnetic field, is absent. This result seems at first sight not in accordance with the principle that we can replace a magnet, as in Ampère's hypothesis, by a system of electric currents; for in this case there would be a finite resultant momentum, while the moment of momentum about any line passing through the fixed electrified point would vanish. The explanation of the discrepancy is that in calculating the momentum due to the magnetic poles, we have supposed that the poles were surrounded by a medium which was non-magnetic, *i. e.* that the magnetic induction was equal to the magnetic force throughout the field. If the magnetic induction is not equal to the magnetic force, the results of our investigation would have to be modified, for the momentum is proportional to the magnetic induction and not to the magnetic force. When we allow for this difference, we see that in the case of the magnets, as well as in that of the currents, the

momentum in the field has a finite resultant passing through the fixed electrified point. For consider a magnetized element AB (fig. 3) and an electrified point at P, then if m is the

Fig. 3.

strength of the pole at A, $-m$ at B, the momentum in the field calculated on the assumption that the magnetic force is equal to the magnetic induction at every point in the field, is made up of a moment of momentum em along AP and a moment of momentum $-em$ along BP. The resultant of these is a moment of momentum $\frac{emAB \sin \theta}{OP}$ at right angles

to OP, O being the middle point of AB and θ the angle AOP. This moment of momentum is equivalent to a

momentum $\frac{emAB \sin \theta}{OP^2}$ through P at right angles to

the plane of the paper downwards and a momentum

$\frac{emAB \sin \theta}{OP^2}$ through O at right angles to the plane of the

paper upwards. In the magnet, however, the magnetic induction is not equal to the magnetic force, so to get the momentum we must add to the preceding momentum a momentum equal to (difference between magnetic induction and magnetic force in the magnet) $\times$ (electric polarization at right angles to AB) $\times$ volume of magnet. The direction of this momentum is at right angles to the plane of the paper downwards. Now the difference between magnetic induction and magnetic force $= 4\pi I$, where I is the intensity of magnetization, the polarization at right angles to AB is

$\frac{e \sin \theta}{4\pi OP^2}$; hence the additional momentum is $\frac{Ie \sin \theta}{OP^2}$ (volume

of magnet), but $I \times (\text{volume of magnet}) = m \cdot AB$. Hence we see that this momentum just balances that acting through

the magnet, and we are left with the momentum through the fixed point, *i. e.* we get the same result as if we replace the magnet by its equivalent currents.

The following illustration will show the difference in the physical effects produced by the differences in the distribution of momentum just considered. Suppose AB is a magnetized piece of soft iron, P the charged point, then we have seen that the momentum in the field has for resultant a momentum I through P perpendicular to the plane containing AB and P. Suppose that AB is demagnetized by tapping, there will then be no momentum in the field; hence by the principle of the conservation of momentum the momentum in the field before the tapping must be transferred to the material system in the field. Thus the charged point will acquire momentum I, but the momentum of AB will remain unchanged. If, however, the distribution of momentum in the field were equivalent to I at P and $-I$ at the magnet, then, when the magnet was tapped I would gain momentum as before; but in this case the magnet would gain an equal momentum in the opposite direction. Thus in this case the system would behave as if the point and magnet were acted on by equal and opposite impulses, while in the former case, which is that given by the rule we have used for finding the momentum, the charged particle would alone be acted upon by an impulse, the magnet remaining undisturbed.

Connexion between the Momentum and the Vector Potential.—From the investigation on page 344, it follows that U, V, W, the components of the momentum due to magnetic force produced by an element of current ids and the electric field due to a charge e' at P, are given by the equations

$$\begin{aligned} U &= \mu e' \frac{1}{R} \frac{dx}{ds} ds - \frac{1}{2} \mu e' \left\{ \frac{dx}{ds} \frac{d^2 R}{dx^2} + \frac{dy}{ds} \frac{d^2 R}{dy^2 dx} + \frac{dz}{ds} \frac{d^2 R}{dz dx dz} \right\} ds \\ &= \mu e' \frac{1}{R} \frac{dx}{ds} ds - \frac{1}{2} \mu e' \left\{ \frac{d}{ds} \frac{dR}{dx} \right\} ds, \end{aligned}$$

with similar expressions for V and W; here R is the distance between the current element and the charged point. Now if we find the momentum at P due to a closed circuit from this expression by integration, we see that the second term vanishes, and $\bar{U}$ the momentum parallel to x acting through P is equal to

$$\mu e' \int \frac{1}{R} \frac{dx}{ds} ds; \quad \text{but} \quad \mu \int \frac{1}{R} \frac{dx}{ds} ds$$

is equal to the x component at P of the vector potential due to the current. Hence we see that the vector potential at any point is the momentum due to the magnetic force produced by the system giving the vector potential and the electric field due to unit charge at the point. We thus get a physical interpretation for the vector potential, and see that instead of being merely an analytical device, it represents a most important physical property of the system. We shall show later on that the principle of the conservation of momentum leads at once to the ordinary equations for the induction of currents.

Momentum due to Two Moving Particles.—Let P and Q be the moving particles, e and e' their charges, R the distance between them, $u_1, v_1, w_1; u_2, v_2, w_2$ the components of their velocities, then from the results proved on page 344 we see that the momentum in the field is equivalent to the following distribution.

A momentum passing through P parallel to the axis of x equal to

$$\frac{2}{3} \frac{e^2 \mu}{a} u_1 + \frac{\mu e e'}{R} u_2 - \frac{1}{2} \mu e' e \left(u_2 \frac{d^2 R}{dx_2^2} + v_2 \frac{d^2 R}{dx_2 dy_2} + w_2 \frac{d^2 R}{dx_2 dz_2} \right),$$

and a momentum through Q parallel to x equal to

$$\frac{2}{3} \frac{\mu e'^2}{b} u_2 + \frac{\mu e e'}{R} u_1 - \frac{1}{2} \mu e e' \left(u_1 \frac{d^2 R}{dx_1^2} + v_1 \frac{d^2 R}{dx_1 dy_1} + w_1 \frac{d^2 R}{dx_1 dz_1} \right);$$

a and b are the radii of the spheres whose centres are P and Q over which the charges are spread, and $x_1, y_1, z_1, x_2, y_2, z_2$ are the coordinates of P and Q respectively. We have of course

$$\frac{d^2 R}{dx_2^2} = \frac{d^2 R}{dx_1^2} \quad \text{and} \quad \frac{d^2 R}{dx_2 dy_2} = \frac{d^2 R}{dx_1 dy_1}.$$

There are, of course, momenta through P and Q parallel to the axes of y and z respectively given by symmetrical expressions. These expressions enable us to find the velocities acquired when a charged particle P at rest in the presence of another, Q, is struck by an impulse. Let us suppose that the particle P initially at rest is struck by an impulse I_x parallel to x , then the momentum in the charged particles and in the field must be equivalent to this impulse. Hence, if $u_1, v_1, w_1, u_2, v_2, w_2$ are the velocities acquired by P and Q

respectively, m and m' the mechanical masses of these particles, we have

$$\begin{aligned} P_x &= mu_1 + \frac{2}{3} \frac{\mu e^2}{a} u_1 + Ru_2 - (\alpha u_2 + \beta v_2 + \gamma w_2), \\ 0 &= \left(m' + \frac{2}{3} \frac{\mu e'^2}{b} \right) u_2 + Ru_1 - (\alpha u_1 + \beta v_1 + \gamma w_1), \\ 0 &= \left(m + \frac{2}{3} \frac{\mu e^2}{a} \right) v_1 + Rv_2 - (\beta u_2 + \delta v_2 + \epsilon w_2), \\ 0 &= \left(m' + \frac{2}{3} \frac{\mu e'^2}{b} \right) v_2 + Rv_1 - (\beta u_1 + \delta v_1 + \epsilon w_1), \\ 0 &= \left(m + \frac{2}{3} \frac{\mu e^2}{a} \right) w_1 + Rw_2 - (\gamma u_2 + \epsilon v_2 + \eta w_2), \\ 0 &= \left(m' + \frac{2}{3} \frac{\mu e'^2}{b} \right) w_2 + Rw_1 - (\gamma u_1 + \epsilon v_1 + \eta w_1); \end{aligned}$$

where R (α , β , γ , δ , ϵ , η) are written for $\mu e e' / R$,

$$\frac{1}{2} \mu e e' \left(\frac{d^2 R}{dx_2^2} \quad \frac{d^2 R}{dx_2 dy_2} \quad \frac{d^2 R}{dx_2 dz_2} \quad \frac{d^2 R}{dy_2^2} \quad \frac{d^2 R}{dy_2 dz_2} \quad \frac{d^2 R}{dz_2^2} \right)$$

respectively.

Writing l, m, n for the direction cosines of the line joining the electrified points and q for $\mu^2 e^2 e'^2 / R^2 M M'$, where

$$M = m + \frac{2}{3} \frac{\mu e^2}{a}, \quad M' = m' + \frac{2}{3} \frac{\mu e'^2}{b},$$

we find

$$\begin{aligned} u_1 &= \frac{P}{M} \left(1 + \frac{l^2 q}{1-q} \right), \\ v_1 &= \frac{P}{M} \frac{lmq}{1-q}, \\ w_1 &= \frac{P}{M} \frac{lnq}{1-q}; \\ u_2 &= -\frac{\mu e e'}{2RM'} \frac{P}{M} \left(1 + \frac{l^2(1+q)}{1-q} \right), \\ v_2 &= -\frac{\mu e e'}{2RM'} \frac{P}{M} \frac{lm(1+q)}{1-q}, \\ w_2 &= -\frac{\mu e e'}{2RM'} \frac{P}{M} \frac{ln(1+q)}{1-q}. \end{aligned}$$

We see that u_2, v_2, w_2 are large compared with v_1, w_1 , so

that the deflexion of the velocity of the particle struck from the line of impulse is not so marked as the motion imparted to the charged particle in its neighbourhood. The deflexion of the first particle may be supposed to occur as follows: when the particle is first acted on by the impulse it starts parallel to the impulse with a velocity P/M ; a pulse of strong electric force spreads out from the particle, after a time this reaches the second particle and starts it impulsively; the sudden starting of the second particle produces pulses of electric force which travel on and strike the first particle *deflecting it from the original direction*.

The momenta of the particles may be expressed in the following way. $U_1, V_1, W_1, U_2, V_2, W_2$ being the components of the momenta of the first and second particles; $u_1, v_1, w_1, u_2, v_2, w_2$ the components of their velocities:—

$$U_1 = Mu_1 + \frac{ee'\mu}{2R}u_2 + \frac{ee'\mu}{2R}l(lu_2 + mv_2 + nw_2),$$

$$V_1 = Mv_1 + \frac{ee'\mu}{2R}v_2 + \frac{ee'\mu}{2R}m(lu_2 + mv_2 + nw_2),$$

$$W_1 = Mw_1 + \frac{ee'\mu}{2R}w_2 + \frac{ee'\mu}{2R}n(lu_2 + mv_2 + nw_2),$$

$$U_2 = M'u_2 + \frac{ee'}{2R}\mu u_1 + \frac{ee'}{2R}\mu l(lu_1 + mv_1 + nw_1),$$

$$V_2 = M'v_2 + \frac{ee'}{2R}\mu v_1 + \frac{ee'}{2R}\mu m(lu_1 + mv_1 + nw_1),$$

$$W_2 = M'w_2 + \frac{ee'}{2R}\mu w_1 + \frac{ee'}{2R}\mu n(lu_1 + mv_1 + nw_1).$$

Take the case when the particles are moving with the same velocity; then, unless the momentum of the system is in the direction of the velocity, the line of action of the resultant momentum will change: thus the moment of momentum in the field about axes of x, y, z will be changing at the rate

$$(W_1 + W_2)v - (V_1 + V_2)w; (U_1 + U_2)w - (W_1 + W_2)u; \\ (V_1 + V_2)u - (U_1 + U_2)v.$$

Since the momentum of the system consisting of the field and particle remains constant, the particle must be acted on by couples about the axes of x, y, z , equal and opposite to the rate of increase of the moment of momentum in the field. Hence, substituting for the values of U, V, W , we find that

when the particles are moving with the same velocities they are acted on by couples whose moments round x, y, z are

$$\frac{ee'\mu}{R}(lu + mv + nw)(mw - nv),$$

$$\frac{ee'\mu}{R}(lu + mv + nw)(nu - lw),$$

$$\frac{ee'\mu}{R}(lu + mv + nw)(lv - mu).$$

These couples vanish if

$$\frac{u}{l} = \frac{v}{m} = \frac{w}{n},$$

i. e., if the particles are moving along the line joining them; and when

$$lu + mv + nw = 0,$$

when the direction of motion is at right angles to the line, in all other cases there is a couple tending to twist the particles so that the line joining them gets parallel to the direction of motion if the charges are of the same sign, or perpendicular to the direction of motion if the charges are of opposite signs.

Kinetic Energy of a System consisting of two moving particles.—If T is the Kinetic Energy,

$$\begin{aligned} T &= \frac{1}{2}(u_1U_1 + v_1V_1 + w_1W_1 + u_2U_2 + v_2V_2 + w_2W_2) \\ &= \frac{1}{2}M(u_1^2 + v_1^2 + w_1^2) + \frac{1}{2}M'(u_2^2 + v_2^2 + w_2^2) + \frac{\mu ee'}{R}(u_1u_2 + v_1v_2 + w_1w_2) \\ &\quad - \frac{\mu ee'}{2}\left(u_1\frac{d}{dx_1} + v_1\frac{d}{dy_1} + w_1\frac{d}{dz_1}\right)\left(u_2\frac{d}{dx_1} + v_2\frac{d}{dy_1} + w_2\frac{d}{dz_1}\right)R. \end{aligned}$$

By Lagrange's equations we have

$$\frac{dT}{dt} \frac{dT}{du_1} - \frac{dT}{dx_1} = \text{force parallel to } x_1 \text{ acting on the first particle.}$$

Hence the effect of the motion may be represented by a force parallel to x_1 equal to

$$\frac{dT}{dx_1} - \frac{d}{dt} \frac{dT}{du_1}.$$

When the velocities of the particles are constant, this may be written as

$$\begin{aligned} \mu ee' \left\{ v_1 \left(v_2 \frac{d}{dx_1} \frac{1}{R} - u_2 \frac{d}{dy_1} \frac{1}{R} \right) - w_1 \left(u_2 \frac{d}{dz_1} \frac{1}{R} - w_2 \frac{d}{dx_1} \frac{1}{R} \right) \right\} \\ - \frac{1}{2} \mu ee' \frac{d}{dx_1} \left(u_2 \frac{d}{dx_1} + v_2 \frac{d}{dy_1} + w_2 \frac{d}{dz_1} \right)^2 R \\ + \mu ee' u_2 \left(u_2 \frac{d}{dx_1} + v_2 \frac{d}{dy_1} + w_2 \frac{d}{dz_1} \right) \frac{1}{R}. \end{aligned}$$

The expressions in the first line represent the usual expressions for the force on the first particle due to its motion in the magnetic field produced by the motion of the second. The terms in the second and third lines reduce to

$$\frac{1}{2} \mu ee' \frac{x_1 - x_2}{R} \frac{1}{R^2} (u_2^2 + v_2^2 + w_2^2) (1 - 3 \cos^2 \theta),$$

where θ is the angle the direction of motion of the second particle makes with the line joining the two particles. Thus these terms represent a force acting along the line joining the particles and equal to

$$\frac{\mu ee'}{2R^2} (u_2^2 + v_2^2 + w_2^2) (1 - 3 \cos^2 \theta).$$

This radial force may be compounded with the force $\frac{ee'}{R^2}$ due to the electrostatic attraction, and we take it into account by supposing that the attraction of the second particle on the first is not

$$\frac{ee'}{R^2}, \text{ but } \frac{ee'}{R^2} \left(1 + \frac{(u_2^2 + v_2^2 + w_2^2)}{2V^2} (1 - 3 \cos^2 \theta) \right),$$

where V is the velocity of light.

The difference between the electric forces due to a charge at rest and in motion is proportional to v^2/V^2 , where v is the velocity of the particle and V that of light; and thus for any individual particle moving slowly compared with light is exceedingly small. Yet, on the view that the structure of the atoms is electrical, the number of charged particles is so enormous, and the velocity of the negative particles so much greater than the positive, it might seem that if any order were to exist in the motion of the negative particles as in the case of a magnet, finite differences of electric force in different directions might be produced: thus, for example, the magnet might behave as if it had a charge of negative

electricity at its ends and positive at its sides. A little closer consideration will show, however, that such an effect is not probable, for suppose the negative particles are describing closed circular orbits, then the average effect will be the same as if the charge were uniformly distributed round the orbit; thus the average effect will be that due to a uniformly electrified ring rotating about an axis through its centre at right angles to its plane, and in this case the electric force due to the motion obviously vanishes.

Electromotive Force due to Induction.—Take the case of a particle fixed by the coordinates x_1, y_1, z_1 , in the presence of any number of charged particles moving about in any manner. Let us fix the position of these particles by coordinates ξ_r, η_r, ζ_r . Then we see from the preceding example that T , the kinetic energy, can be written as follows :

$$T = \frac{M}{2} (u_1^2 + v_1^2 + w_1^2) + (u_1 F + v_1 G + w_1 H) + T_0,$$

where F, G, H are linear functions of the velocities $\dot{\xi}_r, \dot{\eta}_r, \dot{\zeta}_r$ and do not involve u_1, v_1, w_1 ; T_0 is a quadratic function of the velocities $\dot{\xi}_r, \dot{\eta}_r, \dot{\zeta}_r$. The momentum of the first particle parallel to x is

$$Mu_1 + eF.$$

We have seen that F is the vector potential due to the remaining particles at the first particle.

By Lagrange's equation we have

$$\frac{d}{dt}(Mu_1 + eF) - e \frac{d}{dx_1} (Fu_1 + Gv_1 + Hw_1) - \frac{dT_0}{dx_1} = \text{force tending to increase } x_1.$$

Now

$$\frac{d}{dt}(eF) = e \left(\frac{\partial F}{\partial t} + u_1 \frac{dF}{dx_1} + v_1 \frac{dF}{dy_1} + w_1 \frac{dF}{dz_1} \right)$$

and

$$e \frac{d}{dx_1} (u_1 F + v_1 G + w_1 H) = e \left(u_1 \frac{dF}{dx_1} + v_1 \frac{dG}{dx_1} + w_1 \frac{dH}{dx_1} \right);$$

hence

$$\begin{aligned} \frac{d}{dt} Mu_1 = & -e \frac{\partial F}{\partial t} + ev_1 \left(\frac{dG}{dx_1} - \frac{dF}{dy_1} \right) - ew_1 \left(\frac{dF}{dz_1} - \frac{dH}{dx_1} \right) \\ & + \frac{dT_0}{dx_1} + \text{external force tending to increase } x_1. \end{aligned}$$

Thus, in consequence of the variation in the external field

and motion of the point itself, there is a force acting on unit electric charge, *i. e.* an electric force equal to

$$-\frac{\partial F}{\partial t} + v_1 \left(\frac{dG}{dx_1} - \frac{dF}{dy_1} \right) - w_1 \left(\frac{dF}{dz_1} - \frac{dH}{dx_1} \right) + \frac{1}{e} \frac{dT_0}{dx_1};$$

these are the usual expressions for electromotive forces due to the induction of current, the term $\frac{1}{e} \frac{dT_0}{dx_1}$ being included in the term derived from the scalar potential.

*On the Momentum in Electrical Waves.
Pressure of Radiation.*

Let us consider a plane electrical wave in which, as we shall prove later, the electric force and the magnetic force are at right angles to each other, and both in the wave-front. Let H be the magnetic force at a point in the wave-front, then X , the electric force, is in a progressive wave equal to VH , where V is the velocity of light, X and H being measured in electromagnetic units; the electric polarization parallel to X is therefore $H/4\pi V$, so that the momentum which is at right angles both to F and H is along the normal to the wave-front, *i. e.* in the direction of propagation of the wave, and is equal per unit volume to $\frac{H^2}{4\pi V}$. Now $H^2/4\pi$ is the energy (electrostatic + electromagnetic) per unit volume, so that the momentum per unit volume is equal to (energy per unit volume)/(velocity of propagation of the wave). If this wave were to fall on a body by which it is absorbed without reflexion, this momentum would be communicated to the absorbing body, the amount of momentum communicated to the body per unit time would be per unit area of wave-front equal to V times the momentum per unit volume; it would therefore equal the energy per unit volume in the wave. This communication of momentum to the body being proportional to the area of the surface, might be produced by the application of a suitable pressure to the surface, the pressure being equal to the momentum communicated to the body per unit area per unit time. Thus a body absorbing electrical waves behaves as if a pressure acted on its surface in the direction of propagation of the wave equal to the energy per unit volume in the wave. Light waves are a particular case of electrical waves, hence when light falls on an absorbing substance it exerts a pressure on the surface equal to the energy per unit volume in the light wave. The existence of

this pressure of radiation as it is called, was shown by Maxwell to be a consequence of the Electromagnetic Theory of Light ; it has been detected and measured in some beautiful experiments made by Lebedew * and by Nichols and Hull †. The magnitude of this pressure in the case of light waves is very small.

If the light, instead of being absorbed, were wholly reflected, then, when light is incident normally on the surface, the incident light communicates to it per unit time an amount of momentum in the direction of the light equal per unit of surface to E , where E is the energy of the incident light per unit volume ; the reflected light carries away per unit time an amount of momentum in the opposite direction equal to E . Hence the effect on the reflector is the same as if it received $2E$ units of momentum per second in the direction of the incident light ; *i. e.*, as if the reflecting surface were acted on by a pressure equal to $2E$: hence for normal incidence the pressure on a perfectly reflecting surface is twice that on a perfectly absorbing one. If the absorbing body were moving towards the wave with the velocity u , the amount of momentum absorbed in unit time, and therefore the pressure of the radiation, would be increased in the proportion of $V + u$ to V . Radiation will give rise to pressure when it is refracted, for the momentum in the medium being in the direction of propagation, when the latter is changed the momentum in the medium is changed also, and therefore by the principle of the Conservation of Momentum the refracting substance must suffer a change in momentum. Thus, suppose a beam of light is travelling along a curved path in a refracting medium ; then, if ds is an element of the path, $d\theta$ the angle between the direction of the beam at the beginning and end of the element, the direction of momentum is changed by $\delta\theta$ in passing along ds . Thus, if M is the momentum per unit volume in the beam, an amount of momentum $VM\delta\theta$ at right angles to the path of the light must be communicated to the medium per unit time : thus the force per unit volume at right angles to the light will be $VM\frac{\delta\theta}{\delta s}$ or E/ρ , where ρ is the radius of curvature of the beam.

* Rapports présentés au Congrès International de Physique (2) p. 133.

† Proc. American Academy of Arts and Sciences, xxxviii. p. 559.