128. On Gauss' Principle of Least Constraint and the Equations of Mechanics

Author(s): Philip E. B. Jourdain

Source: *The Mathematical Gazette*, Vol. 2, No. 41 (Oct., 1903), pp. 337-340

Published by: Mathematical Association

Stable URL: <http://www.jstor.org/stable/3603057>

Accessed: 12-12-2015 03:43 UTC

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*.

<http://www.jstor.org>

Again, let us find the cube roots of the numbers 2, 3, 4, 5, 6, 7, 9... with a slide rule. Here we have an exact graduation to set the slide to; and we find, for instance, that the P.E. of the result is, say, a unit in the fourth significant figure. Now let us take the cube roots of 2.16, 3.16, 4.16.... An additional source of error arises from the fact that we have to estimate, or interpolate, for the position of the 6. This causes the P.E. of the result to be perceptibly larger. Such examples, which may be multiplied almost indefinitely, give a real insight into the way in which a result is affected by several component sources of error.

There is perhaps no paramount scientific principle so often disregarded as this—always have some idea of the degree of accuracy of which your method is capable, and of the degree of accuracy of your own work.

To take a single case—our whole national system of competitive examination rests on a total disregard of this principle. This, however, is bordering upon topics of controversy.

I have one thing to say in conclusion. If anything in this paper is of value, the credit is largely due to Colonel H. C. Dunlop, R.A., with whom I have constantly discussed slide rule problems for some years.

The Logocyclic Slide Rule.—The special object of the logologarithmic slide is to enable the user to read off powers of numbers in the same way as the common slide rule enables us to read products.

Putting $f(x) = \log x$ in the principal equation, we have at once from the diagram $\log \log y - \log \log x = \log n - \log m$ or $y^m = x^n$, while, as special cases, if $m=1$, $y=x^n$, and if $x=10$, $n=\log_{10} y$. Thus powers and logarithms can be read off at a glance.

Numerous applications at once suggest themselves. As an instance of the use of the slide in getting out empirical formulæ, we may take Dr. Vincent's formula (which, perhaps, ought hardly to be termed empirical), viz.: atomic weight of the n^{th} element, when the elements are arranged in order of their atomic weights, is (in general) $(n+2)^{1.21}$.

It is a singular fact that the logolog scale has repeatedly been invented, forgotten, and reinvented. The first inventor was Dr. Roget. It was rediscovered and patented by Captain Thomson, R.A.; again, in Germany, by M. Blanc (Dyck's *Catalog*, p. 145), and again, almost simultaneously, reinvented by Professor Perry, and reintroduced by Colonel Dunlop and myself. The use of a $\log(-\log x)$ scale for numbers less than unity was one fondly thought new, but in this idea, which Professor Perry also brought forward, we were all anticipated by M. Blanc.

R. M. ACADEMY,
WOOLWICH.

C. S. JACKSON.

MATHEMATICAL NOTES.

128. [R. 6. b.] *On Gauss' Principle of Least Constraint and the Equations of Mechanics.*

The object of this note is to point out the remarkably simple principle to which Gauss' principle leads. The new principle is much simpler—not being an integral principle—than that of Hamilton,

$$\delta \int_{t_0}^{t_1} (T + U) dt = 0,$$

but, like it, is independent of coordinates, and leads at once to equations in generalised coordinates which have important properties.

1. Consider any material particle of mass m_ν , whose coordinates are (x_ν, y_ν, z_ν) at time t , and which forms part of a system. If m_ν were quite

free and (X_ν, Y_ν, Z_ν) were thus the only forces acting on it, its coordinates at time $t+dt$ would be

$$\left. \begin{aligned} x_\nu + \dot{x}_\nu dt + \frac{1}{2} \frac{X_\nu}{m_\nu} dt^2, \\ y_\nu + \dot{y}_\nu dt + \frac{1}{2} \frac{Y_\nu}{m_\nu} dt^2, \\ z_\nu + \dot{z}_\nu dt + \frac{1}{2} \frac{Z_\nu}{m_\nu} dt^2. \end{aligned} \right\} \dots\dots\dots (1)$$

However, owing to the connexions, the actual coordinates at the time $t+dt$ are

$$x_\nu + \dot{x}_\nu dt + \frac{1}{2} \ddot{x}_\nu dt^2. \dots\dots\dots (2)$$

The square of the difference between the positions (1) and (2) is then

$$\frac{dt^4}{4} \left\{ \left(\frac{X_\nu}{m_\nu} - \ddot{x}_\nu \right)^2 + \left(\frac{Y_\nu}{m_\nu} - \ddot{y}_\nu \right)^2 + \left(\frac{Z_\nu}{m_\nu} - \ddot{z}_\nu \right)^2 \right\}.$$

Now, Gauss' principle* states that the sum of these squared deviations, each multiplied by the proper mass, is a minimum. That is to say, the actual accelerations make the sum a minimum; or, for our purpose, the variation, which must affect only the accelerations and not the positions or velocities, must be zero. Thus, if δ denote such a variation, Gauss' principle becomes

$$\delta \sum_\nu \{ (X_\nu - m_\nu \ddot{x}_\nu)^2 + \dots \} = \sum_\nu \{ (X_\nu - m_\nu \ddot{x}_\nu) \delta \ddot{x}_\nu + \dots \} = 0. \dots\dots\dots (3)$$

There is an analogy with the variational form of d'Alembert's principle, but it must be remembered that here the δ 's are of different meaning. Here we have

$$\delta x_\nu = \delta y_\nu = \dots = \delta \dot{x}_\nu = \delta \dot{y}_\nu = \dots = 0,$$

but $\delta \ddot{x}_\nu, \dots$ are not zero; while, with d'Alembert's variations, $\delta x_\nu, \dots$ are not zero.

2. The equation (3) may be written

$$\begin{aligned} \sum_\nu (X_\nu \delta \ddot{x}_\nu + Y_\nu \delta \ddot{y}_\nu + Z_\nu \delta \ddot{z}_\nu) &= \sum_\nu m_\nu (\ddot{x}_\nu \delta \ddot{x}_\nu + \ddot{y}_\nu \delta \ddot{y}_\nu + \ddot{z}_\nu \delta \ddot{z}_\nu) \\ &= \frac{1}{2} \delta \sum_\nu m_\nu (\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2). \dots\dots\dots (4) \end{aligned}$$

If now we denote $\frac{1}{2} \sum_\nu m_\nu (\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2),$

which is independent of coordinates, by $\mathfrak{T}$, in analogy with the kinetic energy T ; and assume, for simplicity, a force-function $\bar{U}$ to exist, so that

$$\delta \bar{U} = \sum_\nu (X_\nu \delta \ddot{x}_\nu + \dots);$$

the equation (4) takes a form independent of coordinates,

$$\delta \bar{U} = \delta \mathfrak{T} \quad \text{or} \quad \delta (\mathfrak{T} - \bar{U}) = 0. \dots\dots\dots (5)$$

This form may now play the part of Hamilton's principle in the deduction of equations of motion in generalised coordinates, but the use of (5) is still simpler than the use of Hamilton's principle.

3. Let $q_1, q_2, \dots, q_n$ be the generalised coordinates of the system; then $\bar{U}$ is a function of $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, \ddot{q}_1, \dots, \ddot{q}_n$, and $\mathfrak{T}$ of these and t , explicitly. Remembering that the δ affects only the accelerations, (5) gives

$$\sum_{\nu=1}^n \left\{ \frac{\partial \mathfrak{T}}{\partial \ddot{q}_\nu} \delta \ddot{q}_\nu - \frac{\partial \bar{U}}{\partial \ddot{q}_\nu} \delta \ddot{q}_\nu \right\} = 0.$$

* "Ueber ein neues allgemeines Grundgesetz der Mechanik" (1829; *Werke*, v., 23-28).

This is equivalent to the system

$$\frac{\partial \mathfrak{T}}{\partial \ddot{q}_v} - \frac{\partial \ddot{U}}{\partial \ddot{q}_v} = 0. \quad (v=1, 2, \dots n.)$$

We at once verify that $\frac{\partial \ddot{U}}{\partial \ddot{q}_v} = \frac{\partial U}{\partial q_v}$;

so that the equations of motion of a system have the simple form

$$\frac{\partial U}{\partial q_v} = \frac{\partial \mathfrak{T}}{\partial \ddot{q}_v}. \quad (v=1, 2, \dots n.) \dots\dots\dots(6)$$

4. Mr. Whittaker remarked to me that the equations (6) had been already given by Appell. In fact, in 1899 and 1900, Appell published* several notes drawing attention to the remarkable fact that the equations (6)—which he obtained in quite a different way from mine—apply to the cases of non-holonomous systems (*i.e.* in which the equations of condition are non-integrable differential equations ; these occur, *e.g.* in cases of rolling motion), in which Lagrange's equations fail. I shall show this in another manner, which appears to me not uninteresting. In fact, I shall show that the equations (6) are equivalent to an extended form of Lagrange's equations†—a form which reduces to the usual form when the system is holonomous.

5. Suppose the equations of condition are given in the non-integrable form

$$dx_r = \sum_{v=1}^n Q_{r,v} dq_v + Q dt, \dots ; \dots\dots\dots(7)$$

so that $\dot{x}_r = \sum_{v=1}^n Q_{r,v} \dot{q}_v + Q$, ... ; and $\frac{\partial \dot{x}_r}{\partial \dot{q}_v} = \frac{\partial \dot{x}_r}{\partial \dot{q}_v}$.

If the equations (7) are integrable, and we can then assert that

$$Q_{r,v} = \frac{\partial x_r}{\partial q_v},$$

we also have

$$\frac{\partial \dot{x}_r}{\partial \dot{q}_v} = \frac{\partial \dot{x}_r}{\partial \dot{q}_v}.$$

Then, putting $w_r^2 = \dot{x}_r^2 + \dot{y}_r^2 + \dot{z}_r^2$,

$$\begin{aligned} \text{we have } \frac{\partial \mathfrak{T}}{\partial \dot{q}_v} &= \frac{1}{2} \sum_r m_r \left(\frac{\partial w_r^2}{\partial \dot{x}_r} \cdot \frac{\partial \dot{x}_r}{\partial \dot{q}_v} + \frac{\partial w_r^2}{\partial \dot{y}_r} \frac{\partial \dot{y}_r}{\partial \dot{q}_v} + \frac{\partial w_r^2}{\partial \dot{z}_r} \frac{\partial \dot{z}_r}{\partial \dot{q}_v} \right) \\ &= \sum_r m_r \left(\dot{x}_r \frac{\partial \dot{x}_r}{\partial \dot{q}_v} + \dots \right) = \sum_r m_r \left\{ \frac{d}{dt} \left(\dot{x}_r \frac{\partial \dot{x}_r}{\partial \dot{q}_v} + \dots \right) - \left(\dot{x}_r \frac{\partial \ddot{x}_r}{\partial \dot{q}_v} + \dots \right) \right\} \\ &= \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_v} \right) - \sum_r m_r \left(\dot{x}_r \frac{\partial \ddot{x}_r}{\partial \dot{q}_v} + \dots \right), \end{aligned}$$

where $T = \frac{1}{2} \sum_r m_r (\dot{x}_r^2 + \dot{y}_r^2 + \dot{z}_r^2)$.

$$\text{Hence } \frac{\partial \mathfrak{T}}{\partial \dot{q}_v} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_v} \right) - \sum_r m_r (\dot{x}_r \dot{Q}_{r,v} + \dot{y}_r \dot{Q}'_{r,v} + \dot{z}_r \dot{Q}''_{r,v}) ;$$

and so Lagrange's equations, generalised for non-holonomous systems, are

$$\frac{\partial U}{\partial q_v} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_v} \right) - \sum_r m_r (\dot{x}_r \dot{Q}_{r,v} + \dots) \dots\dots\dots(8)$$

* *Comptes Rendus*, t. cxxix. (1899), pp. 317-320, 459-460 ; *Crelle's Journal*, Bd. cxxi. (1900), pp. 310-319 ; Bd. cxxii. (1900), pp. 205-209.

† Ferrers : "Extension of Lagrange's Equations," *Quart. Journ.*, vol. xii. (1873), pp. 1-5.

Now if, and only if, the conditions (7) are total differentials,

$$Q_{r,\nu} = \frac{\partial x_r}{\partial q_\nu}, \quad Q'_{r,\nu} = \frac{\partial y_r}{\partial q_\nu}, \quad Q''_{r,\nu} = \frac{\partial z_r}{\partial q_\nu};$$

and the last term of (8) reduces to

$$\frac{\partial T}{\partial q_\nu},$$

and (8) takes the usual Lagrangian form.

The equations (6) are then equivalent to the system (8), and have the advantage of being expressed wholly in generalised coordinates; while Lagrange's equations,

$$\frac{\partial U}{\partial q_\nu} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\nu} \right) - \frac{\partial T}{\partial q_\nu}, \quad (\nu = 1, 2, \dots, n)$$

result from (6) or (8) for the special case of holonomous systems.

PHILIP E. B. JOURDAIN, B.A., Trinity College, Cambridge.

129. [K. 20. e.] *Two Trigonometrical Notes.*

I. There is a large class of formulas by which one length (which may be supposed unknown) is expressed in terms of another length (which may be supposed known), and of the trigonometrical ratios of certain angles. Such formulas are often easily established by the aid of the artifice of expressing the ratio of the unknown to the known as a product of two or more ratios, each of which is either a trigonometrical ratio of a known angle or is the ratio between two sides of a triangle whose angles are known, and therefore expressible as the ratio between two known sines.

For example, take the problem: *To express the in-radius of a triangle in terms of one side and two angles.* With the usual notation we have to express r in terms, say, of a, B, C .

From I , the in-centre, let ID be drawn perpendicular to BC . We have

$$\begin{aligned} \frac{r}{a} &= \frac{ID}{BC} = \frac{ID}{IC} \cdot \frac{IC}{BC} = \sin ICD \times \frac{\sin IBC}{\sin BIC} \\ &= \sin \frac{C}{2} \times \frac{\sin \frac{B}{2}}{\sin \left(180^\circ - \frac{B}{2} - \frac{C}{2} \right)} = \frac{\sin \frac{C}{2} \sin \frac{B}{2}}{\sin \frac{B+C}{2}}. \end{aligned}$$

Hence the well-known result,

$$r = a \sin \frac{B}{2} \sin \frac{C}{2} / \cos \frac{A}{2}.$$

Again take this problem: *Given the distance between two points A, B on a horizontal plane, and the angles of elevation α, β of a point C in the vertical plane containing AB , as seen from A and from B respectively, to find the vertical height of C above the level of AB .*

Let CD be the perpendicular from C on AB produced.

$$\begin{aligned} \text{Then} \quad \frac{CD}{AB} &= \frac{CD}{CA} \cdot \frac{CA}{AB} = \sin CAD \times \frac{\sin CBA}{\sin ACB} \\ &= \sin \alpha \times \frac{\sin (180^\circ - \beta)}{\sin (\beta - \alpha)}; \\ \therefore CD &= AB \sin \alpha \sin \beta / \sin (\beta - \alpha). \end{aligned}$$

It would be easy to multiply examples. These two form part of the usual bookwork of elementary trigonometry, but are often proved in the