On Plane and Spherical Sound- Waves of Finite Amplitude

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X. *On Plane and Spherical Sound-Waves of Finite Amplitude.* By CHARLES V. BURTON, D.Sc.*

PART I.—PLANE WAVES.

1. THE subject of plane waves of finite amplitude has been considered by Riemann†; and so long as we confine our attention to the case where velocity and density are everywhere continuous, his investigation, as is well known, leaves little to be desired. It will not, therefore, be necessary here to make further reference to this aspect of the subject; but there is one part of Riemann's work which Lord Rayleigh has clearly shown to be unsatisfactory, and it is this point which we have now especially to consider. Lord Rayleigh says‡:—

“ It has been held that a state of motion is possible in which the fluid is divided into two parts by a surface of discontinuity propagating itself with constant velocity, all the fluid on one side of the surface of discontinuity being in one uniform condition as to density and velocity, and on the other side in a second uniform condition in the same respects. Now, if this motion were possible, a motion of the same kind

* Read February 24, 1893.

† “Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite,” *Gött. Abhandl.* t. viii. (1860); reprinted in *Werke*, p. 145.

‡ Theory of Sound, vol. ii. § 253, p. 41.

in which the surface of discontinuity is at rest would also be possible, as we may see by supposing a velocity equal and opposite to that with which the surface of discontinuity at first moves, to be impressed upon the whole mass of fluid. In order to find the relations which must subsist between the velocity and density on the one side (u_1, ρ_1) and the velocity and density on the other side (u_2, ρ_2), we notice in the first place that by the principle of the conservation of matter $\rho_2 u_2 = \rho_1 u_1$. Again, if we consider the momentum of a slice bounded by parallel planes and including the surface of discontinuity, we see that the momentum leaving the slice in the unit of time is for each unit of area $(\rho_2 u_2 = \rho_1 u_1) u_2$, while the momentum entering it is $\rho_1 u_1^2$. The difference of momentum must be balanced by the pressures acting at the boundaries of the slice, so that

$$\rho_1 u_1 (u_2 - u_1) = p_1 - p_2 = a^2 (\rho_1 - \rho_2),$$

whence

$$u_1 = a \sqrt{\left(\frac{\rho_2}{\rho_1}\right)}, \quad u_2 = a \sqrt{\left(\frac{\rho_1}{\rho_2}\right)}.$$

The motion thus determined is, however, not possible; it satisfies indeed the conditions of mass and momentum, but it violates the condition of energy expressed by the equation

$$\frac{1}{2} u_2^2 - \frac{1}{2} u_1^2 = a^2 \log \rho_1 - a^2 \log \rho_2."$$

2. The assumed motion here criticised is one in which density and velocity are constant for all points on the same side of the surface of discontinuity, while this surface itself is propagated through the fluid with constant velocity. It is easily shown, however, that the same objection applies when, on either side of the surface, velocity and density vary continuously in the direction of propagation, while the velocity of propagation of the surface is also allowed to vary. For let S (fig. 1) be a surface of discontinuity which is being propagated through the fluid, while the planes A, B, parallel to S and lying on either side of it, are fixed in the fluid. At a given instant let

Fig. 1.

distance of S from A = m ,

„ B „ S = n ;

density and velocity of fluid just to the left of S = ρ_1, u_1 ,
density and velocity of fluid just to the right of S = ρ_2, u_2 ;
velocity with which S is travelling = V .

Then, since A and B are fixed in the fluid, they are approximately moving with the respective velocities u_1, u_2 ; m and n being taken sufficiently small. On the same understanding, the mass of fluid between A and B (referred to unit surface) = $m\rho_1 + n\rho_2$; and since this mass must remain constant,

$$\frac{d}{dt}(m\rho_1 + n\rho_2) = 0;$$

i. e. in the limit, when m and n are infinitesimal,

$$\rho_1 \frac{dm}{dt} + \rho_2 \frac{dn}{dt} = 0,$$

or

$$\rho_1(V - u_1) = \rho_2(V - u_2). \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, if p_1 and p_2 are the pressures corresponding to ρ_1 and ρ_2 , the principle of momentum gives:—

$p_1 - p_2$ = rate of change of momentum between A and B

$$\begin{aligned} &= \frac{d}{dt}(u_1\rho_1 m + u_2\rho_2 n) \\ &= u_1\rho_1(V - u_1) - u_2\rho_2(V - u_2). \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

If the energy per unit volume corresponding to density ρ (in the absence of bodily motion) is called $\chi(\rho)$, the principle of energy would further give

$p_1 u_1 - p_2 u_2$ = rate of change of energy between A and B

$$\begin{aligned} &= \frac{d}{dt} \left\{ m \left(\frac{1}{2} \rho_1 u_1^2 + \chi(\rho_1) \right) + n \left(\frac{1}{2} \rho_2 u_2^2 + \chi(\rho_2) \right) \right\} \\ &= \left\{ \frac{1}{2} \rho_1 u_1^2 + \chi(\rho_1) \right\} (V - u_1) - \left\{ \frac{1}{2} \rho_2 u_2^2 + \chi(\rho_2) \right\} (V - u_2). \quad (3) \end{aligned}$$

Since (1), (2), and (3) involve only the instantaneous values of u_1, ρ_1, u_2, ρ_2 , and V , together with explicit functions of such values, while the space- and time-variations of all these quantities are absent from the equations, it is evident that the conditions to be satisfied at the surface S are the same as if

$u_1, \rho_1, u_2, \rho_2, V$ were absolute constants. We conclude then, that, *with our assumptions*, a surface of discontinuity cannot be propagated through a fluid with any velocity, uniform or variable, except under that special law of pressure for which progressive waves are of accurately permanent type.

3. What, then, becomes of waves of finite amplitude after discontinuity has set in? We may emphasize this difficulty, and at the same time obtain a clue to its solution, by considering the following case (fig. 2):—A is a piston fitting a cylindrical tube (or, if we please, is a portion of an unlimited rigid plane). All the air to the right of A is initially at rest and of uniform density, and then A is impulsively set in motion, and kept moving to the right with uniform velocity v . Consider the speed with which the disturbance generated by A advances into the still air to the right; it is evident that in all cases the front of the disturbance must advance faster than A. Take, then, the case in which

Fig. 2.

$$v > a,$$

where a is the propagation-velocity of infinitesimal disturbances. Two alternatives present themselves:—

(i.) If velocity and density are always either constant or *continuously* variable in the direction of propagation, the rate of propagation at any point will, in accordance with known principles, be

$$\sqrt{\frac{dp}{d\rho}} + u,$$

and therefore at the front of the disturbance, where $u=0$ and $\rho =$ the “undisturbed” density, the velocity of propagation will be simply $=a$; that is, *less* than the velocity with which A is advancing. Obviously this will not do.

(ii.) If velocity and density are *not* always either constant or continuously variable, that is, if one or more surfaces of discontinuity are being propagated through the air, we are met by the difficulty explained in the last section.

4. A simple mechanical analogy will help to indicate the actual motion. A number of equal spheres, of the same

material throughout, are capable of sliding without friction along a straight bar (fig. 3), and are connected together by a number of very weak and exactly similar springs (not shown), so that when there is equilibrium they are equally spaced

Fig. 3.

along the bar. If one of the spheres were moved backwards and forwards through a small range, a disturbance would travel through the whole system, but owing to the weakness of the connecting springs it would travel very slowly. Suppose, now, that the last sphere on the left hand is connected to a movable piston by a spring half the length of the others, but otherwise similar to them; and let this piston be suddenly moved to the right with a considerable velocity which is kept constant, and which we may call unity. The weak connecting spring between the piston and the first sphere produces no sensible effect until the two are almost in contact, when the sphere rebounds with velocity 2. This first sphere then strikes the second, imparting to it the velocity 2, and at the same time coming to rest. The positions of the spheres after successive equal intervals of time are represented in fig. 3, where the number written on any sphere represents its velocity just *after* the impact which it is suffering. No number is written on those spheres which have not so far been affected by the motion. From this it will be evident that when the piston moves to the right with a constant velocity which is very great compared with the propagation-velocity of infinitesimal vibrations of the system, the disturbance advances to the right with twice the velocity of the piston, provided that the diameters of the spheres are excluded from the reckoning.

Now suppose that the spheres are too small and too close together to be individually distinguished; then, at any instant,

the system will appear to be divisible into two parts, in one of which the velocity is unity, while in the other it is zero ; and in the moving part the spheres will appear to be twice as thickly condensed as in the still part. That the constant velocity of the piston is very great compared with the propagation-velocity of small vibrations is of course only a supposition introduced for the sake of simplicity. If, on the other hand, these two velocities are comparable, two adjacent spheres will always remain finitely separated from one another, and the velocity of any individual sphere within the disturbed stretch will never be as small as zero, or as great as twice the velocity of the piston ; the *mean* velocity within the disturbed stretch being equal to that of the piston. When the spheres are very small and very close together, we shall still have apparently an abrupt transition from finite velocity and greater density to zero velocity and smaller density ; and the energy, which is apparently lost as the spheres pass from the latter condition to the former, exists as energy of relative motion and unequal relative displacement amongst the spheres in the disturbed stretch.

5. Let us now compare the case just considered with the case of § 3 (fig. 2) : and first, concerning the nature of the analogy, it should be noticed that the individual spheres are not the analogues of the separate gaseous molecules, but that when both spheres and molecules are very small and very numerous, the apparently continuous properties of the system of spheres correspond to similar properties of the gas. The connecting springs represent the elasticity of the gas, isothermal or adiabatic as the case may be, and the energy of relative motion and unequal relative displacement amongst the disturbed spheres *suggests* that there is a production of heat over and above that which would be due to the (isothermal or adiabatic) change of density ; that is, a *dissipative production of heat*. The motion considered in the last section properly corresponds to the case where there is no conduction of heat, so that the connecting springs are the representatives of *adiabatic* elasticity, and the additional heat generated remains wholly within the more condensed part of the air. If we make the somewhat violent assumption that the temperature of the air remains constant throughout, the additional

heat generated will be conducted away isothermally, and the equivalent energy will be, for our purposes, entirely lost. To represent this case by means of our spheres we should have to regard the connecting springs as representing isothermal elasticity, while the energy of relative motion and unequal relative displacement among the disturbed spheres, as fast as it is produced, is to be consumed in doing work against suitable internal forces.

6. The mechanical system of spheres and springs, having suggested a solution, has served its purpose, and it now remains for us more closely to consider the aerial problem in the light of this suggestion. We may take, first, the case where the temperature is supposed to be invariable; for although such a supposition is necessarily far removed from the truth, it leads to very simple results, which indicate well enough the general character of the motion. Let the piston A (fig. 4) be moving to the right with constant velocity v (which may be either less or greater than a , the velocity of feeble sounds in air).

Assume all the air between A and a parallel plane surface B to have the velocity v

and density ρ_1 , while all the air to the right of B is at rest and has the density ρ_0 . Let the plane B move to the right with velocity V . Then the invariability of mass between A and a plane C fixed in the still air gives

$$\rho_1(V-v) - \rho_0 v = 0; \quad . \quad . \quad . \quad . \quad (4)$$

while from the principle of momentum,

$$\rho_1 v(V-v) = p_1 - p_0; \quad . \quad . \quad . \quad . \quad (5)$$

the pressure p being a function of ρ only, since the temperature is supposed to be constant throughout. If we assume for this case the truth of Boyle's law, so that $p = a^2 \rho$ always, (5) becomes

$$\rho_1(a^2 - Vv + v^2) = \rho_0 a^2, \quad . \quad . \quad . \quad . \quad (6)$$

which together with (4) is sufficient to determine V and ρ_1 when v and ρ_0 are given. Taking all these quantities to

Fig. 4.

remain constant throughout the motion, we see that at each instant the following conditions are satisfied :—

- (i) Every necessary condition between A and B, since density and velocity are there constant with respect to space and time ;
- (ii) Every necessary condition to the right of B, since the air there is at rest and in a constant uniform state ;
- (iii) Equality between the velocity of A and that of the air in contact with it ;
- (iv) At B, the conservation of mass and momentum, which are *necessary* conditions, and which, together with our supposition that the temperature is somehow maintained uniform, are *sufficient* to determine what takes place at B*.

Moreover, if at a time t (reckoned from the instant when A was impulsively started into motion) we take the distance of B from A to be $(V-v)t$, so that initially B coincides with A, the initial conditions are satisfied.

Thus the assumed motion satisfies all the necessary conditions ; it is therefore the actual motion.

7. Let us now examine what occurs when no heat is allowed to pass by conduction or radiation ; a state of things much more nearly realized in practice. Suppose the motion of A and the condition of the undisturbed air to be the same as in the last section, while the (constant) velocity of B is now called V' , and the density and pressure of the air between A and B (called ρ' , p' respectively) are also taken to be uniform and constant. At each instant, in place of (4) and (5), we shall now have

$$\rho'(V'-v) - \rho_0 v = 0, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\rho'v(V'-v) = p' - p_0. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Since we assume that there is no transference of heat by conduction or radiation, the rate at which the *total* energy of the system increases must be equal to the rate at which work is being done upon it by the piston A. Let θ_0 be the abso-

* Energy appears to be lost, because dissipatively produced heat is conducted away isothermally.

lute temperature to the right of B, that between A and B being θ' , and let us further assume for simplicity that

$$\frac{p}{\rho\theta} = \text{a const.};$$

while γ , the ratio of the two specific heats, is also supposed constant. It can then be shown without difficulty that the total energy *per unit mass* between A and B exceeds that to the right of B by

$$\frac{p_0}{(\gamma-1)\rho_0} \frac{\theta' - \theta_0}{\theta_0} + \frac{v^2}{2};$$

and multiplying this by $\rho_0 V'$, the mass of air which crosses one unit of the surface B in each unit of time, we obtain the rate (referred to unit area) at which the system is gaining energy. Again, the rate at which unit area of the piston does work on the system

$$= p'v = p_0 \frac{\rho'\theta'}{\rho_0\theta_0} \cdot v,$$

and equating this to the rate of gain of energy, we obtain

$$p_0 \frac{\rho'\theta'}{\rho_0\theta_0} v = \rho_0 V' \frac{v^2}{2} + \frac{p_0 V'}{(\gamma-1)\theta_0} (\theta' - \theta_0). \quad (9)$$

We may also write equation (8) in the form

$$\rho'v(V' - v) = \frac{p_0}{\rho_0\theta_0} (\rho'\theta' - \rho_0\theta_0); \quad (10)$$

and (7), (9), and (10) will then serve to determine V' , ρ' , θ' when v , ρ_0 , θ_0 are given. Since we have taken all these quantities to remain constant throughout the motion, we see, as before, that at each instant all the necessary conditions are satisfied; the principles of mass and momentum, together with our supposition that there is no exchange of heat, being sufficient to determine what takes place at B. Again, if at a time t from the commencement of the motion we take the distance of B from A to be $(V' - v)t$, so that initially B coincides with A, the initial conditions are satisfied. The assumed motion thus satisfies all the necessary conditions, and is therefore the actual motion.

8. If we compare the results of the last two sections with

those given by Riemann*, we shall find complete accordance so far as § 6 is concerned, though with § 7 the case is different; and this may be easily explained. We cannot in general investigate the motion of a (frictionless) compressible fluid by means of the equations of continuity and momentum, without further making some supposition as to the exchange or non-exchange of heat, and so we usually assume either that the temperature remains constant, or that there is no exchange of heat: in either case (provided the motion is continuous), the pressure is a function of the density only. At a surface of discontinuity there is not only the ordinary heating effect due to compression, but also, as we have seen, a dissipative generation of heat, and so, when applying the equations of continuity and momentum at such a surface, we must know what becomes of this additional heat. Now in all cases Riemann makes the assumption that the pressure is a function of the density only, and this is necessarily equivalent to an assumption concerning the transference of heat. Throughout most of his treatment of waves of discontinuity Riemann assumes that temperature is constant and that Boyle's law holds good; accordingly our § 6 is entirely in harmony with his conclusions, in fact (4) and (5) are only particular forms of equations given by Riemann. Of course the hypothesis that a portion of gas can be instantaneously compressed to a finite extent without any appreciable change of temperature, is not in accordance with experience, *but provided we accept the assumption that the temperature remains constant throughout, all that Riemann says concerning the propagation of waves of discontinuity under Boyle's law will hold good.*

The assumption made in § 7, that there is no appreciable transference of heat, is probably much nearer the truth; but this is not in accordance with any assumption made by Riemann. When pressure is assumed to be a function of density only, and to vary with it according to the adiabatic law, *it is virtually assumed that at the discontinuity just so much heat remains in the gas as would be due to slow adiabatic compression, while the further amount of heat which is dissipa-*

* *Loc. cit.*

tively produced is completely and instantaneously removed by conduction. But though Riemann's results may thus be justified by impossible assumptions concerning the diffusion of heat, we may more reasonably, following Lord Rayleigh, regard them as involving a destruction of energy. The real source of error lies in Riemann's fundamental hypothesis. At the outset he supposes the expansion and contraction of the air to be either purely isothermal or purely adiabatic, and thenceforward he treats the air as a frictionless and mathematically continuous fluid, in which pressure and density are connected by an invariable law. But in general the existence of such a fluid is contrary to the conservation of energy; for as soon as discontinuity arises, energy will be destroyed.

9. It may not be out of place to conclude this portion of the subject by a short reference to a paper by Dr. O. Tumlriz *. This author starts, as Riemann did, with the assumption that the pressure is a function of the density only, the law of pressure being further assumed to be the adiabatic law; and in order to avoid Riemann's error, he explicitly uses the principle of energy applicable to continuous motion, in place of the principle of momentum. But the foregoing discussion will have made it clear, I think, that the solution of the difficulty is not to be sought for in this direction. In addition to the assumptions common to his own work and to that of Tumlriz, Riemann uses only the principle of mass and the principle of momentum; and since by their aid alone he arrives at a completely determinate motion, it follows that any other motion consistent with the same arbitrary assumptions, and with the condition of mass, must violate the condition of momentum. We have seen, in fact, that there is dissipation of energy at a surface of discontinuity, so that the condition of energy applicable to continuous motion ceases to hold good. We are acquainted, too, with other instances where loss of continuity involves dissipation of energy; for example, there is the case of one hard body rolling over another.

As the result of his investigation, Dr. Tumlriz concludes

* "Ueber die Fortpflanzung ebener Luftwellen endlicher Schwingungsweite," *Sitzungsb. der Wien. Akad.* xcv. pp. 367-387 (1887).

that as soon as a discontinuity is formed it immediately disappears again, this effect being accompanied by a lengthening of the wave and a more rapid advance of the disturbance. In this way, therefore, he seeks to explain the increased velocity of very intense sounds, such as the sounds of electric sparks investigated by Mach*. But it has already been pointed out [§ 3 (i.)], that *when density and velocity are everywhere continuous functions of the coordinates*, the *front* of a disturbance advancing into still air must travel forward with the velocity of infinitely feeble sounds. A greater velocity can only ensue when the motion has become discontinuous.

PART II.—SPHERICAL WAVES.

10. When plane waves of finite amplitude are propagated through a frictionless compressible fluid, discontinuity must always occur sooner or later, and a moment's consideration will show that there are at least some cases when the motion in spherical waves becomes discontinuous; the question arises whether in any case it is possible (in the absence of viscosity) for divergent spherical waves to travel outward indefinitely without arriving at a discontinuous state. This question was suggested to me by Mr. Bryan, who at the same time kindly handed me notes of his manner of attacking the problem. His method was to write down the exact kinematical equation for spherical sound-waves, and then to obtain successive approximations to the integral of this equation. If it appears that after any number of approximations the integral would remain convergent for large values of the radius, we may conclude that our equation holds good throughout, and hence that no discontinuity arises. If, on the other hand, the second or any higher approximation becomes divergent for large values of the radius, it is probable that the motion becomes somewhere discontinuous. This method I have not followed out; but by another method which is, I hope, sufficiently conclusive, I shall now endeavour to show that discontinuity must always arise.

* *Sitzungsb. der Wien. Akad.* lxxv., lxxvii., lxxviii. Cf. also W. W. Jacques [On Sounds of Cannon], *Amer. Journ. Sci.* 3rd ser. xvii. p. 116 (1879).

The case in which the motion loses its continuity comparatively early requires no further consideration here ; we have only to concern ourselves with the case in which the initial disturbance has spread out into a spherical shell of very small disturbance whose mean radius is very great compared with the difference between its extreme radii. The equations applicable to the disturbance are then, very approximately,

$$u = a \frac{\delta \rho}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$u \text{ or } a \frac{\delta \rho}{\rho} \propto \frac{1}{r} \text{ for a given part of the wave, } . \quad (12)$$

where ρ is the mean density, $\rho + \delta \rho$ the actual density at a point where the velocity is u , and a is the velocity of infinitely feeble sounds in air of density ρ ; r is as usual the distance of a point from the centre of symmetry. Let us consider two neighbouring points M and N, on the same radius, *each being fixed in a definite part of the wave*, the point M being *behind* N (*i. e.* nearer to the origin), and the air-velocity at M exceeding that at N by Δu . Then, as the wave advances, each part of it will be instantaneously moving forward with (very approximately) the velocity

$$\sqrt{\frac{d\rho}{d\rho}} + u,$$

determined by the corresponding values of ρ and u ; so that M will be gaining on N at the rate

$$\Delta u + \frac{d}{d\rho} \sqrt{\frac{d\rho}{d\rho}} \cdot \frac{d\rho}{du} \Delta u \quad . \quad . \quad . \quad . \quad (13)$$

approximately. We may admit then that the rate at which M gains on N is

$$\text{never} < B \Delta u,$$

where B is a constant suitably chosen.

Again, if $\Delta_0 u$ is the difference between the air-velocities at M and N at the time $t=0$, and r_0 is the corresponding co-ordinate of M, we may admit that

$$\Delta u \text{ is never } < \frac{A r_0}{r_0 + at} \Delta_0 u,$$

where A is a constant not very different from unity. Thus M gains on N at a rate which is

$$\text{never} < AB \frac{r_0}{r_0 + at} \Delta_0 u;$$

and between the times $t=0$ and $t=t_1$ the distance gained by M relatively to N will be

$$\text{at least } AB \Delta_0 u \int_0^{t_1} \frac{r_0 dt}{r_0 + at},$$

$$i. e. \quad \text{at least } AB \Delta_0 u \frac{r_0}{a} \log \frac{r_0 + at_1}{r_0}. \quad . \quad . \quad (14)$$

If B is finite and positive this expression increases indefinitely with the time, so long as the laws of continuous motion hold good. If $\Delta_0 r$ was the distance between M and N at time $t=0$, the time required for M to *overtake** N will be *not greater* than the value of t_1 given by

$$-\Delta_0 r = AB \Delta_0 u \frac{r_0}{a} \log \frac{r_0 + at_1}{r_0};$$

or, when M and N are taken indefinitely close together at starting, by

$$\log \frac{r_0 + at_1}{r_0} = \frac{a}{r_0} \div \left\{ AB \left(-\frac{\partial u}{\partial r} \right)_0 \right\};$$

$$i. e., \text{ we have } t_1 \leq \frac{r_0}{a} \left(e^{\frac{a}{r_0} \div \{ AB (-\frac{\partial u}{\partial r})_0 \}} - 1 \right), \quad . \quad . \quad (15)$$

which gives us a finite upper limit to the time required for discontinuity to set in, provided B is finite. As our assumptions only remain approximately true so long as the motion is continuous, (15) will only give an approximation to the time when discontinuity first commences, and accordingly the relation must be taken to refer to that part of the wave for which its right-hand side is a minimum. If B is negative (which is not the case for any known substance), the appropriate part of the disturbance will be such that $\partial u / \partial r$ is positive.

To determine approximately the value of B , we may refer to (13) and the inequality immediately following. If we

* Cf. Lord Rayleigh, 'Theory of Sound,' vol. ii. p. 36.

assume Boyle's law of pressure, so that $\sqrt{dp/d\rho} = \text{const.}$, we have evidently

$$B=1 \text{ very nearly.}$$

If we assume that the changes of density take place adiabatically, so that $p \propto \rho^\gamma$ and γ is nearly constant, the approximate value of B becomes

$$1 + \frac{d}{d\rho} \sqrt{\frac{dp}{d\rho}} \cdot \rho \div \sqrt{\frac{dp}{d\rho}}$$

by means of (11);

$$= \frac{\gamma + 1}{2}.$$

If, then, viscosity be neglected, we must conclude that under any practically possible law of pressure the motion in spherical sound-waves always becomes discontinuous, and *à fortiori* the same will be true of cylindrical waves. But inasmuch as our result for spherical waves depends on the existence of an infinite *logarithm* in (14) when t_1 is increased without limit, we may conclude that for waves diverging in four dimensions (or, more generally, in any number of dimensions finitely greater than three) there would be some cases where the motion remained always continuous.

11. The general question of spherical sound-waves of finite amplitude is by no means an easy one. In the case of plane waves we can write down at once from Riemann's equations the condition that the disturbance may be propagated wholly in the positive or wholly in the negative direction. The respective conditions are * :—

$$u = \pm \int_{\rho_0}^{\rho} \sqrt{\frac{dp}{d\rho}} \cdot d \log \rho,$$

where ρ_0 is the density of that part of the fluid whose velocity is reckoned as zero. No such simple criterion can be given for the existence of a purely convergent or purely divergent spherical disturbance; a fact which may be readily seen from the equations for waves of infinitesimal amplitude. If ϕ is the potential of a purely divergent system of waves, we have

$$r\phi = f(at - r), \quad . \quad . \quad . \quad . \quad . \quad (16)$$

* Cf. also Lord Rayleigh, 'Theory of Sound,' vol. ii. p. 35 (3).

where f is a function whose form is unrestricted. Let ρ be the ordinary density of the air, and $\rho + \delta\rho$ the actual density at a point where the coordinate is r and velocity u . We have, then, on differentiating (16) the well-known relations

$$u = \frac{\partial\phi}{\partial r} = -\frac{f'(at-r)}{r^2} - \frac{f''(at-r)}{r} \quad \dots \quad (17)$$

and

$$a \frac{\delta\rho}{\rho} = -\frac{1}{a} \frac{\partial\phi}{\partial t} = -\frac{f'(at-r)}{r} \quad \dots \quad (18)$$

From (17) and (18),

$$\left(a \frac{\delta\rho}{\rho} - u\right)r^2 = f(at-r),$$

whence differentiating with respect to r , and neglecting small quantities beyond the first order,

$$\begin{aligned} r^2 \left(\frac{a}{\rho} \frac{\partial\rho}{\partial r} - \frac{\partial u}{\partial r} \right) + 2r \left(a \frac{\delta\rho}{\rho} - u \right) &= -f'(at-r) \\ &= ar \frac{\delta\rho}{\rho} \end{aligned}$$

by (18) ; therefore

$$\frac{a}{\rho} \frac{\partial\rho}{\partial r} - \frac{2u}{r} + \frac{a}{r} \frac{\delta\rho}{\rho} - \frac{\partial u}{\partial r} = 0. \quad \dots \quad (19)$$

If, then, an infinitesimal spherical disturbance is to be purely divergent, this equation must be satisfied for every value of r . But since the left-hand side involves $\delta\rho/\rho$ as well as u , $\partial u/\partial r$, and $\partial(\log \rho)/\partial r$, it is evident that the question whether or not the equation is satisfied for some particular value of r does not depend solely on the state of things in the immediate neighbourhood of this value, but is influenced also by the value of ρ corresponding to the undisturbed air. We must not therefore seek to characterize a purely divergent disturbance by a differential equation expressing that, with respect to the air at each point, the disturbance is wholly propagated in the positive direction of r .

12. Not recognizing this, I had attempted to discover such an equation, and one step of the inquiry is reproduced here, for the sake of any interest which it may have.

It is required to write down the differential equation of an

infinitesimal spherical disturbance, which is superposed on a purely radial steady motion.

Though a steady motion extending inward to the pole would involve a violation of the principle of continuity, we may suppose that throughout a shell of finite thickness the distribution of density and velocity is such as would be consistent with steady motion; the motion within such a shell would then continue steady, provided that its spherical boundaries were constrained to expand or contract in a suitable manner. In the absence of constraints the shell of steady motion would be invaded from without and from within by disturbances emanating from adjoining parts of the fluid, but, at points well within the shell, the character of the steady motion would necessarily be maintained for a finite time.

Let ϕ be the potential of the steady motion.

Let $\phi + \psi$ be the potential of the actual motion so that ψ and its derivatives are small.

Let p, ρ be the pressure and density in the steady motion.

Let $p + \delta p, \rho + \delta \rho$ be the pressure and density in the actual motion, and assume that the pressure is a function of the density only. From the ordinary equations for the motion of compressible fluids we obtain

$$\int^r \frac{dp}{\rho} = -\frac{1}{2} \left(\frac{\partial \phi}{\partial r} \right)^2, \quad (20)$$

$$\begin{aligned} \int^{r+\delta r} \frac{dp}{\rho} &= -\frac{1}{2} \left(\frac{\partial \phi}{\partial r} + \frac{\partial \psi}{\partial r} \right)^2 - \dot{\psi} \\ &= -\frac{1}{2} \left(\frac{\partial \phi}{\partial r} \right)^2 - \frac{\partial \phi}{\partial r} \frac{\partial \psi}{\partial r} - \dot{\psi}, \quad . . . (21) \end{aligned}$$

when small quantities of the second order are neglected. Subtracting (20) from (21),

$$\frac{\delta p}{\rho} = -\dot{\psi} - \frac{\partial \phi}{\partial r} \frac{\partial \psi}{\partial r}. \quad (22)$$

Now

$$\frac{\delta p}{\rho} = \frac{dp}{d\rho} \cdot \frac{\delta \rho}{\rho};$$

therefore

$$\frac{\partial}{\partial t} \frac{\delta \rho}{\rho} = \frac{d\rho}{d\rho} \cdot \frac{1}{\rho} \frac{\partial}{\partial t} \cdot \delta \rho; \quad \dots \quad (23)$$

and the equations of continuity for the steady motion and the actual motion may be written

$$0 = \frac{\partial \rho}{\partial t} = -\rho \nabla^2 \phi - \frac{\partial \phi}{\partial r} \frac{\partial \rho}{\partial r},$$

$$\frac{\partial (\rho + \delta \rho)}{\partial t} = -(\rho + \delta \rho) \nabla^2 (\phi + \psi) - \left(\frac{\partial \phi}{\partial r} + \frac{\partial \psi}{\partial r} \right) \frac{\partial (\rho + \delta \rho)}{\partial r},$$

whence by subtraction

$$\frac{\partial \delta \rho}{\partial t} = -\rho \nabla^2 \psi - \delta \rho \nabla^2 \phi - \frac{\partial \rho}{\partial r} \frac{\partial \psi}{\partial r} - \frac{\partial \delta \rho}{\partial r} \frac{\partial \phi}{\partial r}. \quad (24)$$

Again,

$$\begin{aligned} \frac{\partial \delta \rho}{\partial r} &= \frac{\partial}{\partial r} \left\{ \left(\frac{d\rho}{d\rho} \right)^{-1} \delta \rho \right\} \\ &= \frac{\partial}{\partial r} \left\{ \rho \left(\frac{d\rho}{d\rho} \right)^{-1} \left(-\dot{\psi} - \frac{\partial \phi}{\partial r} \frac{\partial \psi}{\partial r} \right) \right\}; \end{aligned}$$

expanding this and substituting in (24) we get

$$\begin{aligned} \frac{\partial \delta \rho}{\partial t} &= -\rho \nabla^2 \psi - \rho \left(\frac{d\rho}{d\rho} \right)^{-1} \left(-\dot{\psi} - \frac{\partial \phi}{\partial r} \frac{\partial \psi}{\partial r} \right) \nabla^2 \phi - \frac{\partial \rho}{\partial r} \frac{\partial \psi}{\partial r} \\ &\quad - \left[\frac{\partial}{\partial r} \left\{ \rho \left(\frac{d\rho}{d\rho} \right)^{-1} \right\} \left(-\dot{\psi} - \frac{\partial \phi}{\partial r} \frac{\partial \psi}{\partial r} \right) \right. \\ &\quad \left. + \rho \left(\frac{d\rho}{d\rho} \right)^{-1} \left(-\frac{\partial \dot{\psi}}{\partial r} - \frac{\partial \phi}{\partial r} \frac{\partial^2 \psi}{\partial r^2} - \frac{\partial^2 \phi}{\partial r^2} \frac{\partial \psi}{\partial r} \right) \right] \frac{\partial \phi}{\partial r}. \quad (25) \end{aligned}$$

Now differentiate (22) with respect to t and we have, remembering (23),

$$-\dot{\dot{\psi}} - \frac{\partial \phi}{\partial r} \frac{\partial \dot{\psi}}{\partial r} = \frac{1}{\rho} \frac{d\rho}{d\rho} \cdot \frac{\partial \delta \rho}{\partial t}.$$

In this equation we have to substitute the value of $\partial \delta \rho / \partial t$ from (25), and if we then put $\chi \equiv \psi r$, and perform the necessary reductions, we finally obtain as the differential equation satisfied by ψr ,

$$\frac{\partial^2 \chi}{\partial t^2} - (a^2 - u^2) \frac{\partial^2 \chi}{\partial r^2} + 2u \frac{\partial^2 \chi}{\partial r \partial t} + U \frac{\partial \chi}{\partial t} + V \left(\frac{\partial \chi}{\partial r} - \frac{\chi}{r} \right) = 0,$$

where

$\alpha^2 \equiv$ the variable $\frac{d\rho}{d\phi}$ (in the steady motion),

$$u \equiv \frac{\partial \phi}{\partial r} \equiv \frac{\text{constant}}{\rho r^2},$$

$$U \equiv \frac{1}{\rho} \frac{\partial(u\rho)}{\partial r} - \frac{2u}{\alpha} \frac{\partial \alpha}{\partial r} \equiv -\frac{2u}{r} - \frac{2u}{\alpha} \cdot \frac{\partial \alpha}{\partial r},$$

$$V \equiv 2au \frac{\partial}{\partial r} \left(\frac{u}{\alpha} \right) - \frac{\alpha^2 - u^2}{\rho} \frac{\partial \rho}{\partial r}.$$

If the steady motion in question is a state of rest, $u=0$ and ρ is a constant, so that $U=0$, $V=0$, and our equation reduces to the ordinary form for small spherical disturbances,

$$\frac{\partial^2 \chi}{\partial t^2} - \alpha^2 \frac{\partial^2 \chi}{\partial r^2} = 0.$$

If, on the other hand, $r=\infty$, the motion may, through any finite distance, be treated as linear. We shall then have u and ρ both constant, as well as α , and as before $U=0$, $V=0$. In that case

$$\frac{\partial^2 \psi}{\partial t^2} - (\alpha^2 - u^2) \frac{\partial^2 \psi}{\partial r^2} + 2u \frac{\partial^2 \psi}{\partial r \partial t} = 0,$$

and this, by a change of independent variables, is easily seen to be the appropriate form for small plane disturbances of a fluid whose motion is otherwise uniform.

DISCUSSION.

Prof. A. S. Herschel enquired whether the nature of the solution for plane waves of finite amplitude was similar to that for ordinary wave-motion? In the latter case everything depended on the instantaneous impulses, for these alone determined the nature of the wave.

Referring to Mr. Bryan's paper, he (Prof. Herschel) asked if the author could apply his equations to centrifugal fans. A particular kind of double fan had recently been tested, and gave very anomalous results which had not yet been explained.

The President said Mr. Boys's experiments on flying-bullets might have some bearing on Dr. Burton's paper. If the conclusions there stated were correct, then the velocity of the air in front of a bullet should be greater than that of the bullet, even if the latter was travelling faster than ordinary

sound-waves. He now asked Mr. Boys if his photographs gave any evidence of this.

Mr. Boys said the fact that the photographs showed disturbances in front of the bullet proved that the disturbance travelled faster. In one case, where a large bullet was moving at a velocity rather greater than that of ordinary sound in the medium, the front of the disturbance was about half an inch in advance of the bullet. In another instance, where the bullet was smaller and the velocity greater, the distance which the disturbance was in advance of the bullet was somewhat less. In all cases, even when the velocity of the bullet was four times that of sound, the character of the effects remained the same.

Dr. Burton replied to the points raised.

XI. *On a new and handy Focometer.*

By Professor J. D. EVERETT, *F.R.S.**

THIS focometer is designed to permit the distance of the "object" from the screen to be varied, while the lens which is to throw on the screen an image of the "object" is automatically kept midway between the two. This position, as is well known, gives both the sharpest definition and the simplest calculation.

The instrument is constructed on the principle of the well-known toy called *lazy-tongs*. A number of flat bars (fig. 1), all exactly alike, are jointed together in such a way that half

Fig. 1.

Plan.

of them are in one plane and the other half in a superposed plane. With the exception of the end bars, each bar in either plane is jointed to three of the bars in the other plane, one

* Read February 24, 1893.