V.—*On Orthogonal Isothermal Surfaces.* Part I. By Professor TAIT.

[Read January 2d, 1866.—Revised and Improved, January 15th, 1872.]

The following pages contain, in a comparatively compact form, part of the substance of a voluminous paper read to the Society six years ago. Of that paper, which employed ordinary analysis alone, only a few pages had been put in type when I succeeded in overcoming a formidable difficulty which had presented itself in my quaternion treatment of the subject. I therefore withdrew the paper in order that it might have the benefit of the simplification which quaternions always give; but it is only of late that I have found time to complete part of the translation into the new language. From the circumstances under which the paper has thus been produced, i, j, k come forward with undue prominence, a thing to be regarded (in HAMILTON'S words) "as an inelegance and imperfection in this calculus, or rather in the state to which it has hitherto been unfolded." Immense as is the simplification already attained, it is clear that in many places still more is attainable. But I have not postponed my paper till it should receive this final polish, partly because the time I can devote to such inquiries is extremely limited, and partly because I think that several of the results obtained, and of the modes of obtaining them, are new and remarkable. Besides, a question of this order of difficulty is admirably adapted to show in what respects quaternion methods require improvement. There must be some simple mode of deducing (13) and (21) below from (7) without the explicit use of i, j, k , but I have not yet been fortunate enough to discover it.

I append to this introduction, for comparison, a few extracts from the paper in its original form.

α . Let $\left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 + \left(\frac{d\xi}{dz}\right)^2$ be written $\square\xi$,

$$\frac{d\xi}{dx} \frac{d\eta}{dx} + \frac{d\xi}{dy} \frac{d\eta}{dy} + \frac{d\xi}{dz} \frac{d\eta}{dz} \quad . \quad . \quad D(\xi, \eta),$$

and

$$\begin{vmatrix} \frac{d\xi}{dx} & \frac{d\xi}{dy} & \frac{d\xi}{dz} \\ \frac{d\eta}{dx} & \frac{d\eta}{dy} & \frac{d\eta}{dz} \\ \frac{d\zeta}{dx} & \frac{d\zeta}{dy} & \frac{d\zeta}{dz} \end{vmatrix} \quad . \quad . \quad . \quad \Delta(\xi, \eta, \zeta)$$

Then, if

$$D(\xi, \eta) = D(\eta, \zeta) = D(\zeta, \xi) = 0,$$

we have

$$\frac{\frac{d\xi}{dx}}{\frac{d\eta}{dy}, \frac{d\eta}{dz}} = \frac{\frac{d\xi}{dy}}{\frac{d\eta}{dz}, \frac{d\eta}{dx}} = \frac{\frac{d\xi}{dz}}{\frac{d\eta}{dx}, \frac{d\eta}{dy}},$$

or

$$\frac{\frac{d\xi}{dx}}{\frac{d\Delta}{d\left(\frac{d\xi}{dx}\right)}} = \frac{\frac{d\xi}{dy}}{\frac{d\Delta}{d\left(\frac{d\xi}{dy}\right)}} = \frac{\frac{d\xi}{dz}}{\frac{d\Delta}{d\left(\frac{d\xi}{dz}\right)}} = \frac{1}{u} \text{ suppose,}$$

with two similar sets in η and ζ .

b. Since

$$\Delta = \frac{d\xi}{dx} \frac{d\Delta}{d\left(\frac{d\xi}{dx}\right)} + \frac{d\xi}{dy} \frac{d\Delta}{d\left(\frac{d\xi}{dy}\right)} + \frac{d\xi}{dz} \frac{d\Delta}{d\left(\frac{d\xi}{dz}\right)},$$

we have at once

$$\Delta = u \square \xi = v \square \eta = w \square \zeta.$$

Hence

$$u = \frac{\frac{d\Delta}{d\left(\frac{d\xi}{dx}\right)}}{\frac{d\xi}{dx}} = \frac{\Delta}{\square \xi},$$

or

$$\frac{1}{\Delta} \frac{d\Delta}{d\left(\frac{d\xi}{dx}\right)} d\left(\frac{d\xi}{dx}\right) = \frac{\frac{d\xi}{dx} d\left(\frac{d\xi}{dx}\right)}{\square \xi},$$

which gives by integration

$$\log. \Delta = C + \frac{1}{2} \log. \square \xi.$$

Thus, finally,

$$\Delta = \sqrt{\square \xi \square \eta \square \zeta}.$$

c. Thus we have

$$u = \sqrt{\frac{\square \eta \square \zeta}{\square \xi}}, \text{ \&c.}$$

But, identically,

$$\frac{d}{dx} \cdot \frac{d\Delta}{d\left(\frac{d\xi}{dx}\right)} + \frac{d}{dy} \cdot \frac{d\Delta}{d\left(\frac{d\xi}{dy}\right)} + \frac{d}{dz} \cdot \frac{d\Delta}{d\left(\frac{d\xi}{dz}\right)} = 0,$$

or

$$\frac{d}{dx} \left(u \frac{d\xi}{dx} \right) + \frac{d}{dy} \left(u \frac{d\xi}{dy} \right) + \frac{d}{dz} \left(u \frac{d\xi}{dz} \right) = 0,$$

or

$$u \nabla^2 \xi - D(u, \xi) = 0,$$

with similar equations in v, η and w, ζ .

d. Now if $\xi = c$ be one of a system of surfaces isothermal as well as orthogonal, we must have, by the above equation,

$$D(u, \xi) = 0,$$

But the orthogonality gives

$$D(\eta, \xi) = 0,$$

$$D(\zeta, \xi) = 0,$$

and the elimination of ξ among these three equations gives

$$\Delta(u, \eta, \zeta) = 0,$$

i.e., by the property of functional determinants, u is a function of η and ζ alone. Thus we have

$$\sqrt{\frac{\square \eta \square \zeta}{\square \xi}} = f(\eta, \zeta),$$

a well-known relation, &c.

1. Consider the equation

$$T. (\phi + f(h))^{-\frac{1}{2}} \sigma = 1,$$

or, as it may be written,

$$S. \sigma (\phi + f(h))^{-1} \sigma = -1 \quad . \quad . \quad . \quad (1),$$

where ϕ is any self-conjugate linear and vector function, of which i, j, k are the principal vector directions. We assume that the roots of HAMILTON'S equation

$$M_g = 0$$

are finite and different from one another, so that cylinders, surfaces of revolution, &c., are excluded from (1).

For any assigned value of σ , (1) gives in general three values of $f(h)$ and therefore of h . Omitting for the present the consideration that each value of $f(h)$ may give more than one value of h , these values may be any assigned functions of the position of a point in space; because, when they and the function f are assigned, the squares of the constituents of σ (or, what comes to the same thing, the values of σ^2 , $S. \sigma \phi \sigma$, $S. \sigma \phi^2 \sigma$) can at once be found in terms of them, by a system of three *linear* equations.

In this first part I confine myself to cases in which each of these squares is positive, so as to avoid for the present the use of biquaternions.

2. For any assigned constant value of h , (1) represents in general a surface whose normal vector, ∇h , is given by

$$S. \sigma \psi^{-2} \sigma f'(h) \nabla h = 2 \Sigma \left(i S. \frac{d\sigma}{dx} \psi^{-1} \sigma \right) \quad . \quad (2),$$

ψ being written for convenience in place of $\phi + f(h)$.

Now if h_1, h_2 be the other two values of h given by (1) for a particular value of σ , the conditions of orthogonality of the surfaces h, h_1, h_2 , are of the form

$$0 = S. \nabla h \nabla h_1 = \Sigma. S. \frac{d\sigma}{dx} \psi^{-1} \sigma S. \frac{d\sigma}{dx} \psi_1^{-1} \sigma \quad . \quad . \quad (3),$$

where

$$\psi_1 = \phi + f(h_1).$$

3. The three equations (3) may be put in the new form

$$S.\psi^{-1}\sigma\Sigma\left(\frac{d\sigma}{dx}S.\frac{d\sigma}{dx}\psi_1^{-1}\sigma\right) = 0, \text{ \&c.,}$$

whence

$$\Sigma\left(\frac{d\sigma}{dx}S.\frac{d\sigma}{dx}\psi^{-1}\sigma\right) \parallel V.\psi_1^{-1}\sigma\psi_2^{-1}\sigma \quad . \quad . \quad . \quad (4).$$

4. But, by the nature of self-conjugate linear and vector functions,

$$S.\psi^{-1}\sigma\psi_1^{-1}\sigma = S.\sigma\psi^{-1}\psi_1^{-1}\sigma = \frac{1}{f(h_1) - f(h)} S.\sigma(\psi^{-1} - \psi_1^{-1})\sigma = 0 \quad (5),$$

with two other equations of the same kind. These give (when the values of h are different) three equations of the form

$$\psi^{-1}\sigma \parallel V.\psi_1^{-1}\sigma\psi_2^{-1}\sigma \quad . \quad . \quad . \quad (6),$$

where, of course, we may dispense with the V .

5. By (4) and (6) we see that we have three equations of the form

$$\psi^{-1}\sigma \parallel \Sigma\left(\frac{d\sigma}{dx}S.\frac{d\sigma}{dx}\psi^{-1}\sigma\right),$$

and these show at once that

$$\frac{d\sigma}{dx}, \quad \frac{d\sigma}{dy}, \quad \frac{d\sigma}{dz}$$

are rectangular vectors whose tensors are equal. For

$$\tau S.a\beta\gamma = aS.\beta\gamma\tau + \beta S.\gamma a\tau + \gamma S.a\beta\tau$$

is the *only* decomposition of τ parallel to a , β , γ respectively; and we have here the equation

$$\tau \parallel aSa\tau + \beta S\beta\tau + \gamma S\gamma\tau,$$

holding good for the three non-coplanar vectors $\psi^{-1}\sigma$, $\psi_1^{-1}\sigma$, $\psi_2^{-1}\sigma$, and therefore true for all vectors. Hence we must have

$$a \parallel V\beta\gamma, \quad \beta \parallel V\gamma a, \quad \gamma \parallel Va\beta,$$

of which any two include the third as a necessary consequence, and in all three of which the coefficient of proportionality is evidently the same. The only exception to this is when

$$\frac{d\sigma}{dx} \parallel \psi^{-1}\sigma, \text{ \&c.}$$

But, in this case, by (2)

$$\nabla h \parallel i, \text{ \&c.},$$

and we have series of rectangular planes.

6. Hence there must exist a scalar function u , and a quaternion q (which may obviously be taken as a mere versor), such that

$$\frac{d\sigma}{dx} = uqi q^{-1}, \text{ \&c.},$$

or, in one expression,

$$d\sigma = uq d\rho q^{-1} \quad . \quad . \quad . \quad . \quad (7).$$

Thus it appears that, in order that (1) (with the limitations above imposed) may represent a triple series of orthogonal surfaces, σ must be such a function of ρ that, if the extremities of a set of values of ρ form the corners of an indefinitely small cube, those of the corresponding values of σ (drawn from a common origin) form the corners of another such cube; and that, therefore, the passage from ρ to σ is that from one mode of dividing space into indefinitely small cubes to another.

Whatever, therefore, may be thought of the logic of the investigation above, it is worth while to pursue the inquiry thus suggested, by developing the consequences of the equation (7) to which it has led us.

7. From the equations just written we see that if

$$\sigma = i\xi + j\eta + k\zeta, \quad . \quad . \quad . \quad . \quad (8),$$

the direction cosines of qiq^{-1} are

$$\frac{1}{u} \frac{d\xi}{dx}, \quad \frac{1}{u} \frac{d\eta}{dx}, \quad \frac{1}{u} \frac{d\zeta}{dx}.$$

From these, and other six of similar form, we see that the direction cosines of i referred to qiq^{-1} , qjq^{-1} , qkq^{-1} are

$$\frac{1}{u} \frac{d\xi}{dx}, \quad \frac{1}{u} \frac{d\xi}{dy}, \quad \frac{1}{u} \frac{d\xi}{dz},$$

and similarly for those of j and k .

Hence it follows that $\nabla\xi$, $\nabla\eta$, $\nabla\zeta$ form a set of mutually perpendicular vectors whose common tensor is u .

The same result may be obtained as follows:—

$$\begin{aligned} \nabla\xi &= - \left(i \frac{d}{dx} + j \frac{d}{dy} + k \frac{d}{dz} \right) Si\sigma \\ &= - u (iS.iqiq^{-1} + jS.iqjq^{-1} + kS.iqkq^{-1}) \\ &= - u (iS.iq^{-1}iq + jS.jq^{-1}iq + kS.kq^{-1}iq) \\ &= uq^{-1}iq \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9). \end{aligned}$$

Hence we have

$$-d\xi = uS.q^{-1}iqd\rho,$$

of which the condition of integrability is

$$V.\nabla.uq^{-1}iq = 0.$$

Thus u and q must be determined so as to satisfy the equation

$$V.\nabla.uq^{-1}\alpha q = 0 \quad . \quad . \quad . \quad . \quad (10),$$

whatever constant vector be represented by α .

We may state our present conclusions in the following simple form. In order that (1) may represent a triple series of orthogonal surfaces, it is necessary and sufficient that the constituents of σ satisfy three equations of the form (9); *i.e.*, that, when severally equated to constants, they represent three series of surfaces which together cut space into cubes.

8. As a verification of (9) we see that it and the similar expressions for $\nabla\eta$ and $\nabla\zeta$ give

$$\begin{aligned} -d\sigma &= u(iSq^{-1}iqd\rho + \quad . \quad . \quad . \quad .) \\ &= -uqd\rho q^{-1}, \end{aligned}$$

which is equation (7).

9. Performing the operations indicated in (10), it becomes

$$V.\nabla.uq^{-1}\alpha q + uV.\Sigma\left(iq^{-1}\alpha\frac{dq}{dx} - iq^{-1}\frac{dq}{dx}q^{-1}\alpha q\right) = 0,$$

or

$$V.\frac{\nabla u}{u}q^{-1}\alpha q + 2\Sigma V.iV.q^{-1}\alpha qq^{-1}\frac{dq}{dx} = 0,$$

(this simplification being permitted because (§ 6) the tensor of q may be regarded as unity) or, finally,

$$V.\frac{\nabla u}{u}q^{-1}\alpha q - 2q^{-1}\alpha q\Sigma S.iq^{-1}\frac{dq}{dx} + 2\Sigma.(S.q^{-1}\alpha qi)q^{-1}\frac{dq}{dx} = 0;$$

which may be written

$$V.\frac{\nabla u}{u}q^{-1}\alpha q + 2q^{-1}\alpha qS.\nabla q^{-1}q - 2\Sigma.(S.q^{-1}\alpha qi)\frac{dq^{-1}}{dx}q = 0,$$

or

$$V.\frac{\nabla u}{u}q^{-1}\alpha q + 2q^{-1}\alpha qS.\nabla q^{-1}q - 2S.(q^{-1}\alpha q\nabla)q^{-1}.q = 0.$$

Here α has the values i, j, k , so that if we write

$$q^{-1}\alpha q = i', \text{ \&c.},$$

we have three equations of the type

$$-V.i\frac{\nabla v}{u} + 2i'S.\nabla q^{-1}q - 2S(i'\nabla)q^{-1}.q = 0 \quad . \quad . \quad . \quad . \quad (11).$$

From these we have

$$-\Sigma .i \nabla .i \frac{\nabla u}{u} - 6S . \nabla q^{-1} q + 2 \nabla q^{-1} . q = 0 ,$$

or

$$\frac{\nabla u}{u} - 3S . \nabla q^{-1} q + \nabla q^{-1} . q = 0 .$$

Hence

$$\left. \begin{aligned} S . \nabla q^{-1} q &= 0 \\ \frac{\nabla u}{u} &= -V . \nabla q^{-1} q = -\nabla q^{-1} . q = -\frac{\nabla q^{-1}}{q^{-1}} \end{aligned} \right\} \quad (12),$$

or, finally,

$$\nabla . u q^{-1} = 0 \quad . \quad . \quad . \quad . \quad (13).$$

10. But this is not the only relation between u and q . For by (12) we may write (11) in the form

$$-V . (\nabla q^{-1} q q^{-1} a q) q^{-1} - 2S . (q^{-1} a q \nabla) q^{-1} = 0 \quad . \quad (14).$$

It is obvious that, by adding the three equations of this form, each multiplied by a proper scalar, we may derive from them three equivalent ones of the form

$$V(\nabla q^{-1} q i) q^{-1} + 2S(i \nabla) q^{-1} = 0 \quad . \quad . \quad . \quad (15).$$

This may be written by the help of (12) in the form

$$2 \frac{dq^{-1}}{dx} q = -V . i \nabla q^{-1} q = V . i \frac{\nabla u}{u} = V . i \nabla \log . u \quad . \quad (16),$$

and we thus see that the constancy of the tensor of q is recognised.

Differentiating again after multiplying into q^{-1} , we have

$$\begin{aligned} 2 \frac{d^2 q^{-1}}{dx^2} &= V(i \nabla \log . u) \frac{dq^{-1}}{dx} + V . i \frac{d}{dx} \nabla \log . u . q^{-1} \\ &= \frac{1}{2} (V . i \nabla \log . u)^2 q^{-1} + V . i \frac{d}{dx} \nabla \log . u . q^{-1} . \end{aligned}$$

Adding the three equations of this form, we have

$$-2 \nabla^2 q^{-1} = (\nabla \log . u)^2 q^{-1} \quad . \quad . \quad . \quad (17),$$

for obviously $\nabla^2 \log . u$ is a scalar.

But we have also

$$\nabla . u q^{-1} = 0 \quad . \quad . \quad . \quad . \quad (18),$$

which gives

$$\nabla \log . u . q^{-1} + \nabla q^{-1} = 0$$

and

$$\nabla^2 \log . u . q^{-1} + \Sigma . i \nabla \log . u \frac{dq^{-1}}{dx} + \nabla^2 q^{-1} = 0 ,$$

or

$$\nabla^2 \log. u. q^{-1} + \frac{1}{2} \Sigma. i \nabla \log. u \nabla. i \nabla \log. u. q^{-1} + \nabla^2 q^{-1} = 0,$$

which may be simplified into

$$\nabla^2 \log. u. q^{-1} + (\nabla \log. u)^2 q^{-1} + \nabla^2 q^{-1} = 0 \quad (19).$$

Together, (18) and (19) give

$$\left. \begin{aligned} \nabla^2 \log. u. q^{-1} - \nabla^2 q^{-1} &= 0 \\ 2 \nabla^2 \log. u + (\nabla \log. u)^2 &= 0 \end{aligned} \right\} \quad (20).$$

The latter of these equations may be written

$$\nabla(u^{\frac{1}{2}} \nabla \log. u) = 0 = \nabla \left(\frac{\nabla u}{u^{\frac{1}{2}}} \right),$$

or finally

$$\nabla^2(u^{\frac{1}{2}}) = 0 \quad (21).$$

11. Hence u is the square of the potential of some distribution of matter, none of which is contained in the space occupied by the surfaces.

Hence the only strict solution, *i.e.*, the only one which holds at every point of infinite space, is

$$u = \text{constant},$$

and, of course,

$$q = \text{constant}.$$

From this we have

$$\begin{aligned} \nabla \xi &= u q^{-1} i q = u(i a_1 + j b_1 + k c_1) \\ \xi &= c_1 + u(a_1 x + b_1 y + c_1 z). \end{aligned}$$

Thus the constituents of σ , separately equated to constants, give the equations of three series of mutually perpendicular planes cutting space into cubes, for u is the same for all. When we turn the axes so as to be perpendicular to these planes respectively, and adopt a suitable origin, we have

$$\xi = ux, \quad \eta = uy, \quad \zeta = uz,$$

whence

$$\sigma = u\rho,$$

and thus equation (1) gives in this case the confocal surfaces of the second order.

12. We omit for the present, in consequence of the remark at the beginning of last section, other obvious solutions of (21), such as

$$u^{\frac{1}{2}} = S a \rho, \text{ or } u^{\frac{1}{2}} = e^{S(\alpha + \sqrt{-1}\beta)\rho}, \text{ \&c.}$$

But if we admit that at *one* point of space there may be a particle of matter of mass m , we have, of course,

$$u = \frac{m^2}{T\rho^2},$$

so that

$$\nabla q^{-1}q = -\frac{2\rho}{T\rho^2},$$

which gives as a particular integral

$$q^{-1} = U\rho.$$

Hence, in this case,

$$\begin{aligned} d\sigma &= uq d\rho q^{-1} = -\frac{m^2}{T\rho^2}(2U\rho S U\rho d\rho + d\rho) \\ &= +\frac{m^2}{T\rho^4}(2\rho S\rho d\rho + d\rho\rho^2) = -m^2\left(\frac{2\rho dT\rho}{T\rho^3} - \frac{d\rho}{T\rho^2}\right), \end{aligned}$$

or

$$\sigma = m^2\rho^{-1}.$$

The corresponding surfaces are the electric images of the confocal quadrics, taken from the common centre, and include Fresnel's surface of elasticity.

13. It follows, from what we have just proved, that the only orthogonal surfaces which divide all space into indefinitely small cubes are planes and their electric images, or images of images, &c. These are all, therefore, included in a triple series of spheres having a common point, and their centres in three rectangular axes passing through that point.

In fact, if in (7) we put for σ

$$\sigma' = (\sigma + \tau)^{-1},$$

we have

$$\frac{d\sigma'}{dx} = -\sigma' \frac{d\sigma}{dx} \sigma', \text{ \&c.},$$

whence

$$S \frac{d\sigma'}{dx} \frac{d\sigma'}{dy} = \sigma'^4 S \frac{d\sigma}{dx} \frac{d\sigma}{dy} = 0, \text{ \&c.},$$

and

$$T \frac{d\sigma'}{dx} = T \frac{d\sigma'}{dy} = T \frac{d\sigma'}{dz} = \frac{u}{T\sigma'^2}.$$

Hence the electric image of any orthogonal system is also orthogonal; and, if the system cut space into cubes, so does the image.

14. We are now prepared to introduce the conditions that the surfaces (1) shall each be isothermal. If h, h_1, h_2 represent their temperatures, these conditions are simply

$$\nabla^2 h = 0, \quad \nabla^2 h_1 = 0, \quad \nabla^2 h_2 = 0. \quad (22).$$

To express these in another form we must now differentiate equations (2).

15. By (2) we have

$$S. \sigma \psi^{-2} \sigma f'(h) \frac{dh}{dx} = 2S. \frac{d\sigma}{dx} \psi^{-1} \sigma.$$

This gives

$$\begin{aligned} & 2S. \frac{d\sigma}{dx} \psi^{-2} \sigma f'(h) \frac{dh}{dx} - 2S. \sigma \psi^{-3} \sigma \left(f'(h) \right)^2 \left(\frac{dh}{dx} \right)^2 + S. \sigma \psi^{-2} \sigma f''(h) \left(\frac{dh}{dx} \right)^2 \\ & + S. \sigma \psi^{-2} \sigma f'(h) \frac{d^2h}{dx^2} = 2S. \frac{d^2\sigma}{dx^2} \psi^{-1} \sigma + 2S. \frac{d\sigma}{dx} \psi^{-1} \frac{d\sigma}{dx} - 2S. \frac{d\sigma}{dx} \psi^{-2} \sigma f'(h) \frac{dh}{dx}. \end{aligned}$$

Eliminating $\frac{dh}{dx}$ from these equations, we have

$$\begin{aligned} & \frac{4S. \frac{d\sigma}{dx} \psi^{-2} \sigma S. \frac{d\sigma}{dx} \psi^{-1} \sigma}{S. \sigma \psi^{-2} \sigma} - 8S. \sigma \psi^{-3} \sigma \frac{S^2 \frac{d\sigma}{dx} \psi^{-1} \sigma}{S^2 \sigma \psi^{-2} \sigma} + 4S. \sigma \psi^{-2} \sigma \frac{f''(h)}{(f'(h))^2} \frac{S^2 \frac{d\sigma}{dx} \psi^{-1} \sigma}{S^2 \sigma \psi^{-2} \sigma} \\ & + S. \sigma \psi^{-2} \sigma f'(h) \frac{d^2h}{dx^2} = 2S. \frac{d^2\sigma}{dx^2} \psi^{-1} \sigma + 2S. \frac{d\sigma}{dx} \psi^{-1} \frac{d\sigma}{dx} - 4S. \frac{d\sigma}{dx} \psi^{-2} \sigma \frac{S. \frac{d\sigma}{dx} \psi^{-1} \sigma}{S. \sigma \psi^{-2} \sigma}. \end{aligned}$$

Now, whatever vector ω may be, we have by §§ (5), (6)

$$-u^2 \omega = \frac{d\sigma}{dx} S. \frac{d\sigma}{dx} \omega + \frac{d\sigma}{dy} S. \frac{d\sigma}{dy} \omega + \frac{d\sigma}{dz} S. \frac{d\sigma}{dz} \omega,$$

so that, if ω be any other vector,

$$-u^2 S. \omega \omega = \Sigma S. \frac{d\sigma}{dx} \omega S. \frac{d\sigma}{dx} \omega.$$

Adding, then, the three equations, of which that containing $\frac{d^2h}{dx^2}$ is given above, we find

$$\begin{aligned} & -4u^2 \frac{S. \psi^{-2} \sigma \psi^{-1} \sigma}{S. \sigma \psi^{-2} \sigma} + 8u^2 S. \sigma \psi^{-3} \sigma \frac{S. \psi^{-1} \sigma \psi^{-1} \sigma}{S^2 \sigma \psi^{-2} \sigma} - 4u^2 \frac{f''(h)}{(f'(h))^2} \frac{S. \psi^{-1} \sigma \psi^{-1} \sigma}{S. \sigma \psi^{-2} \sigma} \\ & = -2S. \nabla^2 \sigma \psi^{-1} \sigma + 2\Sigma S. \frac{d\sigma}{dx} \psi^{-1} \frac{d\sigma}{dx} + 4u^2 \frac{S. \psi^{-2} \sigma \psi^{-1} \sigma}{S. \sigma \psi^{-2} \sigma}, \end{aligned}$$

where the term in $\nabla^2 h$ of course vanishes by (22). This is seen at a glance to be equivalent to

$$-4u^2 \frac{f''(h)}{(f'(h))^2} = -2S. \nabla^2 \sigma \psi^{-1} \sigma + 2\Sigma S. \frac{d\sigma}{dx} \psi^{-1} \frac{d\sigma}{dx}.$$

The last term here is seen at once, by HAMILTON's beautiful theory of linear and vector functions, to be equivalent to

$$2u^2 \Sigma S. i \psi^{-1} i = -2u^2 \left(\frac{1}{A + f(h)} + \frac{1}{B + f(h)} + \frac{1}{C + f(h)} \right). \quad (23),$$

if A, B, C be the constants of ϕ . Calling the expression in brackets for the present H, we have

$$-H + 2 \frac{f''(h)}{(f'(h))^2} = \frac{1}{u^3} S. \nabla^2 \sigma \psi^{-1} \sigma \quad (24).$$

The left hand member, if multiplied by $f'(h) dh$, is the differential of a function of h only. If, as in § 11 we have

$$\sigma = u\rho, \quad u = \text{constant},$$

the right hand side vanishes, and integration gives

$$C'h = \int \frac{f'(h) dh}{\sqrt{(A + f(h)) (B + f(h)) (C + f(h))}} = \int \frac{df}{\sqrt{M}}.$$

If σ have the value given in § 12, equation (24) is obviously not satisfied. Thus confocal quadrics are the only isothermal orthogonal surfaces included in equation (1) with our present limitations.

16. It is interesting in itself, and will be useful for the second part of this paper, to eliminate σ from (24) by the help of our previous equations. For this purpose we may write (2) in the form

$$\begin{aligned} -T^2 \psi^{-1} \sigma f'(h) \nabla h &= 2\Sigma \left(iS. \frac{d\sigma}{dx} \psi^{-1} \sigma \right) \\ &= 2u\Sigma (iS. iq^{-1} \psi^{-1} \sigma q) \\ &= -2uq^{-1} \psi^{-1} \sigma q \end{aligned} \quad (25),$$

the tensor of which is

$$T\psi^{-1} \sigma f'(h) T \nabla h = 2u.$$

But it is shown in (29) below that

$$\nabla u = q^{-1} \nabla^2 \sigma q,$$

so that (25) gives

$$T^2 \psi^{-1} \sigma f'(h) S. \nabla u \nabla h = 2u S. \nabla^2 \sigma \psi^{-1} \sigma,$$

or, by (24),

$$= 2u^3 \left(-H + 2 \frac{f''(h)}{(f'(h))^2} \right). \quad (26).$$

The three equations of this form give

$$\Sigma. T^2 \psi^{-1} \sigma f'(h) \nabla h S. \nabla h \nabla u = 2u^3 \Sigma. \nabla h \left(-H + 2 \frac{f''(h)}{(f'(h))^2} \right),$$

or, by (25) and (3),

$$-4u^2 \nabla u = 2u^3 \Sigma. f'(h) \nabla h \left(-H + 2 \frac{f''(h)}{(f'(h))^2} \right).$$

Operating by $S. d\rho$, this gives

$$2 \frac{du}{u} = -\Sigma f'(h) dh \left(-H + 2 \frac{f''(h)}{(f'(h))^2} \right),$$

of which the integral, by (23), is

$$C' - 2 \log. u = \Sigma [2 \log. f'(h) - \log. (A + f(h))(B + f(h))(C + f(h))],$$

or, if we write,

$$\frac{f'(h)}{\sqrt{(A + f(h))(B + f(h))(C + f(h))}} = F(h) \quad . \quad . \quad (27),$$

then

$$\frac{C''}{u} = F(h)F(h_1)F(h_2) \quad . \quad . \quad . \quad . \quad . \quad (28).$$

17. The following is the first quaternion method that occurred to me. I give it here, though it is considerably more prolix than the preceding, because it exhibits, incidentally, many curious properties of the system σ , u , q above defined.

Starting again with equation (7), we see that it gives

$$-\nabla^2 \sigma = q \nabla u q^{-1} - 2uq \Sigma \left(V. i V q^{-1} \frac{dq}{dx} \right) q^{-1},$$

or,

$$= -q \nabla u q^{-1} \quad . \quad . \quad . \quad . \quad . \quad (29),$$

to which value, as we shall see immediately, it may be reduced.

18. From (7) we obtain at once

$$\begin{aligned} \frac{1}{u} \frac{d^2 \sigma}{dx dy} &= \frac{dq}{dy} i q^{-1} - q i q^{-1} \frac{dq}{dy} q^{-1} + \frac{1}{u^2} \frac{du}{dy} \frac{d\sigma}{dx}, \\ &= \frac{dq}{dy} i q^{-1} - q i q^{-1} \frac{dq}{dy} q^{-1} + \frac{1}{u} q i q^{-1} \frac{du}{dy}, \\ \frac{1}{u} \frac{d^2 \sigma}{dy dx} &= \frac{dq}{dx} j q^{-1} - q j q^{-1} \frac{dq}{dx} q^{-1} + \frac{1}{u} q j q^{-1} \frac{du}{dx}. \end{aligned}$$

Comparing the last two values, we have

$$\frac{1}{u} q^{-1} \frac{d^2 \sigma}{dx dy} q = -2 V. i V q^{-1} \frac{dq}{dy} + \frac{i}{u} \frac{du}{dy} = -2 V. j V q^{-1} \frac{dq}{dx} + \frac{j}{u} \frac{du}{dx} \quad (30).$$

Operating by $S.k$, we have

$$-S. j q^{-1} \frac{dq}{dy} = S. i q^{-1} \frac{dq}{dx}.$$

From this, and other equations similar to it, it is obvious that

$$S. i q^{-1} \frac{dq}{dx} = S. j q^{-1} \frac{dq}{dy} = S. k q^{-1} \frac{dq}{dz} = 0 \quad . \quad . \quad (31).$$

For we should find, as their common value, the expression of their *sum*

$$S.iq^{-1}\frac{dq}{dx} + S.jq^{-1}\frac{dq}{dy} + S.kq^{-1}\frac{dq}{dz}.$$

19. Also, from (30) and the other similar equations, we have the following series of values—

$$\left. \begin{aligned} \frac{1}{u}\frac{du}{dy} &= 2S.kq^{-1}\frac{dq}{dx} \\ \frac{1}{u}\frac{du}{dz} &= 2S.iq^{-1}\frac{dq}{dy} \\ \frac{1}{u}\frac{du}{dx} &= 2S.jq^{-1}\frac{dq}{dz} \\ -\frac{1}{u}\frac{du}{dx} &= 2S.kq^{-1}\frac{dq}{dy} \\ -\frac{1}{u}\frac{du}{dy} &= 2S.iq^{-1}\frac{dq}{dz} \\ -\frac{1}{u}\frac{du}{dz} &= 2S.jq^{-1}\frac{dq}{dx} \end{aligned} \right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (32).$$

These give three equations of the form

$$V.q^{-1}\frac{dq}{dx} = \frac{1}{2u}\left(j\frac{du}{dz} - k\frac{du}{dy}\right),$$

which enable us to make the transformation assumed in § 17 above, and which may be all summed up in the following—in which the omission of the V is due to the remark in § 6 that the tensor of q may be assumed constant—

$$2V.q^{-1}dq = 2q^{-1}dq = -V.(d\rho\nabla)\log.u \quad (33).$$

This is the equation determining the quaternion which gives the position of a rigid body in terms of the vector-axis of instantaneous rotation. (*On the Rotation of a Rigid Body about a Fixed Point.* Trans. R.S.E. 1868–69.)

20. But by § 6, we have

$$\left. \begin{aligned} \left(\frac{d\sigma}{dx}\right)^2 &= \left(\frac{d\sigma}{dy}\right)^2 = \left(\frac{d\sigma}{dz}\right)^2 = -u^2, \\ S\frac{d\sigma}{dx}\frac{d\sigma}{dy} &= S\frac{d\sigma}{dy}\frac{d\sigma}{dz} = S\frac{d\sigma}{dz}\frac{d\sigma}{dx} = 0 \end{aligned} \right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (34).$$

From these

$$S\frac{d\sigma}{dx}\frac{d^2\sigma}{dx^2} = -u\frac{du}{dx},$$

$$S\frac{d\sigma}{dy}\frac{d^2\sigma}{dx^2} = -S\frac{d\sigma}{dx}\frac{d^2\sigma}{dx dy} = u\frac{du}{dy},$$

$$S \frac{d\sigma}{dz} \frac{d^2\sigma}{dx^2} = -S \frac{d\sigma}{dx} \frac{d^2\sigma}{dx dz} = u \frac{du}{dz},$$

which give

$$\begin{aligned} -u^2 \frac{d^2\sigma}{dx^2} &= u \left(-\frac{du}{dx} \frac{d\sigma}{dx} + \frac{du}{dy} \frac{d\sigma}{dy} + \frac{du}{dz} \frac{d\sigma}{dz} \right) \\ &= -2u \frac{du}{dx} \frac{d\sigma}{dx} + u \left(\frac{du}{dx} \frac{d\sigma}{dx} + \frac{du}{dy} \frac{d\sigma}{dy} + \frac{du}{dz} \frac{d\sigma}{dz} \right). \end{aligned}$$

From the three equations of this form we obtain, first,

$$u^2 \nabla^2 \sigma = u \left(\frac{du}{dx} \frac{d\sigma}{dx} + \frac{du}{dy} \frac{d\sigma}{dy} + \frac{du}{dz} \frac{d\sigma}{dz} \right). \quad (35);$$

and, secondly, three equations of the form

$$\frac{d}{dx} \left(\frac{1}{u^2} \frac{d\sigma}{dx} \right) = -\frac{1}{u^3} \left(\frac{du}{dx} \frac{d\sigma}{dx} + \frac{du}{dy} \frac{d\sigma}{dy} + \frac{du}{dz} \frac{d\sigma}{dz} \right).$$

These are summed up in

$$\frac{d}{dx} \left(\frac{1}{u^2} \frac{d\sigma}{dx} \right) = \frac{d}{dy} \left(\frac{1}{u^2} \frac{d\sigma}{dy} \right) = \frac{d}{dz} \left(\frac{1}{u^2} \frac{d\sigma}{dz} \right) = -\frac{1}{u^2} \nabla^2 \sigma \quad (36),$$

which express some of the conditions of *orthogonality* of the three series of surfaces given by equation (1).

21. To obtain the others, remark that by (30) we have

$$\frac{1}{u} q^{-1} \frac{d^2\sigma}{dx dy} q = -2V \cdot i V q^{-1} \frac{dq}{dy} + \frac{i}{u} \frac{du}{dy} = -2V \cdot j V q^{-1} \frac{dq}{dx} + \frac{j}{u} \frac{du}{dx},$$

or

$$\frac{1}{u} q^{-1} \frac{d^2\sigma}{dx dy} q = V \cdot i V j \nabla \cdot \log. u + \frac{i}{u} \frac{du}{dy} = V \cdot j V i \nabla \cdot \log. u + \frac{j}{u} \frac{du}{dx},$$

and that each of the two latter expressions may be written

$$\frac{i}{u} \frac{du}{dy} + \frac{j}{u} \frac{du}{dx}.$$

Hence

$$\frac{d^2\sigma}{dx dy} = \frac{1}{u} \left(\frac{d\sigma}{dx} \frac{du}{dy} + \frac{d\sigma}{dy} \frac{du}{dx} \right) \quad (37),$$

and there are, of course, other two vector equations of a similar form.

From these we have nine equations of the form

$$\frac{d^2\xi}{dx dy} = \frac{1}{u} \left(\frac{d\xi}{dx} \frac{du}{dy} + \frac{d\xi}{dy} \frac{du}{dx} \right) \quad (38).$$

Now

$$\frac{d}{dy} \left(\frac{1}{u^2} \frac{d\xi}{dx} \right) = \left(\frac{1}{u^2} \frac{d^2\xi}{dx dy} - \frac{2}{u^3} \frac{du}{dy} \frac{d\xi}{dx} \right)$$

$$= \frac{1}{u^3} \left(\frac{d\xi}{dy} \frac{du}{dx} - \frac{d\xi}{dx} \frac{du}{dy} \right).$$

Symmetry shows from this that

$$\frac{d}{dy} \left(\frac{1}{u^2} \frac{d\xi}{dx} \right) = - \frac{d}{dx} \left(\frac{1}{u^2} \frac{d\xi}{dy} \right) \quad (39),$$

which is one of another set of nine equations, three each for ξ , η , ζ .

22. Now, by (36), it is obvious that we may write, ϖ_1 being a new variable,

$$\frac{1}{u^2} \frac{d\xi}{dx} = \frac{d^2 \varpi_1}{dy dz}, \quad \frac{1}{u^2} \frac{d\xi}{dy} = \frac{d^2 \varpi_1}{dz dx}, \quad \frac{1}{u^2} \frac{d\xi}{dz} = \frac{d^2 \varpi_1}{dx dy} \quad (40),$$

and thus (39) becomes

$$\frac{d^3 \varpi_1}{dy^2 dz} = - \frac{d^3 \varpi_1}{dx^2 dz} \quad (41),$$

with two others in ϖ_1 and three each in ϖ_2 , ϖ_3 .

Putting

$$\omega_1 = \frac{d^3 \varpi_1}{dx dy dz},$$

these give by differentiation

$$\begin{aligned} \frac{d^2 \omega_1}{dy^2} &= - \frac{d^2 \omega_1}{dx^2} \\ \frac{d^2 \omega_1}{dz^2} &= - \frac{d^2 \omega_1}{dy^2} \\ \frac{d^2 \omega_1}{dx^2} &= - \frac{d^2 \omega_1}{dz^2}, \end{aligned}$$

so that all three quantities vanish. Hence we have

$$\omega_1 = \frac{d^3 \varpi_1}{dx dy dz} = 2hxyz + 2lyz + 2mzx + 2nxy + 2ax + 2by + 2cz + e,$$

where h, l, m, n, a, b, c, e are absolutely constant. From this, and (40), we have

$$\left. \begin{aligned} \frac{1}{u^2} \frac{d\xi}{dx} &= hxy^2z + 2lxyz + mx^2z + nx^2y + ax^2 + 2bxy + 2cxa + ex + f_1(y, z), \\ \frac{1}{u^2} \frac{d\xi}{dy} &= hxy^2z + ly^2z + 2mayz + nxy^2 + 2axy + by^2 + 2cyz + ey + f_2(z, x), \\ \frac{1}{u^2} \frac{d\xi}{dz} &= hxyz^2 + lyz^2 + mz^2x + 2nxyz + 2azx + 2byz + cz^2 + ez + f_3(x, y), \end{aligned} \right\} (42).$$

Applying (26) to these, we obtain

$$\left. \begin{aligned} hx^2z + 2lxz + nx^2 + 2bx + \frac{df_1}{dy} &= -hy^2z - 2myz - ny^2 - 2ay - \frac{df_2}{dx} \\ hxy^2 + 2mxy + ly^2 + 2cy + \frac{df_2}{dz} &= -hxz^3 - lz^2 - 2nzx - 2bz - \frac{df_3}{dy} \\ hyz^2 + 2nyz + mz^2 + 2az + \frac{df_3}{dx} &= -hx^2y - 2lxy - mx^2 - 2cx - \frac{df_1}{dz} \end{aligned} \right\} (43).$$

The elimination of f_2 from the two first, by differentiation, gives

$$h(x^2 - y^2) + 2lx - my + \frac{d^2f_1}{dydz} = \frac{d^2f_3}{dxdy} + h(z^2 - y^2) - 2my + 2nz,$$

and the third gives

$$hz^2 + 2nz + \frac{d^2f_3}{dxdy} = -\frac{d^2f_1}{dydz} - hx^2 - 2lx,$$

so that we have

$$hx^2 + 2lx + \frac{d^2f_1}{dydz} = hy^2 + 2my + \frac{d^2f_2}{dzdx} = hz^2 + 2nz + \frac{d^2f_3}{dxdy} = 0. \quad (44),$$

which proves that h, l, m, n are separately zero, and that each of the f 's is the sum of two separate functions, each containing one of the constituent variables only, *i.e.*,

$$\left. \begin{aligned} f_1(y, z) &= Y_1 + Z_1 \\ f_2(z, x) &= Z_2 + X_2 \\ f_3(x, y) &= X_3 + Y_3 \end{aligned} \right\} \quad . \quad . \quad . \quad (45).$$

But by (43) and (45), we have

$$2bx + \frac{dY_1}{dy} = -2ay - \frac{dX_2}{dx}, \text{ \&c.}$$

whence

$$\frac{dY_1}{dy} = -p''' - 2ay$$

$$\frac{dX_2}{dx} = p''' - 2bx, \text{ \&c.,}$$

giving

$$\begin{aligned} Y_1 &= -ay^2 + C - p'''y \\ X_2 &= -bx^2 + C' + p'''x, \text{ \&c.,} \end{aligned}$$

so that, finally,

$$\left. \begin{aligned} \frac{1}{u^2} \frac{d\xi}{dx} &= a(x^2 - y^2 - z^2) + 2bxy + 2czx + ex + g_1 - p'''y + p''z \\ \frac{1}{u^2} \frac{d\xi}{dy} &= 2axy + b(y^2 - x^2 - z^2) + 2cyz + ey + g_2 - p'z + p'''x \\ \frac{1}{u^2} \frac{d\xi}{dz} &= 2azx + 2byz + c(z^2 - x^2 - y^2) + ez + g_3 - p''x + p'y \end{aligned} \right\} (46).$$

If, in these, we write

$$\begin{aligned}ia + jb + kc &= \gamma \\ig_1 + jg_2 + kg_3 &= \gamma_1 \\ip' + jp'' + kp''' &= \gamma_2,\end{aligned}$$

we obtain, by multiplication by i, j, k respectively and addition,

$$\frac{1}{u^2} \nabla \xi = e\rho - \rho\gamma\rho + \gamma_1 + V\gamma_2\rho \quad . \quad . \quad . \quad . \quad . \quad (46'),$$

which is equivalent to the three equations (46), and may be put in the form

$$\frac{1}{u^2} \nabla \xi = (e - 2S\gamma\rho)\rho + \gamma\rho^2 + \gamma_1 + V\gamma_2\rho, \quad . \quad . \quad (46'').$$

23. It was shown above, § (7), that

$$\nabla \xi, \nabla \eta, \nabla \zeta$$

form a rectangular system of vectors whose common tensor is u . Hence, by (46'') we have three equations of the form

$$\begin{aligned}-\frac{1}{u^2} &= (e - 2S\gamma\rho)^2\rho^2 + \gamma^2\rho^4 + \gamma_1^2 + S^2\gamma_2\rho - \rho^2\gamma_2^2 \\&+ 2\rho^2S\gamma\rho(e - 2S\gamma\rho) + 2S\gamma_1\rho(e - 2S\gamma\rho) + 2\rho^2S\gamma\gamma_1 + 2\rho^2S.\gamma\gamma_2\rho + 2S.\gamma_1\gamma_2\rho,\end{aligned}$$

expressing the equality of the tensors; and three others of the form

$$\begin{aligned}0 &= (e - 2S\gamma\rho)(e' - 2S\gamma'\rho)\rho^2 + \rho^2(e' - 2S\gamma'\rho)S\gamma\rho + (e' - 2S\gamma'\rho)S\gamma_1\rho \\&+ (e - 2S\gamma\rho)\rho^2S\gamma'\rho + \rho^4S\gamma\gamma' + \rho^2S\gamma'\gamma_1 + \rho^2S.\gamma'\gamma_2\rho \\&+ (e - 2S\gamma\rho)S\gamma_1'\rho + \rho^2S\gamma_1'\gamma + S\gamma_1'\gamma_1 + S.\gamma_1'\gamma_2\rho \\&+ \rho^2S.\gamma\gamma_2'\rho + S.\gamma_1\gamma_2'\rho + S\gamma_2'\rho S\gamma_2\rho - S\gamma_2'\gamma_2\rho^2.\end{aligned}$$

Here the constants in $\nabla \eta, \nabla \zeta$, are expressed by the application of one, and two, dashes respectively to those of $\nabla \xi$.

In the first set of three, the terms in the various powers of $T\rho$ must be equal. This gives the following sets

$$\begin{aligned}\gamma^2 &= \gamma'^2 = \gamma''^2 \\S(V\gamma\gamma_2 - e\gamma)\rho &= S(V\gamma'\gamma_2' - e'\gamma')\rho = . \quad . \quad . \quad . \\(e^2 - \gamma_2^2 + 2S\gamma\gamma_1)\rho^2 + S^2\gamma_2\rho - 4S\gamma\rho S\gamma_1\rho &= . \quad . \quad . \quad . \quad . \\S(V\gamma_1\gamma_2 + e\gamma_1)\rho &= . \quad . \quad . \quad . \quad . = . \quad . \quad . \quad . \quad . \quad . \\ \gamma_1^2 &= \gamma_1'^2 = \gamma_1''^2.\end{aligned}$$

In the other set of three, we have by the same process

$$0 = S\gamma\gamma' = S\gamma'\gamma'' = S\gamma''\gamma$$

$$0 = S\rho\{V\gamma'\gamma_2 + V\gamma\gamma'_2 - e'\gamma - e\gamma'\} = \dots$$

$$0 = \rho^2\{ee' + S\gamma'\gamma_1 + S\gamma'_1\gamma - S\gamma_2\gamma'_2\} - 2S\gamma'\rho S\gamma_1\rho - 2S\gamma\rho S\gamma'_1\rho + S\gamma_2\rho S\gamma'_2\rho = \dots$$

$$0 = S\rho\{V\gamma'_1\gamma_2 + V\gamma_1\gamma'_2 + e'\gamma_1 + e\gamma'_1\} = \dots = \dots$$

$$0 = S\gamma_1\gamma'_1 = S\gamma'_1\gamma''_1 = S\gamma''_1\gamma_1.$$

We might easily have obtained this last set of equations from that which preceded, by a species of differentiation, ρ being constant, and $d\gamma = \gamma'$, $d\gamma' = \gamma''$, &c.

24. From these we conclude that, if they exist at all, γ , γ' , γ'' , and γ_1 , γ'_1 , γ''_1 , form rectangular systems with equal tensors. In terms of them we obtain

$$-\gamma_2 = \kappa\gamma_1 - e''\gamma'_1 + e'\gamma''_1 = c\gamma + e''\gamma' - e'\gamma''$$

$$-\gamma'_2 = e''\gamma_1 + \kappa\gamma'_1 - e\gamma'_1 = -e''\gamma + c\gamma' + e\gamma''$$

$$-\gamma''_2 = -e'\gamma_1 + e\gamma'_1 + \kappa\gamma''_1 = e'\gamma - e\gamma' + c\gamma'',$$

where κ and c are scalar constants to be determined.

Expressing, from these, γ_1 in terms of γ , γ' , γ'' , we have

$$\gamma_1 = -\gamma + \frac{c+\kappa}{D}\{(\kappa^2 + e^2)\gamma + (ee' + \kappa e'')\gamma' + (ee'' - \kappa e')\gamma''\},$$

where

$$D = \begin{vmatrix} \kappa, & -e'', & e' \\ e'', & \kappa, & -e \\ -e', & e, & \kappa \end{vmatrix} = \kappa\{\kappa^2 + e^2 + e'^2 + e''^2\}.$$

Now, the above expressions for γ_2 , &c., show that

$$T\gamma_1 = T\gamma, \text{ \&c.,}$$

hence by expanding and simplifying

$$0 = \frac{c+\kappa}{D}(\kappa^2 + e^2)\left(2 - \frac{c+\kappa}{\kappa}\right).$$

This admits of no values but

$$\kappa = \mp c$$

and

$$\kappa = e = e' = e'' = 0.$$

The first of these three values of κ gives

$$\gamma_1 = -\gamma, \text{ \&c.,}$$

and thence, by the equations at the end of § 23, leads to an impossibility, which

requires that all three sets of vectors $\gamma, \gamma_1, \gamma_2$ shall be null, and thus gives no solution. A similar nugatory result is obtained from the second.

The third thus shows that the only solution is

$$\frac{1}{u^2} \nabla \xi = \gamma_1, \text{ \&c.}$$

Hence

$$\nabla \frac{1}{u^2} = 0,$$

and, finally,

$$\sigma = -u^2 (iS\gamma_1\rho + jS\gamma_1'\rho + \kappa S\gamma_1''\rho),$$

where $\gamma_1, \gamma_1', \gamma_1''$ form a rectangular system with equal tensors, whose common value must obviously be the reciprocal of u . But we have seen that

$$\frac{d\sigma}{dx}, \frac{d\sigma}{dy}, \frac{d\sigma}{dz},$$

also form a rectangular set of vectors with equal tensors. Hence

$$\begin{aligned} (Si\gamma_1)^2 + (Si\gamma_1')^2 + (Si\gamma_1'')^2 &= (Sj\gamma_1)^2 + (Sj\gamma_1')^2 + (Sj\gamma_1'')^2 = \dots \\ 0 &= Si\gamma_1 Sj\gamma_1 + Si\gamma_1' Sj\gamma_1' + Si\gamma_1'' Sj\gamma_1'', \text{ \&c.} \end{aligned}$$

These equations also are satisfied identically, and we therefore have, as before,

$$\sigma = uq\rho q^{-1}$$

where u and q are each constant.