The present result thus lies fairly close to the values of Weber and of Lees, the difference between it and each of these (extrapolating to 20° C. from the numbers given above) being about three per cent.

In conclusion, we hope to be able shortly to apply our method to two problems upon which it seems desirable that more work should be done—the temperature-coefficient of water and the properties of solutions. Each of these measurements amounts to the comparison of the thermal conductivities of two liquids in different physical conditions, and the experience now gained of the working of our apparatus points to its being particularly well adapted to such comparisons.

III. *Reflexion and Refraction of Elastic Waves, with Seismological Applications.* *By* Professor C. G. KNOTT, *D.Sc., F.R.S.E.**

AT Lord Kelvin's suggestion I reproduce, with additions and extensions, a paper I published eleven years ago in the 'Transactions' of the Seismological Society of Japan. This Society ceased to exist some years ago; a fact which may serve as a further reason for reproducing a paper, in which the problem of the behaviour of an elastic wave incident on the interface of rock and water was for the first time fully worked out. In that paper also, I believe, the sound method of treating the general problem when the two media are elastic solids was first explicitly stated (see below, pp. 71, 92).

For convenience I have divided the present communication into three parts.

Part I. is a reproduction of my seismological paper of 1888 with a few verbal corrections. Footnotes added now are enclosed in square brackets.

Part II contains detailed numerical calculations for rock-rock interface and for rock-air interface, similar to the calculations for rock-water interface in Part I.

Part III. gives the mathematical investigation and the various sets of formulæ on which these calculations are based.

PART I.†

EARTHQUAKES AND EARTHQUAKE SOUNDS: *as Illustrations of the General Theory of Elastic Vibrations.*

The first systematic application of the theory of vibrations to the problems of earthquake motion was made, I believe,

* Communicated by the Author.

† [Read at Tokyo before the Seismological Society of Japan, February 23rd, 1888, and published in their 'Transactions' of that year.]

by Hopkins in his "Report on the Theories of Elevation and Earthquakes," presented to the British Association in 1847. During the forty years which have elapsed since then, our knowledge of earthquake phenomena has steadily grown. The labours of Mallet have been largely supplemented by the observations and experiments of a small army of enthusiasts, who have pitched their tents on the trembling soil of Italy and Japan. Their energies have been mainly directed to the perfecting of seismographs and seismometers, to the registering of all kinds of earth-movements, to the study of the effects of these on buildings, and, in a limited degree, to the measurement of the velocities of propagation of disturbances due to artificial earthquakes. With all this activity on the experimental side, we have to confess that theoretic views have hardly advanced beyond the stage in which Hopkins left them in 1847. G. H. Darwin's discussion of the strains due to continental areas, and Lord Rayleigh's investigation into a special case of surface-waves on an elastic solid, are perhaps the only mathematical pieces of work that have any distinct bearing on seismic phenomena. The former gives an obvious *raison d'être* for the existence of seismically sensitive regions within the earth's crust, but, being an equilibrium problem, can throw no light on that progress of the state of strain which constitutes earthquake motion. Lord Rayleigh's results will be referred to hereafter in due course. Meanwhile, as it is my object to discuss in a general way how far earthquakes and their accompanying effects may be explained as disturbances in an elastic or subelastic medium, it will be convenient to reproduce here much that may be found in authoritative earthquake literature, such as Hopkins' and Mallet's 'Reports,' Mallet's 'Neapolitan Earthquake,' Milne's 'Earthquakes,' and so on.

From the general theory of the vibrations of homogeneous elastic solids, we know that there are three types of wave propagated with different velocities. If we confine our attention to an isotropic elastic solid these types reduce to two, which are kinematically easily distinguished by the relation which the direction of vibration of any particle bears to the direction of propagation of the wave. Thus, in the one type the vibrations are *normal* to the wave-front; in the other they are *transverse* or *tangential*. Dynamically, the types may be distinguished as the *condensational* and *distortional* waves. The former is of essentially the same character as ordinary sound-waves in air; and the latter may be compared, so far as direction of motion is concerned, to waves of light in the luminiferous æther. In the condensational wave the vibrating particles move to and fro in lines parallel to the direction of motion of the wave. In the distortional wave the particles move to and fro in lines perpendicular to the wave's direction of motion.

In all cases these two types of wave are propagated with different velocities, which depend upon the density and the elastic constants of the material. For an isotropic elastic solid there are two independent elastic moduli, known respectively as the *bulk-modulus*, or resistance to compression, and the *rigidity*, or resistance to distortion. The velocity of the distortional wave depends on the ratio of the rigidity to the density. The velocity of the condensational wave, however, is not so simply related to the other modulus, but depends for its value upon the rigidity as well.

Take, for example, a uniform cylindrical rod of iron. By giving the one end of this rod a slight twist we may set up a series of torsional vibrations, whose velocity of propagation along the rod is to be measured by the square root of the ratio of the rigidity to the density. The velocity of propagation of longitudinal vibrations, which may be supposed to be given by an impact on the end, is to be measured by the square root of the ratio of the so-called Young's Modulus to the density. Young's Modulus is a definite function of the principal moduli already mentioned, being given by the formula

$$9nk/(3k+n),$$

where k is the resistance to compression and n is the rigidity.

Again, if we consider the case of plane waves in an infinite solid, we find that here also the velocity of propagation of the distortional wave is given by the ratio $\sqrt{n}/\sqrt{\rho}$; while that of the condensational waves is measured in terms of a mixed modulus which is not necessarily the same as Young's Modulus. Its value is $k+\frac{4}{3}n$, which is equal to Young's Modulus only if $3k=2n$.

According to Navier's and Poisson's theory of elasticity we should have $3k=5n$. This is usually expressed by saying that, when a bar is stretched under a longitudinal pull, its linear contraction at right angles to the pull is one quarter of the elongation in the direction of the pull. So far as experiments with hard metals go, this ratio may vary from ·2 to ·4. Nevertheless ·25 may be taken to be a pretty fair mean value.

If we write m instead of $k+\frac{4}{3}n$*, we obtain for the value of the Poisson ratio the expression

$$s=\frac{m-2n}{2(m-n)}.$$

The possible values of s range from $+\frac{1}{2}$ to -1; the former being its value in an incompressible elastic body, the latter its value in a body of infinite rigidity but finite compressibility. The luminiferous æther appears to be a substance of infinite resistance to compression; but of the other limiting kind of elastic material we have no example.

The velocities of the condensational and distortional waves are given respectively by the expressions $\sqrt{m/\rho}$ and $\sqrt{n/\rho}$, ρ being the density of the material.

There are experimental methods for measuring the quantities m and n; and from them the two velocities can easily be calculated. Or, if the two velocities are known, it is possible to calculate from them the two moduli. Now it is quite obvious that m must always be greater than n; the ratio indeed varies from ∞ for the case of the incompressible body to $\frac{4}{3}$ for the case of the infinitely rigid body. Of course, in the latter case, both waves travel with an infinite speed; but the speed of the distortional wave can never become equal to the speed of the condensational wave, however large it is made to be.

In deducing the true values of m and n from the two wave-velocities, we must know the density of the material. The only values I have been able to find for wave-velocities of both types in rocks are those given by Messrs. Milne and Gray. These velocities were originally obtained from direct measurements of the elastic moduli of the rocks in question. The moduli themselves Professor Milne has recently furnished me with. In the following table they are given †, expressed in C.G.S. units, along with the Poisson ratio s.

* This m is not the same as the m used by Thomson and Tait; but for our present purpose it is convenient to use one symbol for the mixed modulus which determines the speed of the condensational wave.

† In the notes given me by Professor Milne the numbers here tabulated under m are headed "Young's Modulus." This, I am inclined to think, is a mistake. Professor Milne himself, not having the complete records in possession is doubtful. At any rate, these numbers give the velocities of the normal vibrations as tabulated by Messrs. Milne and Gray (see Phil. Mag., November 1881). Further, if they really were Young's Moduli, we should have in granite and marble examples of substances which expand when compressed!

Rock.	*m.*	*n.*	*s.*
Granite	$4\cdot68\times10^{11}$	$1\cdot44\times10^{11}$	$+\cdot28$
Marble	$4\cdot35\times10^{11}$	$1\cdot3\ \times10^{11}$	$+\cdot29$
Tuff	$2\cdot44\times10^{11}$	$1\cdot31\times10^{11}$	$-\cdot08$
Clay-Rock	$3\cdot66\times10^{11}$	$1\cdot94\times10^{11}$	$-\cdot07$
Slate	$6\cdot07\times10^{11}$	$2\cdot45\times10^{11}$	$+\cdot16$

Two of the ratios come out negative, which means physically that, if the substance be perfectly elastic, an extension of the substance by a pull in a given direction is accompanied by an extension at right angles to this direction. It also means that the ratio of the velocities of the two waves is distinctly smaller than in the other cases. This diminished ratio, it will be noticed, exists along with a diminished resistance to compression, while the rigidity continues to have much the same value as those which hold for the other rocks. In the cases of the tuff and clay-rock we may have to do with either a considerable compressibility, or a sluggishness in recovery due to the viscosity of the material. Such a viscosity might well show itself more distinctly in compression than in distortion.

If we calculate from Milne's values of wave-velocities obtained from his experiments on artificial earthquakes, we find for the ratio s in two different cases the values $+\cdot154$ and $-\cdot152$, and for the corresponding ratios of m to n the values $2\cdot43$ and $1\cdot76$.

In the calculations to be described presently, I have taken the following values of the several constants involved as a fair approximation to what might reasonably be regarded as somewhat near the truth, when the elastic properties of fairly solid rock are to be considered.

Density	ρ	$=3$
Rigidity	n	$=1\cdot5\times10^{11}$
Ratio of the wave-moduli	m/n	$=3$
Poisson's ratio	s	$=\cdot25$

We now pass to the consideration of the transmission of waves in an elastic solid; and first I desire to call attention to Lord Rayleigh's short paper "On Waves Propagated along the Plane Surface of an Elastic Solid" *. To show

* 'Proceedings of the London Mathematical Society,' vol. xvii. (1885–86).

that the paper deserves the special attention of members of the Society I need but quote the two concluding sentences :—" It is not improbable that the surface-waves here investigated play an important part in earthquakes, and in the collision of elastic solids. Diverging in two dimensions only, they must acquire at a great distance a continually increasing preponderance,"—that is, I presume, as compared to waves diverging in three dimensions.

The purpose of the paper is " to investigate the behaviour of waves upon the plane free surface of an infinite homogeneous isotropic elastic solid, their character being such that a disturbance is confined to a superficial region of thickness comparable with the wave-length. The case is thus analogous to that of deep-water waves, only that the potential energy here depends upon elastic resilience instead of upon gravity."

Starting with the usual equations of motion of a vibrating elastic solid, Lord Rayleigh obtains a general solution on the assumptions that the displacements are harmonic functions of the time and the two coordinates parallel to the plane free surface, but are exponential functions of negative multiples of the distance from this plane. The boundary equations are then introduced; and from the conditions for the equilibrium of a surface-element the various constants of integration are determined in terms of the circumstances of the assumed motion. Two cases are discussed in detail—those, namely, of an incompressible elastic solid, and of a solid for which the Poisson ratio has the value one-fourth. For both cases the results are very similar. Thus, if the displacements are supposed to be confined to one plane, a particle at the surface moves in an elliptic orbit whose major axis is perpendicular to the plane surface of the solid. For the incompressible solid the major axis is nearly twice as great as the minor axis; and for the other case it is about one and a half times as great. The displacement parallel to the plane surface penetrates but a short distance into the solid—to about one-eighth of a wave-length for the incompressible substance, and to about one-fifth for the other case. On the other hand, there is no finite depth at which the motion perpendicular to the plane vanishes. The surface-waves are propagated at a slightly slower rate than a purely distortional plane wave would be.

It would appear then that vertical motion on a level surface over which a disturbance is passing cannot exist alone. Associated with it there must always be a distinctly smaller horizontal motion, which vanishes completely at a short depth below the surface. Lord Rayleigh's formulæ also show that the amplitude of the displacement is directly as the wave-length; so that for vibrations of short period the surface motions are proportionally small.

If we consider the features of earthquake motions, we find that the vertical motion when it is appreciable is always very much smaller than the horizontal motion. Hence we cannot have here merely the surface disturbance discussed by Lord Rayleigh. If his investigation touches upon any earthquake phenomenon, this phenomenon is never met with by itself alone. Horizontal displacements exist, at any rate along with it, of a magnitude greater far than Lord Rayleigh's result requires. The simple conclusion is that ordinary earthquakes cannot be regarded as due to the propagation of surface-waves. Milne has, at various times, speculated upon the existence of such surface-waves outstripping the vibrations transmitted through the mass. There never has seemed to me sufficient reason for calling in the aid of these surface-waves, as distinct from the mass-waves. Lord Rayleigh's investigation shows besides that the velocity of a surface disturbance is somewhat less than the velocity of the distortional plane wave travelling through the mass. There is no evidence of a quickened velocity. These two facts, namely, the comparative minuteness of the vertical motion in all earthquakes, and the somewhat slower speed of Lord Rayleigh's surface-wave, seem to show that we can expect very little towards the elucidation of earthquake phenomena, by taking into account the so-called surface-wave.

I now pass to the consideration of the reflexion and refraction of plane waves at the surface of separation of two elastic media. In doing so I shall direct more especial attention to the case in which the one medium is rock and the other water. The case in which both media are solid substances is a good deal more troublesome to deal with; and so far I have not had time to work out any detailed calculations concerning it*. A few general considerations will show the nature of the problem.

The reflexion and refraction of plane waves at the bounding surface of two media have been very closely studied by many mathematicians. Especially have their efforts been directed towards the explanation of the ordinary phenomena of light upon a purely dynamic basis. Cauchy, Green, Maccullagh, Lorenz, Rayleigh, Thomson [Kelvin], may be mentioned in this connexion. It is sufficient here to point out that, when the problem is worked out for the case of two incompressible

* [Now worked out below, Part II.].

elastic substances of equal rigidities but different densities, results are obtained in fair accordance with observation. The media being incompressible, no wave of condensation can be propagated through them. Distortional waves only can exist. Thus an incident distortional wave falling on the bounding surface will, in general, be broken up into two waves—one reflected into the first medium, and the other refracted into the second medium. But although distortional waves alone exist in the media, the correct solution of the problem in elastic solids requires us to take account of something existing at the bounding surface of the nature of a condensational wave. We must bear in mind, indeed, what the physical meaning of incompressibility is. It is not that the condensational wave vanishes, but that it is transmitted with infinite velocity. By taking this surface disturbance into account—this *pressural* wave as Thomson [Kelvin] has called it—we are able partially to explain certain phenomena of reflexion and refraction of polarized light in terms of the theory of elastic solids.

Now in this special problem we begin with a distortional wave incident on the bounding surface; and, although the media are taken as incompressible, we must not neglect the effect of the pressural wave. Hence, if our methods of attack are to be the same in all cases, we must admit the possibility of true waves of compression being started in media of finite compressibility, when upon their boundary a single distortional wave impinges. In other words, an incident distortional wave may be broken up into four parts :—a reflected distortional, a refracted distortional, a reflected condensational, and a refracted condensational. In like manner, an incident condensational wave will in general give rise to reflected and refracted distortional waves as well as to reflected and refracted condensational waves.

The various angles of reflexion and refraction are easily calculated in terms of the angles of incidence, it being noted that the surface trace is common to all the waves. In other words, each wave velocity is, so to speak, the component in its direction of the velocity of propagation of the surface trace.

Thus, let a condensational wave be incident at an angle θ to the normal to the bounding surface; let m, n, ρ, and m', n', ρ', be the wave moduli and densities of the two media, in the first of which the incident wave is given. Then if θ' be the angle of refraction of the condensational wave, and ϕ ϕ' the angles of reflexion and refraction of the distortional waves, the above condition gives these equations :—

$$\frac{m}{\rho}\operatorname{cosec}^2\theta=\frac{n}{\rho}\operatorname{cosec}^2\phi=\frac{m'}{\rho'}\operatorname{cosec}^2\theta'=\frac{n'}{\rho'}\operatorname{cosec}^2\phi'.$$

Now, as n is less than m, there will always be a reflected distortional wave, except of course at normal incidence when $\theta=0°$, or at a grazing incidence when $\theta=90°$. There will be refracted waves at all except the limiting incidences if m/ρ is greater than m'/ρ'. If m/ρ should be intermediate in value to m'/ρ' and n'/ρ' there will always be a refracted distortional wave, but for angles of incidence higher than a certain critical value a refracted condensational wave is impossible. Further, if m/ρ should be less than n'/ρ', then, for each refracted wave, there is a special critical angle of incidence at and above which the wave vanishes. When the critical value corresponding to the refracted distortional wave is reached, there will be total reflexion, and the whole energy of the incident wave will be divided between the two reflected waves.

If the incident wave is a distortional wave, there must always be a critical angle of incidence for and above which the reflected condensational wave vanishes. The existence of such critical angles for the refracted waves will depend upon the relative values of the quantities n/ρ, m'/ρ', n'/ρ',—the condition for the possibility of total reflexion being that n/ρ is less than n'/ρ'.

If one of the media is a fluid, there can, of course, be no distortional wave in it. It is this somewhat simple case I propose to discuss in detail. I shall not here enter into the purely mathematical method * by which the energies of the various possible waves have been determined. It is sufficient to say that it is the usual mode of treatment of plane waves, an harmonic form being assumed and the constants determined so as to satisfy the equations of motion and the boundary conditions.

We shall take then, as the one medium, water; and, as the other, rock of density 3, rigidity $1{\cdot}5\times10^{11}$ and Poisson ratio ·25. The density of the water is taken as unity and the value of the bulk-modulus, which in this case is also the wave-modulus, $2{\cdot}2\times10^{10}$. The quantities are given in C.G.S. units. The manner in which, for different angles of incidence in the rock, the energy of the incident wave is distributed amongst the reflected and refracted waves is shown in the following tables. The first refers to the case of the incident wave being condensational; the second to the case of the incident wave being distortional. The quantities A A_1 A' represent the energies of the incident, reflected, and refracted condensational waves; B B_1 B' the energies of the similar set

* [The mathematical investigation and formulæ are given below, Part III.]

of distortional waves. The corresponding angles of incidence, reflexion, and refraction are given in contiguous columns—θ referring to the condensational and ϕ to the distortional waves.

Incident Wave Condensational.

Incident.		Reflected.	Refracted.		Reflected.	
θ.	A.	A_1.	θ.	A′.	ϕ_1.	B_1.
0°	1	·599	0°	·401	0°	·000
10°	1	·536	3° 49′	·397	5° 45′	·071
20°	1	·377	7° 32′	·370	11° 23′	·254
30°	1	·195	11° 2′	·333	16° 35′	·456
40°	1	·056	14° 15′	·293	21° 47′	·660
50°	1	·006	17° 4′	·244	26° 15′	·753
60°	1	·014	19° 22′	·206	30°	·775
70°	1	·031	21° 5′	·188	32° 15′	·783
80°	1	·000	22° 9′	·182	34° 39′	·818
89°	1	·616	22° 31′	·069	35° 16′	·314
90°	1	1		·000		·000

Incident Wave Distortional.

Incident.		Reflected.	Reflected.		Refracted.	
θ.	B.	B_1.	θ_1.	A_1.	θ'.	A′.
0°	1	·1	0°	·0°	0°	·0
10° 23′	1	·711	20°	·253	7° 32′	·036
21° 47′	1	·222	40°	·656	14° 15′	·126
30°	1	·014	60°	·779	19° 22′	·206
34° 39′	1	·027	80°	·815	22° 9′	·157
35° 16′	1	·679	89°	·311	22° 31′	·007
35° 6′	1	1	90°	·000	22° 31′	·000
36°	1	·584		non-existent.	22° 56′	·415
40°	1	·461			25° 14′	·539
50°	1	·504			30° 33′	·495
60°	1	·506			35° 4′	·494
70°	1	·520			38° 34′	·480
80°	1	·634			40° 47′	·366
89° 45′	1	·818			41° 34′	·183
90°	1	1				·000

In the first table B and B′ of course do not appear; and in the second table A and B′ do not appear.

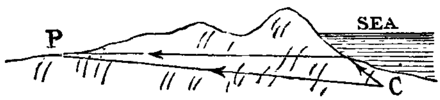
It should be mentioned that each wave-energy is calculated independently; and a test of the accuracy of the calculations is afforded by the condition that the energy of the incident wave must be fully accounted for. That is, since, in every case the incident wave (either A or B) is taken as unity, the sum of all the others must be unity.

The chief peculiarities embodied in these tables are shown graphically in the corresponding curves (figs. 5 and 6, p. 86). Any one curve represents the manner in which the energy of each wave depends on the angle of incidence. The angles of incidence are measured off along the horizontal line; and the corresponding energies are represented by the ordinates perpendicular thereto. The energy of the incident wave is of course represented by a straight line at unit distance from the line along which the angles of incidence are measured off.

The first set of curves shows the state of things for an incident condensational wave. For the sake of brevity, we shall occasionally refer to the different waves by the letters A A_1 A′ B B_1 chosen to represent their energies. At perpendicular incidence condensational waves only are started at the bounding surface; and as the angle of incidence increases the energies of both of these diminish. A′, which we may also call the water wave, seems to fall off continuously until it vanishes at grazing incidence. The A-wave, however, vanishes at two distinct incidences, and after 80° is reached begins to increase till at 90° it attains unity. The behaviour of this reflected condensational wave is extremely curious, the wave being practically non-existent for incidences between 50° and 80°. The greater part of the energy of the incident wave is then accounted for by the B_1 or reflected distortional wave. For incidences higher than 45°, three-quarters of the whole incident energy is so transformed. It will be noticed that up to pretty high angles of incidence the energy of the water-wave does not suffer any very great falling off.

Turning now to the second set of curves, which show the state of things for an incident distortional wave, we meet with some very curious relations. For reasons already discussed, the A_1-wave cannot exist for incidences higher than a certain critical value, which depends only on the rock itself. The energy of this wave, however, attains a considerable maximum value for an angle of incidence slightly below this critical value. Almost for the same incidence, the energy of the B_1-wave falls to a very low minimum, almost vanishing indeed. Comparing this first portion of the second set of curves with the first set of curves as a whole, we see a general resemblance between the two. That is, the energy of the reflected wave of the same type as the incident wave rapidly falls off to a minimum as the angle of incidence grows, while that of the reflected wave of the other type rapidly increases to a maximum. Finally the energy of the reflected wave of the same type, in both cases quite abruptly, runs up to equality with the incident wave. In the second set of curves this happens at the angle of total reflexion; for, not only does the A_1-wave vanish, but so also does the A'-wave *—which indeed never attains any great significance at the lower incidences. After the critical angle of incidence is passed, however, the energy of the A'-wave soon reaches a maximum, being then of greater value than that of the B_1-wave, and gradually falls away to zero, while the energy of the B_1-wave as gradually rises to unity.

In trying in some way to bring these results into correspondence with earthquake phenomena, we notice first of all that, if an earthquake is to be regarded as a progressive wave in an elastic solid, the angles of emergence of the waves will generally be small—that is, the angles of incidence large. Hence we need pay but little attention to the state of affairs at the lower incidences. For higher incidences we see that whether the incident wave is condensational or distortional, the energy is reflected either wholly or almost wholly in the distortional wave form. Suppose for example that a disturbance begins at some region below the bottom of the sea, say at the point C in the figure; and let us assume that what starts from C is a simple wave of compression—that is a condensational wave. Then to any point P suitably placed, there will come not only a purely condensational but also a distortional wave produced by reflexion from some part of the surface separating the sea and the land.



It is easy to see, however, that this transformation of condensational into distortional straining will accompany all similar cases of reflexion at the boundary of two different media, whether the one medium is water or some other substance—air, say, or mud, or rock. Also we may safely assume that during *refraction* across a boundary separating two media, both being of the category of elastic solids, an incident condensational wave will give rise to a distortional as well as to a condensational refracted wave. In the light of these results, then, it is little wonder that no definite relation has ever been shown to exist between the manner of motion of a particle and the direction of propagation of an earthquake.

* This seems to be a result as novel as it is curious from a purely theoretical point of view, although it has no special bearing on earthquake phenomena. [See below, p. 95.]

I should also be inclined to regard as absolutely futile any attempt to infer the nature of the movement in which the shock originates from the nature of the motion of any surface particle.

Even in the extremely simple case of an isotropic elastic solid, we see how a single reflexion (and probably refraction) is sufficient to alter the type of wave motion, or rather to bring into existence the other type. How much more will this be true in such a heterogeneous mass as we know the earth's crust to be! And if the large earth shiftings, which certainly mean straining beyond the limits of elasticity, differ essentially from the purely elastic disturbances we have just been considering, it will not be in the direction of simplicity. It seems reasonable to expect in these also somewhat analogous, although much more complex, relations. Hence it may safely be concluded that the existence of distortional or transverse waves does not of necessity imply a *faulting* of rocks, any more than that the existence of the other type necessarily points to a *rupture* or an explosive increase of pressure. In short, as observation has only too plainly demonstrated, it seems vain to look for any certain separation of the normal and transverse types of vibration. Only when the origin of the disturbance is within a few miles of us, and is at an insignificant depth below the earth's surface, can we reasonably expect to find an appreciable separation of the two types of waves*.

At this stage we may very fitly consider the general import of the assumption of the existence of these two types of wave in earthquake motion. The assumption is tantamount to regarding the earth's crust as isotropic. Such a characteristic may safely be applied to surface soil; so that, in artificial earthquake experiments, such as Milne has carried out, it may be an easy matter to distinguish the normal vibrations as their wave outstrips that of the transverse vibrations. But it is altogether out of the question to regard any stratified rock as isotropic; while as for non-stratified rocks, their heterogeneity makes a theoretical discussion of their elastic properties impossible. By consideration then of the elastic properties of homogeneous isotropic media, we can only hope to get at best a glimpse into the seismic darkness. And small though the present contribution may be to the vast problem of earthquake motion, it surely will have some value if only it opens our eyes to the

* [This statement requires modification in the light of recently acquired knowledge regarding the transmission at great speeds of tremors through thousands of miles of the earth's material.]

vanity of expecting the study of surface motions to throw much light on the question of earthquake origin.

And now let us pass to the discussion of the refracted water-wave. Here a glance at the two sets of curves shows that the incident distortional wave is, at the higher incidences, much more efficient than the condensational wave in creating a progressive disturbance in the water. The angle of refraction can never exceed 42° ; so that even for very high incidences the water wave will travel upwards to the surface tolerably directly. Here I think we may have the explanation of the curious bumpings which have sometimes been felt at sea. These must not be confounded with the so-called tidal waves so frequently the companions of earthquakes, and due almost without a doubt to large displacements of the ocean bottom. What I refer to here are the jerks or shakings (sometimes accompanied by sounds) discussed by Milne in the opening paragraph of chapter ix. of his book on 'Earthquakes.' Sounds of course will be heard if the periodic time of any of the components in the wave-motion is short enough, and if at the same time the intensity is sufficient to give rise to *audible* sound waves in the air, either directly, or indirectly through the medium of such a solid as a ship. According to Colladon's experiments at the Lake of Geneva, the speed of sound in water at 8°·1 C. is 1435 metres per second. This gives 14·35 metres (or about 8 fathoms) for the wave-length of a wave whose pitch is 100 vibrations per second. A slower vibration will of course give a longer wave-length ; and a quicker a shorter. But enough has been said to show that in such a wave of condensation we have something quite fitted to affect even a large ship as a whole.

Now all that has been said regarding the transference of vibrations from rock to water will, in a general way, hold true of their transference from rock to air. For all angles of incidence in the rock, the angle at which the refracted ray passes out into the air is very small. Thus, returning to the equation

$$\frac{m}{\rho} \operatorname{cosec}^2 \theta = \frac{m'}{\rho'} \operatorname{cosec}^2 \theta,$$

and giving m', the wave-modulus in air, the value $1{\cdot}41 \times 10^6$, and ρ' the value $\cdot 0013$, we find, with the same values as formerly for the rock constants,

$$\operatorname{cosec}^2\theta = \cdot 00242 \operatorname{cosec}^2 \theta'.$$

Hence if $\theta = 90°$, $\theta' = 2° \; 50'$ nearly.

In the same way, calculating for the incident distortional wave, we obtain

$$\operatorname{cosec}^2\phi = \cdot 00726 \operatorname{cosec}^2\theta'.$$

Hence if $\phi = 90°$, $\theta' = 4° \ 53'$ fully. Thus, whatever the incidence, the refracted wave goes off in a direction never more than 5° removed from the normal.

Into a detailed calculation regarding the distribution of the energy, it is not necessary to go*. The amount of energy which gets into the air as a condensational wave is extremely small compared to the vibratory energy existing in the rock. With the constants as given above it is doubtful if for any incidence as much as the thousandth part of the original energy is so transmitted into the air. For most incidences it is distinctly less.

It is thus easy to see why in earthquakes which may be accompanied by considerable mechanical violence, there may be no audible sound phenomena. The essential condition for the production of earthquake sounds is a sufficiently pronounced vertical motion with a sufficiently rapid period. According to Professor Sekiya's recent analysis †, vertical motion as measured on the seismographs is absent from most of the earthquakes that shake Japan. When vertical motion is apparent, it is in the more intense shocks. We cannot assume of course that the vertical motion is absent in those cases in which the seismograph shows no trace of it. It is always much smaller than the horizontal motion, being on the average only one-sixth of it. Hence when the horizontal motion is itself very small, as in the weaker shocks, the vertical motion may be too small to affect the seismograph. Or, as is more than likely, it may have too short a period to make itself felt, even though its amplitude may be large enough to be otherwise apparent. We must be careful indeed not to confuse the seismograph indications with the rapid elastic vibrations which seem a necessity for the production of sound phenomena. That the quick short-period motions that precede the big wave as shown on our seismographs may co-exist along with vertical vibrations sufficiently rapid to cause audible sounds is highly probable ; but in no other sense can they be regarded as "connected" with those sounds, as seems to be suggested by Milne. These rapid sinuosities appear on all the best diagrams showing the

* [This calculation is now given below, Part II. Case (4).]

† See Transactions Seism. Soc. of Japan, vol. xii. ; also the Journal of the College of Science, Imperial University, vol. ii.

horizontal motion ; but, as I believe, it is the vertical motion we must look to specially.

Another point brought out strongly by Sekiya's analysis is that in no case has he found the vertical motion precede the horizontal motion. The vertical seems always to show itself later. It is certain, however, that earthquake sounds are often heard before the earthquake shock is felt. This simply means that the big earth shiftings which affect our seismographs are preceded by rapid vibratory motions which, however large they may be, cannot have any mechanical effect on the instruments. The case is exactly similar to what happens if we pass alternating electric currents through the coil of an ordinary galvanometer. No matter how sensitive the galvanometer, or how intense the alternating current,—so long as the alternation is rapid enough, no effect is observed on the galvanometer needle. So it cannot fail to be with ordinary seismographs as regards rapid vibrations. I doubt if a seismograph, mechanically capable of registering vibrations occurring at even so slow a rate as 10 per second, has been as yet imagined. It is very questionable also if those sinuous records which the seismograph tracings show as precursors of the large slow waves really indicate what is taking place in the soil. For, exactly as very rapid vibrations will not show at all on the seismograph trace, so somewhat less rapid vibrations will not show to their full. There must always be a lagging of the record behind the motion recorded. Thus before a given motion has its full effect on the seismograph, the rapid reverse motion may set in and prevent anything like a complete record. Not until a comparatively slow "swing-swang" of the ground takes place can we hope to have a tracing even approximately true as to amplitude. It is therefore well, I think, that seismologists should bear this point in mind. It is highly probable that an earthquake is preceded by rapid vibr tory motions. That we should expect ; and the early sinuosities of earthquake tracings certainly suggest the same. But that these sinuosities can be taken as an approximate representation of the amplitudes or periods of the rapid vibration to which they are due may well be matter of grave doubt.

In conclusion, I would draw special attention to the following point which seems to be of some importance.

In the discussion of the propagation of seismic disturbances through the earth's crust, a clear distinction should be drawn between purely elastic and QUASI-*elastic phenomena.* So long as the materials co stituting the earth's crust are not strained distinctly beyond the limits of elasticity, we have to do with purely elastic vibrations. These, generally speaking, will be transmitted with considerable speed, comparable to that of sound in steel wires. Such high speeds have indeed been observed, their existence depending upon a small compressibility (or high rigidity) combined with a comparatively small density. The destructive effects of earthquakes are, however, due to the propagation of *quasi*-elastic disturbances. In them the material is distinctly strained beyond the limits of elasticity; or, at all events, so strained as to bring about conditions in which other strain-coefficients than the usual ones of rigidity and compressibility play the important part. It is quite to be expected that these *quasi*-elastic disturbances should travel much more slowly than the purely elastic ones. The investigation given above into the effect upon the type of elastic waves as they suffer reflexion at the boundary of two isotropic elastic media suggests the existence of analogous effects in the propagation of all seismic disturbances. The æolotropy and discontinuity of the earth's crust will transform a disturbance of an originally simple type into one or more of excessive complexity. Furthermore, wherever a *quasi*-elastic disturbance suffers transformation at some region of discontinuity, it will give rise to a new set of elastic disturbances. And again, as the *quasi*-elastic disturbances lose energy per unit volume, partly because of radiation, partly because of dissipation, they will gradually lose their *quasi*-character, and become of a purely elastic nature. It is quite conceivable, then, that under certain circumstances the speed of a disturbance might undergo strange variations, appearing even to be accelerated as its intensity diminished. Such a phenomenon was observed by Lieut.-Col. Abbot at Flood Rock explosion in 1885. Of course a peculiar change of this kind might easily be due to the different elastic properties of successive portions of rock travelled through. It is quite possible, however, that the other explanation is the true one. It is known that a very intense sound travels faster in air than one of less intensity; and the same will be true of vibrations in elastic solids. But there must be a superior limit to the intensity, for intensities above which this relation will cease to hold. Viscosity, friction, and the little understood effects of permanent strain will make themselves more and more strongly felt as the strains increase beyond the limits of elasticity. I understand, indeed, that in the case of cannon-reports the sound has been observed to travel somewhat less rapidly during its early than during its after stages. Here the very large initial aërial disturbances bring in conditions, either thermodynamic or elastic, under which the ordinary theory fails even in approximate application. If such a phenomenon is met with in the comparatively simple case of sound-waves in air, similar phenomena are certain to exist in the more complex cases that correspond to earth shakings.

Another point which this explicit recognition of purely elastic and *quasi*-elastic disturbances suggests is in relation to the measurement of earthquake velocities by comparison of the effects at distant stations. Thus the purely elastic tremors felt at stations far distant from the centre of seismic disturbance have probably not come as such directly therefrom. They are, so to speak, the feebler descendants of the *quasi*-elastic disturbances, which may have caused havoc at localities nearer to the seismic centre. The initial elastic tremors felt at these nearer stations will reach the further distant ones with intensities so diminished as no longer to be appreciable. Thus in the very usual method of timing the arrival of a tremor by the blurring of an image reflected from the surface of mercury, it is evident that the speed, as estimated between two stations in the line of propagation of the disturbance, must be somewhat smaller than the true value. For before the particular tremors which sufficiently blurred the image at the first station have reached the second one, their intensity has become diminished. Hence the sufficient blurring of the image at the second station is due to the diminished violence of tremors which passed through the first station subsequently to the blurring of the image there. Now the same reasoning will apply with even greater force to other than mere tremors; and especially will it apply to the case of the propagation of the quasi-elastic disturbances which constitute dangerous earthquakes.

If the views so far expressed are correct, there is no difficulty in understanding the nature of earthquake-sounds. As already pointed out, they are to be traced to rapid vertical vibrations of the ground, so rapid as to be inappreciable on our seismographs. Sometimes they may be due to transverse vibrations of walls caused by horizontal displacements of the ground; or, as suggested by Mallet, they may be transmitted through the framework of the body. That these sounds should frequently precede the coming of the true earthquake-shock is simply due to the running ahead of the purely elastic waves. The nature of the rock or soil through which these waves proceed will have a powerful influence upon their final intensity. Thus in soft rock viscosity will soon destroy the vibrations of short period. In such circumstances there will be less chance of hearing earthquake-sounds than when the rock is hard and solid. Very frequently earthquake-sounds die away before the earth-swayings have ceased—a fact which is probably connected with the short-lived character of the vertical motion as compared with the horizontal motions traced out by our seismographs.

With the air pulsations, which if rapid enough constitute audible sounds, the following curious effect of earthquakes may have some connexion. I am indebted to Professor Sekiya for the information. It seems that at the time an earthquake-shock passes, or it may be a little sooner, birds flying in the air have been seen to drop suddenly, as if for an instant paralysed, and then to recover themselves. This effect might be sufficiently explained as due to a momentary mental paralysis produced by fear. Perhaps, however, we have a sufficient physical cause in the air pulsations, a slight change of density being enough to disturb the delicate poise of the hovering bird.

Part II.

Additional Examples of Reflexion and Refraction of Elastic Waves in the Earth's Substance.

To the detailed numerical results formerly published for the case of rock and water, I now add the corresponding results for rock and rock, for rock and air, for solid rock and fluid rock, assuming certain relations among the densities and rigidities. The hypothetical case of rigidities equal and densities as 1 to 2 was worked out several years ago; the others have been worked out quite recently. In calculating these I have taken as the angles of incidence the angles for which the cotangent has values ∞, 4, 2, 1, 0·6, 0·3, 0·1. When expedient I have introduced other angles, especially when there were *critical* angles corresponding to cases of total reflexion. The angles of reflexion and refraction corresponding to these are given in the tables only to the nearest degree. The angle of incidence or reflexion of the condensational wave is represented by θ, and θ' is the angle of refraction of the wave of the same type. The symbols ϕ and ϕ' represent the angles of incidence (or reflexion) and refraction of the waves of distortional type. The letters A_1, A', represent the derivative condensational waves; and B_1, B', the derivative distortional. The tabulated numbers give the energies, the energy of the incident wave, A or B as the case may be, being taken as unity.

1. *Condensational-rarefactional Wave Incident on the Interface of two Elastic Solids; of which the Rigidities and Compressibilities are equal and the Densities as* 1 *to* 2.

(1) Incident Wave in the less dense medium.

θ.	A.	A.	θ'.	A'.	ϕ.	B_1.	ϕ'.	B'.
0°	1	·03		·97		0		0
14° 2'	1	·024	10°	·970	8°	·004	5°	·003
26° 34'	1	·016	18°	·958	15°	·016	11°	·011
45°	1	·001	30°	·930	24°	·039	17°	·029
59° 2'	1	·011	37°	·881	30°	·062	21°	·047
73° 18'	1	·128	43°	·743	34°	·071	23°	·055
84° 17'	1	·511	45°	·410	35°	·044	24°	·035

(2) Incident Wave in the denser medium.

θ.	A.	A_1.	θ'.	A'.	ϕ.	B_1.	ϕ'.	B'.
0°	1	·03		·97				
14° 2'	1	·025	20°	·966	8	·004	11°	·006
26° 34'	1	·015	39°	·939	15°	·018	21°	·025
42° 16'	1	·020	72°	·767	23°	·093	33°	·119
45°	1	·444	90°	0	24°	·244	35°	·311
45° 17'	1	·463	imaginary.		24°	·237	36°	·302
48°	1	·514	,,		25°	·210	37°	·266
59° 2'	1	·631	,,		30°	·144	44°	·167
73° 18'	1	·825	,,		34°	·084	52°	·089
84° 17'	1	·936	,,		35°	·033	54°	·032

The chief points to be noted here are :—(*a*) The manner in which the energy of the reflected condensational waves, always small at the lower incidences, passes through a minimum value, and the suddenness with which, in the second case, it increases when the angle of total reflexion is reached ; (*b*) the simultaneous rapid increase, in the region of this critical angle, of the energy of the derivative distortional waves ; (*c*) the fairly large values of these distortional waves when the condensational wave is incident on the denser medium, more than half the energy taking this form for incidences in the neighbourhood of 45°.

2. *Distortional Wave Incident on the Plane Interface of two Elastic Solids, of which the Rigidities and Compressibilities are equal and the Densities as* 2 *to* 3.

(1) Incident Wave in the denser medium.

ϕ.	B.	B_1.	ϕ'.	B'.	θ.	A_1.	θ'.	A'.
0°	1	·011		·989		·0		·0
14° 2'	1	·01	17°	·978	25°	·005	31°	·007
26° 34'	1	·012	33°	·912	51°	·047	72°	·029
45°	1	·01	60°	·99	imaginary.		imaginary.	
54° 24'	1	1·00	imaginary.		,,		,,	

For all higher incidences there is total reflexion.

(2) Incident Wave in the less dense medium.

ϕ.	B.	B_1.	ϕ'.	B'.	θ.	A_1.	θ'.	A.
0°	1	·011		·989				
14° 2'	1	·008	11°	·984	25°	·005	20°	·004
26° 34'	1	·027	21°	·943	51°	·019	39°	·015
45°	1	·01	35°	·99	imaginary.		imaginary.	
59° 2'	1	·01	44°	·99	,,		,,	
73° 18'	1	·088	51°	·912	,,		,,	
90°	1	1·00	54°	0·00	,,		,,	

Similar peculiarities present themselves here. In case (1), for example, the refracted distortional wave accounts for nearly all the energy until the critical angle is reached, when the energy in the reflected distortional wave suddenly increases to the final value unity. The comparative smallness in the energies of the derived condensational waves is due to the densities being taken as 2 to 3, that is, more nearly equal than in the previous case.

3. *Behaviour of Waves at the Plane Interface of the Slate and Granite, whose Elastic Constants are as given in the Table on* p. 68, *the Densities being assumed to be equal.*

(1) Distortional Wave incident in the Slate.

ϕ.	B.	B_1.	ϕ'.	B'.	θ.	A_1.	θ'.	A'.
0°	1	·016		·984				
14° 2'	1	·006	11°	·981	22°	·006	20°	·008
26° 34'	1	·001	20°	·960	45°	·005	39°	·032
39° 25'	1	·008	29°	·787	90°	0	62°	·2
45°	1	·020	33°	·922	imaginary.		81°	·058
47° 56'	1	·02	34°	·98	,,		90°	0
59° 2'	1	·085	42°	·915	,,		imaginary.	
73° 18'	1	·144	48°	·856	,,		,,	
84° 17'	1	·386	50°	·614	,,		,,	

(2) Distortional Wave incident in the Granite.

ϕ.	B.	B_1.	ϕ'.	B'.	θ.	A_1.	θ'.	A'.
0°	1	·016		·984				
5° 43′	1	·014	7°	·983	10°	·001	12°	·002
14° 2′	1	·004	18°	·977	26°	·006	30°	·013
26° 34′	1	·009	35°	·920	54°	·0002	65°	·071
29° 48′	1	·075	39°	·793	62°	·132	90°	0
33° 31′	1	·026	45°	·974	90°	0	imaginary.	
45°	1	·11	66°	·890	imaginary.		,,	
48° 1′	1	·145	74°	·855	,,		,,	
49° 43′	1	1·000	90°	0·000	,,		,,	

(3) Condensational-rarefactional Wave incident in the Slate.

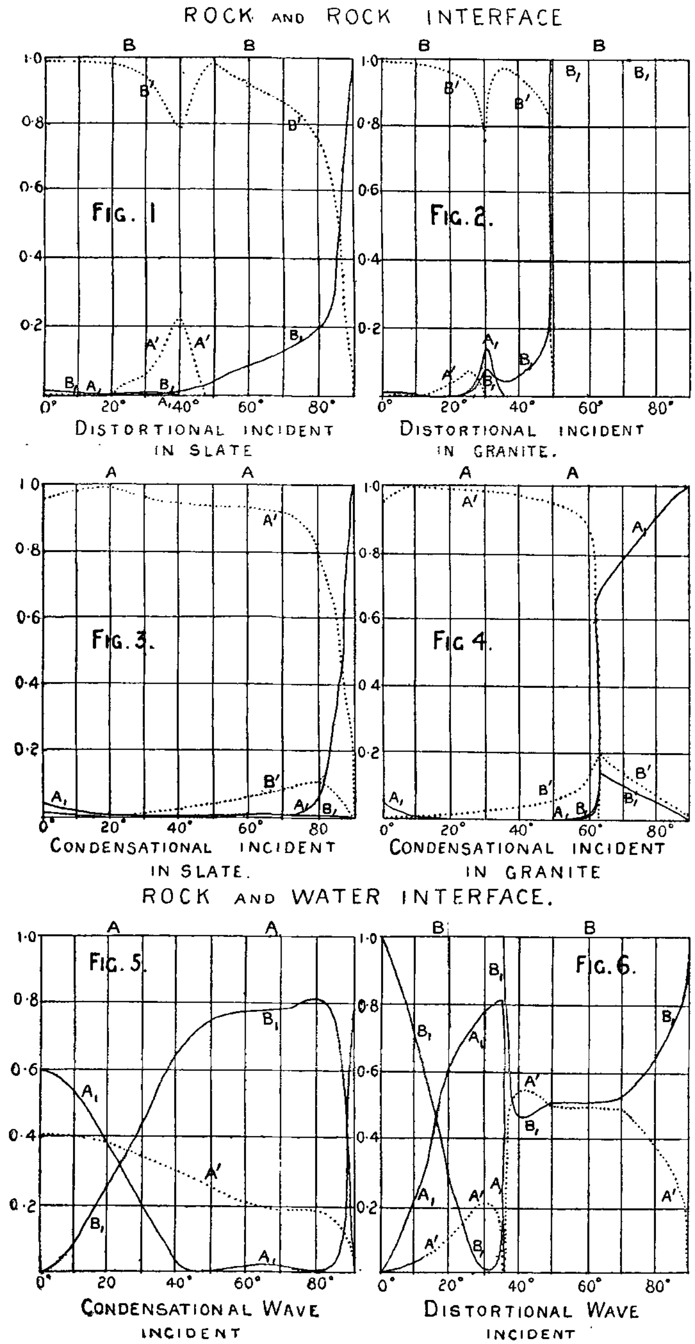
θ.	A.	A_1.	θ.	A'.	ϕ.	B_1.	ϕ'.	B'.
0°	1	·041		·959				
14° 2′	1	·002	12°	·992	9°	·003	7°	·003
26° 34′	1	·004	22°	·979	16°	·008	12°	·011
45°	1	·003	32°	·949	24°	·011	19°	·037
59° 2′	1	·004	50°	·940	33°	·0002	25°	·056
73° 18′	1	·006	58°	·904	37°	·004	28°	·083
84° 17′	1	·247	62°	·668	39°	·009	29°	·077
87° 17′	1	·500	62°	·441	39°	·006	29°	·053

(4) Condensational-rarefactional Wave incident in the Granite.

θ.	A.	A_1.	θ'.	A'.	ϕ.	B_1.	ϕ'.	B'.
0°	1	·041		·959				
5° 43′	1	·004	6°	·994	3°	·001	4°	·001
14° 2′	1	·002	16°	·991	8°	·003	10°	·004
26° 34′	1	·043	30°	·978	14°	·007	19°	·015
45°	1	·003	53°	·948	23°	·005	30°	·044
59° 2′	1	·006	75°	·890	29°	·004	38°	·100
62° 33′	1	·664	90°	0	30°	·131	39°	·198
73° 18′	1	·813	imaginary.		32°	·081	43°	·106
84° 17′	1	·932	,,		34°	·028	45°	·041
87° 17′	1	·962	,,		34°	·014	46°	·023

These four cases are shown graphically in figs. 1, 2, 3, and 4. The curves which refer to the second medium are dotted.

Undoubtedly the most peculiar are cases (1) and (2). For nearly all incidences in the one case, and for incidences up to the critical angle for complete total reflexion in the other, the refracted distortional wave, B′, is by far the most important. Except for a limited region in the vicinity of the angle at


ROCK AND ROCK INTERFACE
FIG. 1
DISTORTIONAL INCIDENT IN SLATE
FIG. 2.
DISTORTIONAL INCIDENT IN GRANITE.
FIG. 3.
CONDENSATIONAL INCIDENT IN SLATE.
FIG 4.
CONDENSATIONAL INCIDENT IN GRANITE
ROCK AND WATER INTERFACE.
FIG. 5.
CONDENSATIONAL WAVE INCIDENT
FIG. 6.
DISTORTIONAL WAVE INCIDENT

which the reflected condensational wave vanishes, 90 per cent. of the whole initial energy is transferred to the second medium in the distortional form. The curious fall in the B′ curve, and the corresponding abrupt rise in the A′ or A_1 curves (or in the B_1 curve) are very characteristic. Meanwhile the energy of the reflected distortional wave remains very small; and not until the condensational derivatives have become imaginary does it attain any appreciable value. It then increases, for the higher incidences, at an acceleratingly rapid rate, finally becoming equal to unity—at grazing incidence in the one case, at the critical angle of 49° 43′ in the other.

It is obvious that, with greater divergences among the values of the constants than those here chosen, the condensational waves in cases (1) and (2) and the distortional waves in cases (3) and (4) would have become relatively more important; but the example brings out very clearly the complex nature of the whole phenomenon of reflexion at the plane interface of two elastic solids.

4. *Behaviour of Waves at the Plane Interface of Rock and Air, the Elastic Constants of the rock being taken as on* p. 68, *with the exception of the Density, which is taken* 2000 *times that of air.*

(1) Distortional Wave incident in Rock.

ϕ.	B.	B_1.	θ.	A_1.	θ'.	A′.
0	1	1				
14° 2′	1	·534	25°	·466	1°·6	·00002
26° 34′	1	·025	51°	·975	3°	·00006
33° 40′	1	·003	74°	·997	3°·7	·00006
35° 13′	1	1	90°	0	3°·8	0·00000
39° 48′	1	1	imaginary		4°·3	·00019
45°	1	1	,,		4°·7	·00016
59° 2′	1	1	,,		5°·7	·00014
73° 18′	1	1	,,		6°·3	·00014
84° 17′	1	1	,,		6°·6	·00006

(2) Condensational Wave incident in Rock.

θ.	A.	A_1.	θ'.	A'_1.	ϕ'.	B_1.
0	1	1		·00013		
14° 2′	1	·828	0°·9	·00013	8°	·172
26° 34′	1	·464	1°·7	·00011	15°	·536
45°	1	·079	2°·7	·00009	24°	·921
59° 2′	1	·0002	3°·3	·00007	30°	1
73° 18′	1	·003	3°·7	·00006	34°	·997
84° 17′	1	·091	3°·8	·00005	35°	·909

The energy which escapes into the air is so small that practically the whole energy remains in the rock. The general behaviour of the phenomenon is very similar to what was found in the case of rock and water, the differences being differences of degree and not of kind. Thus we may make the graphs for rock and water serve in a rough way for rock and air by imagining a few slight changes to be made. In the graph for the incident condensational wave, imagine the distortional-energy curve, B_1, to run up into practical contact with A when the angle of incidence is about 60°, to remain very near to it till about 75°, and then to fall rapidly away to zero at 90°. At the same time, because of the great minuteness of the refracted energy, A′, the reflected condensational energy A_1 begins, at zero incidence, with practically unit value and is to a very close approximation the inversion of B_1. In like manner, the incident distortional wave is, for incidences between 0° and the critical angle 35° 13′, practically represented by the two reflected waves. The distortional energy begins and ends with value unity, passing through a small minimum value immediately before the critical angle is reached; while the condensational energy begins and ends with zero and passes through a maximum which is practically unity just before the critical angle is reached. The refracted condensational energy, A′, is very small throughout, and could not be shown graphically with the others unless it were drawn to a scale of at least 1000 to 1. Immediately after the critical angle is past the condensational energy in the air rises abruptly to the greatest value it ever attains, and falls off steadily with increasing incidences until it vanishes at grazing incidences. Practically the whole energy is retained in the solid in the purely distortional form.

5. *Behaviour of Waves at the Plane Surface of Rock and Fluid, whose Density and Compressibility are equal to those of the Rock.*

(1) Distortional Wave incident in the Rock.

ϕ.	B.	B_1.	θ.	A_1.	θ'.	A′.
0	1	1				
14° 2′	1	·725	25°	·144	18°	·131
26° 34′	1	·318	51°	·261	35°	·421
35° 13′	1	1	90°	0	48°	0
45°	1	·147	imaginary.		66°	·853
50° 44′	1	1		,,	90°	0
Higher incidences.	1	1		,,	imaginary.	

(2) Condensational-rarefactional Wave incident in the Rock.

θ.	A.	A_1.	θ'.	A'.	ϕ.	B_1.
0	1	·021	0°	·979		
14° 2'	1	·007	10°	·937	8°	·055
26° 34'	1	·003	19°	·839	15°	·158
45°	1	·104	32°	·646	24°	·256
59° 2'	1	·240	40°	·503	30°	·256
73° 18'	1	·268	46°	·471	34°	·261
84° 17'	1	·039	48°	·618	35°	·344
90°	1	1	48°·2		35°·3	0

The chief peculiarity in the first of these hypothetical cases is, perhaps, the vanishing at *two* incidences of the refracted wave in the fluid. It vanishes at the critical angle (35° 13') at which the reflected condensational wave disappears; and then it has its own critical angle (50° 44').

Between these limits its energy rises to a pronounced maximum. In these respects there is a broad similarity between this case and the cases of rock and water, and rock and air. The differences are only differences of degree, depending on the different relations among the densities.

The bearing of this investigation upon the question of Earthquake sounds has been discussed above. There is, however, another point on which some light seems to be thrown by these calculations. I refer to the Preliminary Tremors and (comparatively) Large Waves, which were first observed in 1889 and are now recorded on many delicate seismographs in countries which are not, in the ordinary sense, subject to earthquakes. The discussion of these forms the bulk of the last British Association Report of the Seismological Committee: no one doubts that they have come from a distant earthquake origin, the preliminary tremors outrunning the big waves as they pass through the earth. The origin being known, it is an easy matter to calculate approximately the average velocity of the swiftest of these tremors. Following out one of Milne's suggestions, I have found * that the square of this average speed may be represented by the formula

$$v^2 = 2{\cdot}9 + {\cdot}026\,d,$$

where d is the average depth of the chord joining the earthquake origin with the station where the tremors are recorded, the units being miles and seconds.

This involves an increase of about 1·2 per cent. per mile

* See "The New Seismology" in the 'Scottish Geographical Magazine,' January 1899.

descent below the earth's surface in the value of the elastic coefficient which determines the speed of transmission of the preliminary tremors.

But these preliminary tremors, from the moment they begin to show themselves, continue until the large wave-motion sets in; and are probably continuous with the tremors which survive after the large waves have died away. If we regard the first recorded tremors as having passed from the earthquake origin to the station at which they are being recorded by the path of shortest time through the earth, the subsequent tremors may be regarded as having arisen in one of two ways. They may be sent off as forerunners from the wave-front of the Large Waves, especially when this wave-front is passing across surfaces of discontinuity; or they may come by more or less circuitous paths, after it may be several reflexions from fissures or other surface barriers. A very distinct change in the elastic constants or in the densities of the materials in contact is sufficient to make the interface, for certain incidences, a practical barrier to the transmission of waves. Milne's recent discovery of reverberations, that is, the recurrence of the same groups of waves in the tremor record, seems to demonstrate the existence of reflexion of waves within the body of the earth.

Part III.

Theoretical Discussion of the Behaviour of Elastic Waves at the Plane Interface of Solids and Fluids.

The equations of motion for plane waves in an elastic solid are expressible in the form

$$\left.\begin{aligned}(m+n)\nabla^2\phi &= \rho\frac{d^2\phi}{dt^2}\\ n\nabla^2\psi &= \rho\frac{d^2\psi}{dt^2}\\ n\nabla^2\zeta &= \rho\frac{d^2\zeta}{dt^2}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I.})$$

in which, the plane XY being taken perpendicular to the wave-front, the displacements in directions X, Y, Z are respectively

$$\xi=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y},\quad \eta=\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x},\quad \zeta=\zeta,$$

and in which ρ is the density, n the rigidity, and $(m-\frac{1}{3}n)$ the bulk-modulus or resistance to compression.

The stress-components have the values

$$\left.\begin{aligned}
P&=(m+n)\nabla^2\phi-2n\frac{\partial\eta}{\partial y}, & S&=n\frac{\partial\zeta}{\partial y}\\
Q&=(m+n)\nabla^2\phi-2n\frac{\partial\xi}{\partial x}, & T&=n\frac{\partial\zeta}{\partial x}\\
R&=(m-n)\nabla^2\phi, & U&=n\left(2\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)
\end{aligned}\right\} \quad \text{(II.)}$$

where P, Q, R, S, T, U have the same meanings as in Thomson and Tait's 'Natural Philosophy.'

The components of stress on the plane whose normal has direction-cosines λ, μ, ν, are

$$\left.\begin{aligned}
F&=P\lambda+U\mu+T\nu\\
G&=U\lambda+Q\mu+S\nu\\
H&=T\lambda+S\mu+R\nu
\end{aligned}\right\} \quad \ldots\ldots \text{(III.)}$$

The waves of the ϕ type are condensational-rarefactional waves travelling with a speed equal to $\sqrt{(m+n)/\rho}$. The waves of the ψ type are purely distortional waves travelling with speed $\sqrt{n/\rho}$, the vibrations being in the plane XY.

The ζ displacement belongs also to a purely distortional wave, the vibrations being at right angles to the plane XY.

Let the plane interface between two media, $mn\rho$ and $m'n'\rho'$, be perpendicular to the X-axis, then any incident wave of the ζ type will, in general, break up into two waves, a reflected wave and a refracted wave, of the same type.

But any incident wave of the ϕ type will, in general, break up into *four* parts, two distortional (reflected and refracted) as well as two condensational-rarefactional (reflected and refracted).

Similarly any incident wave of the ψ type will also, in general, break up into four parts, two condensational-rarefactional as well as two distortional.

The necessity for this duplication of reflected and refracted waves may be easily shown by a simple consideration of the boundary conditions which must be satisfied at the interface.

Even in the simple case of a solid bounded by an impassable barrier, we must assume the two reflected waves as derived from the one incident wave, or we encounter an absurdity.

This important principle does not seem to have been explicitly recognized in the literature of the subject, although Green's treatment of what Kelvin calls the *pressural* wave involves it *. In 1887 I introduced the complete discussion of the problem in my lectures on Elasticity to my advanced students in the Imperial University of Japan; and made some definite calculations in regard to earthquakes in a paper which was read before the Seismological Society of Japan in February 1888 and published in their 'Transactions.' This is the paper which is reproduced above.

In the September number of the 'Philosophical Magazine' for 1888, Lord Kelvin gives the formulæ for the case of the incident distortional waves, and discusses a similar problem in the February number for this year (p. 179).

I propose now to give the complete solutions for the different cases, the meanings of which are brought out by the definite numerical calculations given above.

I. *Distortional Wave at the Interface of two Elastic Solids.*

The solution is of the form

$$\left.\begin{aligned} \psi &= B\epsilon^{ib(cx+y+\omega t)} + B_1\epsilon^{ib(-cx+y+\omega t)}, \\ \phi &= A_1\epsilon^{ib(-\gamma x+y+\omega t)}, \end{aligned}\right\} \text{ in medium } mn\rho\,;$$

$$\left.\begin{aligned} \psi' &= B'\epsilon^{ib(c'x+y+\omega t)}, \\ \phi' &= A'\epsilon^{ib(\gamma' x+y+\omega t)}, \end{aligned}\right\} \text{ in medium } m'n'\rho'.$$

The equations of motion in the two media give the relations

$$\left.\begin{aligned} n(c^2+1) &= \rho\omega^2 = (m+n)(\gamma^2+1), \\ n'(c'^2+1) &= \rho'\omega^2 = (m'+n')(\gamma'^2+1), \end{aligned}\right\}. \quad . \quad . \quad (1)$$

The quantities $c\ c'\ \gamma\ \gamma'$ are evidently the cotangents of the angles of incidence, refraction, and reflexion of the various waves.

The conditions to be satisfied at the surface, $x=0$, are

(1) Equality of normal displacements on each side of the

* To show how completely the principle was neglected, I need but refer to Question 9 on p. 378 of Ibbetson's 'Elasticity' (1887), in which a plane so-called "sound-wave" is assumed to give rise to reflected and refracted waves of like type *only*, when it impinges on the plane interface of two elastic solids. It is taken for granted that only the *normal* components of displacement and of stress in the two media are equal at every point of the interface. But no reason is even hinted at why the *tangential* components should be treated as of no account. In fact, for two solids in slipless contact, all four conditions must be satisfied.

interface, or

$$\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}=\frac{\partial\phi'}{\partial x}+\frac{\partial\psi'}{\partial y}\quad\text{when } x=0.$$

(2) Equality of tangential displacements on each side of the interface, or

$$\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x}=\frac{\partial\phi'}{\partial y}-\frac{\partial\psi'}{\partial x}\quad\text{when } x=0.$$

(3) Equality of normal stresses on each side of the interface, or

$$(m+n)\nabla^2\phi-2n\left(\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\psi}{\partial x\partial y}\right)=(m'+n')\nabla^2\phi'-2n'\left(\frac{\partial^2\phi'}{\partial y^2}-\frac{\partial^2\psi'}{\partial x\partial y}\right)$$

when $x=0$.

(4) Equality of tangential stresses on each side of the interface, or

$$n\left(2\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)=n'\left(2\frac{\partial^2\phi'}{\partial x\partial y}+\frac{\partial^2\psi'}{\partial y^2}-\frac{\partial^2\psi'}{\partial x^2}\right)$$

when $x=0$.

These lead to the equations

$$\left.\begin{aligned}-\gamma A_1+\qquad X &= \gamma'A'+\qquad B'\\ A_1-\qquad cY &= A'-\qquad c'B'\\ (c^2-1)A_1+\qquad 2cY &= \frac{n'}{n}(c'^2-1)A'+\frac{2n'}{n}c'B'\\ 2\gamma A_1+(c^2-1)\,X &= -\frac{2n'}{n}\gamma'A'+\frac{n'}{n}(c'^2-1)B'\end{aligned}\right\}(2)^*$$

where $X=B+B_1$, and $Y=B-B_1$.

Multiplying together the first and third of these and also the second and fourth, and then taking the difference of the two equations so obtained, we get

$$(c^2+1)\gamma A_1^2-(c^2+1)cXY=-\frac{n'}{n}(c'^2+1)\gamma'A'^2-\frac{n'}{n}(c'^2+1)c'B'^2,$$

which by (1) becomes

$$\gamma\rho A_1^2+\gamma'\rho'A'^2+c'\rho'B'^2=c\rho XY=c\rho B^2-c\rho B_1^2. \quad . \quad . \quad . \quad (3)$$

This is the energy equation showing how the original energy ($c\rho B^2$) of the incident wave is distributed among the four waves into which it breaks up at the interface. In the

* These correspond to Kelvin's formulæ Nos. 39–42 in his paper of 1888 (Phil. Mag. xxvi. p. 422).

detailed numerical calculation of the ratios of the A and B quantities in any particular case, the energy equation supplies an important criterion of the accuracy of the work.

The equations (1) give the relations among the quantities $c\ c'\ \gamma\ \gamma'$ for any assumed values of the densities and elastic constants. Hence for any chosen value of c, that is for any chosen angle of incidence, the corresponding values of c', γ, γ' are readily calculated; and the numerical values of the coefficients in equations (2) can be filled in. We are thus left with four simple numerical linear equations from which any four of the five quantities can be determined in terms of the fifth *. B and B′ follow at once; and finally, by calculating the terms in the energy equation (3) and dividing throughout by $c\rho B^2$, we obtain numbers showing the partition of energy among the reflected and refracted waves.

When any one of the quantities c', γ, γ', becomes zero or imaginary, there is no wave of that type. In such cases the A and B quantities may work out in the form

$$p+q\sqrt{-1},$$

and we must then take the expression p^2+q^2 as the number on which the energy depends.

II. *Distortional Wave at the Interface of an Elastic Solid and a Fluid.*

There is no distortional wave in the second medium. The term in B′ has therefore no existence; and we have only three surface conditions. Obviously the second condition in the general problem is the one that must be dropped; while in the fourth condition the right-hand side becomes zero.

We may work out the solution *ab initio*, or we may get the necessary equations by putting $B'=0$, $n'=0$, and $c'^2=\infty$ in the first, third, and fourth of equations (1).

Also, c' being infinite,

$$\frac{n'}{n}(c'^2-1)=\frac{\rho'}{\rho}\frac{c^2+1}{c'^2+1}(c'^2-1)=\frac{\rho'}{\rho}(c^2+1).$$

* I have found it both quicker and more accurate to fill in the numerical values in the equations as they stand, and then solve the equations for every individual case, than to write out the several algebraic expressions for each ratio and then substitute. Except when n and n' are equal, or when either vanishes, the expressions are unwieldy. With a table of squares and square roots and with Crelle's *Rechentafeln* at hand, the four equations with numerical coefficients can be worked out with great ease.

Thus we find

$$\left.\begin{array}{lll} -\gamma A_1 + & X = & \gamma' A' \\ (c^2-1)A_1 + & 2cY = \frac{\rho'}{\rho}(c^2+1)A' \\ 2\gamma A_1 + (c^2-1)X = & & 0 \end{array}\right\} \quad . \quad . \quad (4)$$

These give

$$\left.\begin{array}{l} X = \frac{2\gamma'}{c^2+1}A' = \frac{2\gamma}{1-c^2}A_1 \\ Y = \frac{1-c^2}{2c}A_1 + \frac{\rho'}{\rho}\frac{1+c^2}{2c}A' \end{array}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

For numerical calculations these expressions are simple enough to be used as they stand.

This is one of the cases I worked out in my paper of February 1888.

The vanishing of everything except the reflected distortional wave when the cotangent of the angle of reflexion of the condensational-rarefactional part is zero, is clearly brought out both in the Table on page 73, and in the curve (fig. 6).

The condition is that $\gamma=0$ in equations (4).

Hence also

$$X = 0 \text{ and } X = A'\gamma',$$

so that A' also vanishes if γ' has a value differing from zero. This gives

$$B = -B_1,$$

and

$$Y = 2B = -\frac{c^2-1}{2c}A_1 \; ;$$

but the energy associated with the wave A_1 is proportional to the product $\rho\gamma A_1^2$, and therefore vanishes with γ. Thus at the critical angle of incidence at which the reflected condensational-rarefactional wave runs along the interface, the *refracted* condensational wave also vanishes, whatever its angle of refraction may be. The whole energy is found in the reflected distortional wave.

III. *Condensational-Rarefactional Wave at the Interface of two Elastic Solids.*

The solution is of the form

$$\left.\begin{aligned}\phi&=\mathrm{A}\epsilon^{ib(cx+y+\omega t)}+\mathrm{A}_1\epsilon^{ib(-cx+y+\omega t)}\\ \psi&=\mathrm{B}_1\epsilon^{ib(\gamma x+y+\omega t)}\end{aligned}\right\}\text{ in medium } mn\rho\,;$$

$$\left.\begin{aligned}\phi'&=\mathrm{A}'\epsilon^{ib(c'x+y+\omega t)}\\ \psi'&=\mathrm{B}'\epsilon^{ib(\gamma' x+y+\omega t)}\end{aligned}\right\}\text{ in medium } m'n'\rho'.$$

The equations of motion give the relations

$$\left.\begin{aligned}(m+n)(c^2+1)&=\rho\omega^2=n(\gamma^2+1)\\ (m'+n')(c'^2+1)&=\rho'\omega^2=n'(\gamma'^2+1)\end{aligned}\right\}\quad . \quad . \quad (1')$$

The conditions to be satisfied at the surface ($x=0$) are identical with those for the incident distortional wave, and lead to the equations

$$\left.\begin{aligned}\mathrm{B}_1+cY&=\mathrm{B}'+c'\mathrm{A}'\\ \gamma\mathrm{B}_1+X&=-\gamma'\mathrm{B}'+\mathrm{A}'\\ -2\gamma\mathrm{B}_1+(\gamma^2-1)X&=2\frac{n'}{n}\gamma'\mathrm{B}'+\frac{n'}{n}(\gamma'^2-1)\mathrm{A}'\\ (\gamma^2-1)\mathrm{B}_1-2cY&=\frac{n'}{n}(\gamma'^2-1)\mathrm{B}'-2\frac{n'}{n}c'\mathrm{A}'\end{aligned}\right\}\quad (2')$$

where $X=A+A_1$ and $Y=A-A_1$.

By taking the difference of the products of the first and third and of the second and fourth of these, and by suitable substitution according to (1′), we get the energy equation in the form

$$c\rho\mathrm{A}^2-c\rho\mathrm{A}_1^2=\gamma\rho\mathrm{B}_1^2+c'\rho'\mathrm{A}'^2+\gamma'\rho'\mathrm{B}'^2.\quad . \quad . \quad (3')$$

IV. *Condensational-Rarefactional Wave at the Interface of an Elastic Solid and a Fluid; incident in the Solid.*

Here again we must drop the second boundary condition; and, as in case No. II., are led to the simplified equations,

$$\left.\begin{aligned}\mathrm{B}_1+cY&=c'\mathrm{A}'\\ -2\gamma\mathrm{B}_1+(\gamma^2-1)X&=\frac{\rho'}{\rho}(\gamma^2+1)\mathrm{A}'\\ (\gamma^2-1)\mathrm{B}_1-2cY&=0\end{aligned}\right\};\quad . \quad (4')$$

whence

$$\left.\begin{aligned}Y&=\frac{\gamma^2-1}{2c}\mathrm{B}_1=\frac{c'}{c}\cdot\frac{\gamma^2-1}{\gamma^2+1}\mathrm{A}'\\ X&=\left(\frac{4c'\gamma}{\gamma^4-1}+\frac{\rho'}{\rho}\frac{\gamma^2+1}{\gamma^2-1}\right)\mathrm{A}'\end{aligned}\right\}\quad . \quad . \quad . \quad (5')$$

V. *Condensational-Rarefactional Wave at the Interface of a Fluid and an Elastic Solid ; incident in the Fluid.*

We get the solution from the general case by putting $B_1=0$, $n=0$, $\gamma=\infty$, and

$$\frac{n}{n'}(\gamma^2-1)=\frac{\rho}{\rho'}\frac{\gamma'^2+1}{\gamma^2+1}(\gamma^2-1)=\frac{\rho}{\rho'}(\gamma'^2+1),$$

and the result is

$$\left.\begin{aligned} cY &= B'+c'A' \\ \frac{\rho}{\rho'}(\gamma'^2+1)X &= 2\gamma'B'+(\gamma'^2-1)A' \\ 0 &= (\gamma'^2-1)B'-2cA' \end{aligned}\right\}; \quad . \quad (4'')$$

whence

$$\left.\begin{aligned} B' &= \frac{2c'}{\gamma'^2-1}A' \\ Y &= \frac{c'}{c}\frac{\gamma'^2+1}{\gamma'^2-1}A' \\ X &= \frac{\rho'}{\rho}\left(\frac{4c'\gamma'}{\gamma'^4-1}+\frac{\gamma'^2-1}{\gamma'^2+1}\right)A' \end{aligned}\right\} \quad . \quad . \quad . \quad (5'')$$

IV. *Leakage of Electricity from Charged Bodies at Moderate Temperatures.*

By J. C. Beattie, *D.Sc., F.R.S.E., Professor of Applied Mathematics and Physics, South African College, Cape Town* *.

§ 1. THE conditions in which a charged body retains its charge have been investigated with great thoroughness in many directions. The effect of heat, of light, of Röntgen rays, of uranium rays on the insulation ; the effect of the nature of the charged body; the effect of the surrounding atmosphere—its constitution and pressure; the effect of fumes from flames, have been made the subject of experiment. In quite recent years the subject has also been investigated from the point of view of what becomes of the electricity which leaks away from an insulated body in certain conditions. The object of the present series of papers is to communicate a number of results obtained by the writer on the leakage of electricity from metallic plates covered with various substances when the plates were placed in an atmo-

* Communicated by Lord Kelvin, having been read before the Royal Society of Edinburgh, May 1, 1899.