THE PHYSICAL REVIEW.

ON THE DIFFERENTIAL TRANSFORMER.

BY A. TROWBRIDGE.

THE principle of the differentially wound current detecting instrument, of which the differential galvanometer is the type, has been adopted by H. Ho,¹ and later by Duane and Lory,² for the comparison of inductances and capacities by means of alternating currents.

The instrument described by each of these authors is a telephone the bobbin of which is wound in twin in such a way that, when equal and oppositely directed currents are sent through the coils constituting the twin winding, no magnetic field which reaches as far as the diaphragm of the telephone is set up. This we may call the first condition which the instrument must fulfil.

The second condition is that the coefficients of self-induction of the two coils must be equal.

Both conditions are fulfilled if the two coefficients of self-induction are equal to each other and either is equal to the coefficient of mutual induction between the two circuits, since in this case no magnetic field whatever would be set up if equal currents were to flow in opposite direction in the two circuits. If, moreover, the ohmic resistances of the circuits are equal, or are made equal by the addition of suitable non-inductive resistance in series with the coils, then when an alternating electromotive force is applied to the two circuits

¹ Electrical World and Engineer, May, 1903.

² PHYS. REV., Vol. XCVII., p. 275, 1904.

in parallel the currents are the same in each and the telephone behaves as a perfectly non-inductive resistance and gives out no sound even when carrying relatively strong currents through its two coils.

Thus an instrument which fulfills this last double condition relative to its three coefficients of induction becomes an exceedingly sensitive device to detect whether or not two alternating currents are equal in amplitude and are in phase with each other, for only in the case of equality both of amplitude and phase is the instrument silent.

H. Ho was able, it seems, to construct a telephone which fulfilled the condition of equality between the three coefficients of induction of the twin coils and for this reason might be treated as perfectly inductionless, thus making it applicable to a great variety of uses, but the accuracy he reported was very much less than that obtained by Duane and Lory, who constructed a telephone for which the condition of equality of the three coefficients of induction was only approximately fulfilled.

These authors, finding great difficulty in winding the twin coils so that they might be exactly equivalent in their magnetic effect on the diaphragm, placed in series with one of the coils a small movable loop consisting of a few turns of wire, and by adjusting the position of this loop with respect to the diaphragm were able to bring the telephone to silence when the same current flowed through the coils in opposite directions.

In this manner the first condition was fulfilled, but it was found that the two coefficients of self-induction were not equal and, of course, the coefficient of mutual induction differed slightly from either one of these.

By adding a small adjustable inductance in series with each of the twin coils, the self-inductance of the two systems thus formed could be brought to equality, and so an alternating electromotive force applied to the two systems in parallel would send equal currents through each, provided the resistances were adjusted to be equal, and these two currents would be in phase with each other, though not in phase with the applied electromotive force.

The fact that the inductances of the systems are not zero when in the balance condition somewhat restricts the usefulness of the instru-

ment, but Duane and Lory reported a sensibility of $\frac{1}{50}$ per cent. in the comparison of self-inductions, while H. Ho only reported 1 per cent. However, the latter states in a subsequent paper that he attributes the higher degree of accuracy obtained by Duane and Lory to their experimental skill and not to their method of applying the differential principle, and justly points out that his method of winding the coils is superior to theirs, since in his instrument the coils were non-inductive in the state of balance.

Having used the Duane and Lory type of differential telephone for some time and having found it a remarkably sensitive device and one very easy to use, it seemed well worth while to attempt to obviate what appeared to be its defects, preserving at the same time its great sensibility.

The most serious defect appeared to me to be the inductance of the coils in the state of balance and this I have been able to overcome, while still maintaining an accuracy at least equal to that obtained by Duane and Lory, and thus about fifty to a hundred times that reported by H. Ho.

It seemed to me a mistake to attempt to make the instrument portable, for in raising it to the ear and lowering it again, it is liable to get out of adjustment and further, unless the lead wires are perfectly non-inductively wound, their position relative to the diaphragm influences the balance point and thus an irregular source of error is introduced. A better plan seemed to be to wind a Faraday ring with a double primary in twin and a fine wire secondary to connect, by means of non-inductive lead wires, to an ordinary telephone. In this manner all current carrying lead wires could be fixed once for all and the essential part of the apparatus could be made larger, and therefore more easy to adjust without at the same time making the instrument too bulky. Figure 1 shows the connections including a slide wire and convenient switch board device for inserting in the circuits the inductances, capacities, or combinations of inductances and capacities to be compared, and one which permits one or both of the twin coils to be shunted with non-inductive resistance. In the figure AB' represents the primary consisting of the twin coils AB and $A'B'$ which are wound on an iron core and consist of about 1,000 turns each of No. 24 copper

wire — with a little care in winding, their coefficients of self-induction may be brought equal to within one part in 20,000.

The secondary S connects with an ordinary telephone. The iron of the transformer core need not be soft nor is it necessary to laminate it, since at the balance there are no eddy current or hysteresis losses.

After setting up the instrument an alternating e.m.f. is applied to be the terminals 1 and 2 and all the plugs except d , f and e' are removed from the switch board; this arranges the connections so that the same alternating current flows through the two coils in the opposite direction.¹ If properly wound no sound should be heard in the telephone even when a current of a couple of amperes is flowing. In one of the instruments I have made there was a slight inequality in magnetic effect of the two coils due to carelessness in winding, but the inequality could be compensated by a single loop of wire in series with either coil brought to within three or four centimeters of the secondary. By twisting this loop between the fingers the sound could be made to cease in the telephone, and, on further twisting in the same direction, to begin again.

Fig. 1.

After testing for equal magnetic effect plugs e and e' are removed from the switch board, all the others being in place, and the slide wire is moved until the telephone is silent. With careful winding it should be possible to get a complete extinction of the sound even when the current through each coil is as great as two amperes. At any rate, a very sharp minimum is always obtainable even with rather careless winding.

If the instrument is found to satisfy the conditions both of equal-

¹ Resistance coils are inserted in the openings a , b , c , a' , b' and c' so that the equivalent length of the slide wire may be varied. When the plugs are removed from the openings d , f , d' and f' the circuit is broken at these points.

ity of magnetic effect and equality of inductance, it is ready for use in the comparison of inductances, either by the multiple arrangement of connections described by Duane and Lory, or by the series arrangement described by H. Ho. I will not enter into the details of these methods as they may be obtained from the original papers, but by way of an example to show the degree of accuracy attainable with the differential transformer, I will give the results of a series of observations taken by me in the comparison of a fixed inductance of about 25 milli-henries with a variable standard manufactured by Leeds and Northrop. The series was obtained adopting the multiple arrangement of connection in which the variable standard was placed in series with one of the twin coils and the fixed inductance in series with the other coil.¹ In order to increase the accuracy in the setting of the standard, a mirror was placed horizontally on the prolongation of the vertical axis of the standard and the setting was observed by means of a telescope and scale placed about ten feet from the instrument—a string passing over a pulley on the axis of the movable coil of the variable standard made it possible to vary gradually the amount of inductance of the standard until the sound was extinguished in the telephone.

Having previously determined that 1 degree on the scale of the variable standard corresponded to 1.3 cm. displacement of the scale of the optical pointer, I made twenty consecutive settings for balance when the variable standard was in series with the coil $A'B'$ and the unknown in series with the coil AB . In this case the greatest departure from the mean was 1.4 cm. or about one tenth of a degree of the scale of the standard, the setting corresponding to the mean being 85.17° —that is, the greatest departure from the mean was in the neighborhood of one tenth per cent. The mean of a similar set of observations with the variable standard in series with AB and the unknown in series with $A'B'$ was 85.16° which shows that the twin coils were symmetrical to within one part in 8,500, or to within one eighty-fifth of one per cent.

Another double series of ten observations taken when the laboratory was free from disturbing sounds gave practically the same mean value of the unknown inductance, and a maximum departure

¹ In the openings d and d' respectively.

of any single observation from the mean of only one fortieth per cent.

It is thus very easy to compare inductances by means of the differential transformer to an accuracy of one tenth per cent. and with a little care an accuracy of a few hundredths of a per cent. may be attained.

H. Ho has pointed out that if the differential measuring instrument satisfies the condition of equality between its coefficients of induction, it may be employed to advantage in the comparison of capacities, since the twin coil in series with the greater capacity may be shunted with a non-inductive resistance, and the law of division of currents which holds for non-inductive circuits may be rigidly applied to obtain an expression for the ratio of the two capacities when silence in the telephone indicates equality of current and phase in the two twin coils.

The following data were obtained with the differential transformer in the comparison of two subdivided mica condensers, having a range of from 0.05 to 1.00 microfarads. Both condensers were made by Elliott, the one called C_1 having been in use four years, and that called C_2 being a new instrument with a capacity nearly 2 per cent. greater than the other.

The ratio of the capacities was found to be $C_2/C_1 = 1.01735$ when C_2 was in series with the twin coil AB which had, therefore, to be shunted with 860 ohms, as its resistance was 14.91 ohms. On exchanging the positions of C_1 and C_2 , the ratio was found to be

$$\frac{C_2}{C_1} = 1.01745.$$

A separate determination taken some weeks later gave values 1.01765 and 1.01785. These values show that the asymmetry of the transformer coils is not greater than .02 per cent. while the mean of the two separate sets of observations agree to within about .03 per cent. The ratio of the same capacities determined by Gott's method was 1.018, the degree of accuracy obtainable being only about one tenth per cent.

As a check on the value obtained for ratio C_2/C_1 , a separate series of observations was made on the ratios of the component

parts of C_2 and C_1 , and from the data obtained the ratio between the capacities C_2 and C_1 was calculated and found to differ from that directly observed by less than one part in ten thousand.

When we consider that the differential transformer was constructed of low resistance so that large currents might be employed in the comparison of inductances, these results obtained in the comparison of capacities are remarkably good. It should be easy to increase the accuracy in the comparison of capacities by using a transformer the twin coils of which consisted of a great many turns of fine wire.

The condition of resonance either in a series or parallel arrangement of capacity and inductance is determined not only by the values of the capacity, inductance and resistance, but also by the frequency of the alternating current. In employing a telephone to detect this resonance condition, it is advisable to adopt some scheme of connections which will make the resonance condition practically independent of the frequency, especially if the alternating electromotive force employed is not a simple sine function of the time, since otherwise perfect silence at the balance cannot be secured owing to the presence of harmonics in the current.

The following scheme of connections for use with the differential transformer (or telephone) makes it possible to offset the lead due to the capacity C by the lag due to the inductance L so that the current will flow in phase with the impressed electromotive force (as through a non-inductive resistance), and in addition the balance condition is sensibly independent of the frequency.

Fig. 2.

This desirable end is accomplished by a slight loss in sensibility which seems to be amply compensated for by the resulting agreeable sharpness of the balance.

In Fig. 2, A represents the source of alternating electro-motive force, L_1 and L_2 , the twin coils of the differential transformer, a and b the portions of the variable resistance which may be put in series with the coils L_1 and L_2 respectively, R a non-inductive resistance in series with the coil L_1 , r_1 and r_2 other variable non-inductance resist-

ances which serve as shunts to the capacity C , and to a variable inductance L of constant resistance ρ respectively. The twin coils l_1 and l_2 are here shown separate but in point of fact are superimposed. The secondary coil of the differential transformer and its connections to the telephone are not shown in the figure.

The complex expression for the admittance of the capacity C shunted by the resistance r_1 is

$$\frac{1}{Z} = \frac{1}{r_1} + \frac{1}{j\epsilon} \quad (1)$$

where $\epsilon = \text{the capacitance} = 1/C\omega$ and $j = \sqrt{-1}$. The expression for the admittance of the variable inductance shunted by the non-inductive resistance r_2 is

$$\frac{1}{Z'} = \frac{1}{r_2} + \frac{1}{\rho - jx} \quad (2)$$

where $x = L\omega$. From these expressions the impedances may be calculated

$$Z = \frac{r_1\epsilon^2}{r_1^2 + \epsilon^2} + j\frac{r_1^2\epsilon}{r_1^2 + \epsilon^2} \quad (3)$$

$$Z' = \frac{\rho(r_2 + \rho)r_2 + r_2x^2}{(r_2 + \rho)^2 + x^2} - j\frac{r_2x^2}{(r_2 + \rho)^2 + x^2}. \quad (4)$$

It is not necessary to complicate the problem by the introduction of the inductances of the twin coils, since in the state of balance each behaves as a non-inductive resistance.

In order that the current through the circuit containing the coil l_2 may be in phase with the impressed electromotive force, and hence in phase with the current through the coil l_1 , the coefficients of the j components must be equal

$$\frac{r_2^2x}{(r_2 + \rho)^2 + x^2} = \frac{r_1^2\epsilon}{r_1^2 + \epsilon^2}. \quad (5)$$

This is a necessary but not a sufficient condition for silence in the telephone.

The other necessary condition is that the currents in the two coils l_1 and l_2 shall be equal in magnitude at every instant. That is, the real components of Z and Z' added to the ohmic resistance

of the coil l_2 and the resistance b in series with it shall equal the resistance $R + l_1 + a$.

$$\frac{r_1 c^2}{r_1^2 + c^2} + \frac{\rho(r_2 + \rho)r_2 + r_2 x^2}{(r_2 + \rho)^2 + x^2} = a + l_1 + R - l_2 - b = R'. \quad (6)$$

From these two conditions it follows that

$$x = (r_2 + \rho) \frac{r_1^2 c}{c^2 r_1 + (r_2 - R')(r_1^2 + c^2)} \quad (7)$$

or

$$L = (r_2 + \rho) \frac{r_1^2 C}{r_1 + (r_2 - R')(r_1^2 C^2 \omega^2 + 1)}. \quad (8)$$

The term $r_1^2 C^2 \omega^2 + 1$ which appears as a factor of a part of the denominator does not differ in practice much from unity, for, assigning the most unfavorable values to r_1 , C and ω , say 200 ohms, 1 microfarad and 120π respectively, the term in question is 1.006. Even in the most unfavorable case, an error of .6 per cent. in the value of the term $(r_2 - R')$ would be introduced by neglecting $r_1^2 C^2 \omega^2$ with respect to unity. Now the term $(r_2 - R')$ is in practice always small with respect to r_1 . In the most unfavorable case

$$\frac{r_2 - R'}{r_1} = \frac{1}{50} \quad (9)$$

so that the error in the value of L introduced by neglecting $r_1^2 C^2 \omega^2$ is always extremely small, and we may write

$$L = \frac{(r_2 + \rho)r_1^2 C}{r_1 + r_2 - R'}. \quad (10)$$

Now since, to a close order of approximation,

$$R' = r_1 + \frac{r_2 \rho}{r_2 + \rho}, \quad (11)$$

we have

$$L = \frac{(r_2 + \rho)^2 r_1^2 C}{r_2^2} \quad (12)$$

and since r_1 , r_2 and ρ may be accurately determined, this permits a calculation of L in terms of C .

It is evident that the smaller $(r_2 - R')$ is the less will be the disturbing effect of the term containing ω .

If ρ is small compared to r_1 and r_2 then R' is about equal to r_1 and thus r_2 should be about equal to r_1 for the greatest sharpness of balance.

Using the differential transformer described in this paper, I was able to compare inductances ranging from 7 to 40 millihenries with capacities ranging from 0.05 to 1.00 microfarad to an accuracy of a little better than one per cent. for a single observation.

The reason the sensibility is so low compared to that reported in the case of the comparison of inductances or of capacities is due to the fact that the differential transformer used was designed for use with relatively large currents, while the method of comparison of inductance and capacity necessitates the use of relatively high resistances r_1 , r_2 and R , and hence, of weak currents.

I do not report any series of observations at this time because I intend to repeat the experiment with the aid of a transformer especially designed for the purpose, since I am confident that an accuracy may be attained in the comparison of inductance and capacity by this method as great as that which I have reported for the comparison of inductances.

I base this statement on a series of observations, which my colleague Mr. A. H. Taylor very kindly secured for me, adopting a Wheatstone bridge and sensitive telephone. Three arms of the bridge consisted of non-inductive resistance, and the fourth consisted of the combination of resistance, inductance and capacity which is shown in Fig. 2 in series with the twin coil L_2 . An alternating electromotive force of 110 volts was used and a complete extinction of sound could be attained on properly adjusting the values of C , L , r_1 , r_2 and R . With 500 ohms in each of the proportional arms of the bridge a series of ten observations showed a maximum departure from the mean of one twenty-fifth of one per cent.; with 200 ohms in each of the proportional arms a maximum departure of one forty-second of one per cent. No disturbing effect, due to the complex wave form of the alternating electromotive force used, was noticed. This fact is, of course, explained by the absence of ω from the equation expressing the condition of balance which for the

bridge arrangement is substantially the same as for the differential transformer. The values of the inductance obtained by this bridge method were in accord with those obtained by the differential transformer to within the limit of error of the latter instrument. The limit of error of the bridge method, which is sufficiently small for most purposes, can, I am sure, be equaled by employing a properly designed differential transformer.

The question naturally arises as to whether or not the resistance r_2 (Fig. 2) could not be suppressed and the circuit still retain its property of being non-reactive for all frequencies. Treating the problem in a manner similar to that given above and employing the same symbols, the equations

$$L = \frac{r_1^2 C}{1 + C^2 \omega^2 r_1^2} \quad (13)$$

and

$$\rho + \frac{\frac{r_1}{C^2 \omega^2}}{r_1^2 + \frac{1}{C^2 \omega^2}} = R' \quad (14)$$

may be deduced.

The first of these shows that the term $C^2 \omega^2 r_1^2$ may not be neglected with respect to unity with anything like the same degree of accuracy as in the other case, where $1 + C^2 \omega^2 r_1^2$ was multiplied by a very small quantity ($r_2 - R'$). Other things being equal, this would result in a tone minimum at the balance instead of practically silence as in the other case.

The second of the above equations shows that the resistance R' , required to balance R is less than $r_1 + \rho$, which is the value R' would have if steady currents and a differential galvanometer were employed instead of the variable currents and differential transformer.

This diminution in the effective resistance of the circuit amounts to 1.3 ohms (or about one half of one per cent.) if we assume the same unfavorable values of r_1 , C and ω mentioned above.

If the resistance r_2 is not suppressed the effective resistance differs from the true resistance by a much smaller quantity. This is shown when we form the difference, $\Delta R'$, between R' as given in equation

6 and $r_1 + r_2\rho/(r_2 + \rho)$, which is the true resistance of the portion of the circuit in question.

$$\Delta R' = \frac{r_1^3 C^2 \omega^2}{r_1^2 C^2 \omega^2 + 1} - \frac{r_2^2 L^2 \omega^2}{(r_2 + \rho)((r_2 + \rho)^2 + L^2 \omega^2)}. \quad (15)$$

A close approximation of this value may be obtained if we substitute the value of L give by equation 12 and assume r_1 equal to r_2 which is the condition for maximum sensibility

$$\text{Thus} \quad \Delta R' = \frac{r_1^3 C^2 \omega^2}{r_1^2 C^2 \omega^2 + 1} - \frac{r_1^2 (r_1 + \rho) C^2 \omega^2}{1 - (r_1 + \rho)^2 C^2 \omega^2}. \quad (16)$$

Since $r_1^2 C^2 \omega^2$ is always small with respect to unity it is evident that $\Delta R'$ tends toward zero with decreasing value of ρ . In the experiments described ρ was ten ohms and so, if we again assign the same unfavorable values to r_1 , C and ω we obtain $\Delta R' = 0.07$ ohm. or about $\frac{3}{100}$ of one per cent.

It is thus evident that the function of the resistance r_2 is to make both L and R' practically independent of the frequency of the alternating current employed and this result is well confirmed by experiment as it was never possible to obtain silence in the telephone when the resistance r_2 was suppressed.

PHYSICAL LABORATORY, UNIVERSITY OF WISCONSIN,
November, 1904.