ON MODES OF CONVERGENCE OF AN INFINITE SERIES OF FUNCTIONS OF A REAL VARIABLE

By E. W. HOBSON.

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IT is well known that a series of functions of a real variable, each of which is continuous in a given interval, and such that for every point in the interval the sum of n terms of the series converges as n is increased indefinitely to a definite number, has a continuous function for its sum, provided the convergence of the series is uniform in the interval, but that this condition, though sufficient, is not necessary. A less stringent condition which is also sufficient, but not necessary, for the continuity of the sum-function, was introduced by Dini, and has been adopted by other writers: this condition is that the series must be simply uniformly convergent; in the first part of the present paper some remarks are made upon this mode of convergence. The necessary and sufficient conditions that the sum-function may be continuous throughout its domain have been given by Arzelà.* They consist in the necessity for a mode of convergence of the series of a less stringent character than either of the above modes: this mode is called "uniform convergence by intervals" and includes the above-mentioned modes as special cases. A principal object of the present communication is to show that the necessity and sufficiency of this mode of convergence may be established in a simple manner by means of an application of the Heine-Borel theorem concerning an infinite set of intervals. The latter part of the paper is concerned with the determination of the necessary and sufficient conditions that the sum-function may be an integrable function, the terms of the series being no longer necessarily continuous, but being assumed to be integrable; the main condition is expressed by saying that the series must have a mode of convergence called "uniform convergence by intervals in general" — a term due also to Arzelà. The conditions that an integrable sum-function is such that its integral is obtained as the sum of the integrals of the terms of the series are not dealt with here. Special cases have been treated by Osgood,† by myself,‡ and by W. H. Young,§ and the matter has been treated more

* See his memoir in two parts "Sulle serie di funzioni," *Memorie della R. Accad. degli Sci. di Bologna*, Series 5, Vol. VIII., 1900.

† *American Journal*, Vol. XIX.

‡ *Proc. London Math. Soc.*, Vol. XXXIV.

§ *Supra*, p. 89.

generally by Arzelà.* A criticism is given of Arzelà's proof of the conditions of integrability of the sum-function; the proof given in the present paper depends upon an application of the Heine-Borel theorem.

Simple Uniform Convergence.

1. Let us suppose that $u_1(x)$, $u_2(x)$, ..., $u_n(x)$, ... denote an unending sequence of functions each of which is defined for every point of the continuous interval (a, b) of the real variable x; moreover, suppose that the sum $s_n(x) \equiv u_1(x)+u_2(x)+\ldots+u_n(x)$, for each value of x in (a, b), converges as n is increased indefinitely to a definite number $s(x)$ which depends upon the value of x; the difference $s(x)-s_n(x) = R_n(x)$, we call the "remainder function."

The series $u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots$ is said to converge uniformly in the interval (a, b), to the limiting sum $s(x)$, when corresponding to every arbitrarily chosen positive number σ an integer m independent of x can be found such that $|R_m(x)|$, $|R_{m+1}(x)|$, $|R_{m+2}(x)|$ ... are all less than σ, whatever value x may have. When the condition of uniform convergence in the interval is not satisfied the mode of convergence may be that which has been termed by Dini "simple uniform convergence." Dini's† definition is as follows:—The series is said to be simply uniformly convergent in the interval (a, b) when corresponding to every arbitrarily chosen positive number σ as small as we please and to every integer m', only one or several integers m exist which are not less than m', and are such that, for all values of x in the interval (a, b), the $|R_m(x)|$ are $<\sigma$.

The condition of simple uniform convergence is less stringent than that of uniform convergence, in that in the latter case all the remainders after a certain one are numerically less than σ, whereas in the former case one or several, but not all, the remainders are numerically less than σ.

As regards Dini's definition, it may in the first place be remarked that, if there is one value of m which satisfies the prescribed condition, there must be an infinite number of such values, because we have only to ascribe to m' a series of values increasing indefinitely, and for each of these there is a value of m; any one of this infinite series of values of m may be taken to correspond to the smallest value of m'.

Moreover the definition contains a redundancy; it will here be shown that, if, corresponding to every σ, a value of n can be found such that $|R_n(x)| < \sigma$, independently of x, then there must be an indefinitely great

* See the paper already referred to.

† See *Grundlagen*, by Lüroth and Schepp, p. 137.

number of such values of n. Let us denote by $\overline{R}_s$ the upper limit* of $|R_s(x)|$ for the whole interval (a, b); $\overline{R}_s$ may be either finite or indefinitely great. If $|R_n(x)| < \sigma$, for all values of x, we have $\overline{R}_n \leqslant \sigma$: take a positive number ϵ_1 less than $\overline{R}_n$ and also less than the least of the numbers $\overline{R}_1, \overline{R}_2, \ldots, \overline{R}_{n-1}$; then by hypothesis a number n_1 can be found such that for all values of x, $|R_{n_1}(x)| < \epsilon_1$. This number n_1 cannot be one of the numbers $1, 2, 3, \ldots, n$, for it is always possible to find a value of x for which $|R_s(x)|$ is arbitrarily near its upper limit $\overline{R}_s$, and is thus greater than ϵ_1; hence a number $n_1 > n$ has been shown to exist such that, for all the values of x, $|R_{n_1}(x)| < \sigma$. Similarly, it may be shown that a number $n_2 > n_1$ exists which has the same property; thus there are an indefinite series of values of n such that $|R_n(x)| < \sigma$.

It thus appears that the definition may be more simply stated as follows:—The series is said to converge simply uniformly in the interval if, corresponding to every arbitrarily chosen number σ as small as we please, a number n can be found such that, independently of x, $|R_n(x)| < \sigma$.

If the series converges simply uniformly, but not uniformly, there must also be an indefinitely great number of values of n for which the condition $|R_n(x)| < \sigma$, for all values of x, is not satisfied; for, if there were only a finite number of such values of n, we could take a value of n greater than all of them, and, starting from this value of n, the condition of uniform convergence would be satisfied.

2. Let $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ be a sequence of diminishing positive numbers which converges to zero: if the series is simply uniformly convergent, we can find n_1 so that $|R_{n_1}(x)| < \epsilon_1$, for all values of x; we can then find n_2 a number greater than n_1, so that $|R_{n_2}(x)| < \epsilon_2$; then $n_3 > n_2$ such that $|R_{n_3}(x)| < \epsilon_3$; and so on. If now we amalgamate the first n_1 terms into one, then those after the first n_1 up to and including $u_{n_2}(x)$ into one term, and so on, the series may be written

$$s_{n_1}(x) + [s_{n_2}(x) - s_{n_1}(x)] + [s_{n_3}(x) - s_{n_2}(x)] + \ldots;$$

in this form the series is uniformly convergent. It thus appears that a simply uniformly convergent series can be changed into a uniformly convergent series by bracketing the terms suitably, and amalgamating the functions within each bracket. Conversely, a uniformly convergent series may be replaced by one which is only simply uniformly convergent; if each term $u_n(x)$ of a uniformly convergent series be replaced by the sum of r_n functions, such that $u_n(x) = U_{n,1}(x) + U_{n,2}(x) + \ldots + U_{n,r_n}(x)$, the new series $U_{1,1}(x) + U_{1,2}(x) + \ldots + U_{1,r_1}(x) + U_{2,1}(x) + \ldots$ is not necessarily

* The trivial case in which $R_n(x) = 0$ for every x is omitted from consideration.

convergent, but, if the functions U are so chosen that the series converges for every value of x in (a, b), then the series converges at least simply uniformly. It thus appears* that the distinction between uniform convergence and simple uniform convergence is an unessential one.

Conditions for the Continuity of the Sum-Function at a Point.

3. Let us now suppose that all the functions $u_1(x), u_2(x), \dots, u_n(x), \dots$ are continuous throughout the interval (a, b); it is known that, if $s(x)$ is discontinuous at a point x, the series is non-uniformly convergent in any neighbourhood $(x-\delta, x+\delta')$ of the point x, but that, if it is so non-uniformly convergent, the function $s(x)$ is not necessarily discontinuous at the point x. It is frequently convenient to replace the remainder $R_n(x)$ by the transformed remainder-function† $R(x, y)$, obtained by taking $y = 1/n$, and considering $R(x, y)$ as a function of the two variables x, y; the function $R(x, 0)$ is defined to be zero for all values of x. At a point x of uniform continuity of the series the function $|R(x, y)|$ is continuous at the point $(x, 0)$ with regard to (x, y); at a point x of non-uniform continuity of the series the function $|R(x, y)|$ is discontinuous at $(x, 0)$ with regard to (x, y), the saltus of the function at the point giving the measure of non-uniform convergence at that point. To find the necessary and sufficient conditions that $x_1(P)$ is a point of continuity of $s(x)$, let us first assume this to be the case: then a neighbourhood of P can be found such that, if p be any point whatever, in that neighbourbood, $|s(p)-s(P)| < \frac{1}{3}\epsilon$, where ϵ is arbitrarily chosen. Now let n be so chosen that $|R_n(P)| < \frac{1}{3}\epsilon$: this is possible on account of the convergency condition of the series at the point P. Since $s_n(x)$ is continuous, a neighbourhood of P can be found such that, if p is any point in it, $|s_n(p)-s_n(P)| < \frac{1}{3}\epsilon$; take now that neighbourhood of P which is the common part of the two neighbourhoods which have been already chosen, denote this by $(x-h, x+k)$; then, if p is any point in $(x-h, x+k)$, we have both $|s(p)-s(P)| < \frac{1}{3}\epsilon$ and $|s_n(p)-s_n(P)| < \frac{1}{3}\epsilon$. Since

$$R_n(p)-R_n(P) = \{s(p)-s(P)\} - \{s_n(p)-s_n(P)\},$$

we have $$|R_n(p)-R_n(P)| \leqslant |s(p)-s(P)| + |s_n(p)-s_n(P)|;$$

hence, if p is any point in $(x-h, y+k)$, we have $|R_n(p) - R_n(P)| < \frac{2}{3}\epsilon$, and, since $|R_n(P)| < \frac{1}{3}\epsilon$, we have $|R_n(p)| < \epsilon$. It has thus been shown

* I find that the substance of Art. 2 is contained in a paper by Arzelà in the *Bologna Rendiconti* for 1899. However, I retain Art. 2 in the present paper for the sake of completeness.

† See *Proc. London Math. Soc.*, Vol. XXXIV.

that, if $s(x)$ is continuous at x, and n has a sufficiently great value, viz., one such that $|R_n(x)| < \frac{1}{3}\epsilon$, a neighbourhood $(x-h, x+k)$ of x can be found such that everywhere in it $|R_n(x)| < \epsilon$; this condition is necessary for the continuity of $s(x)$ at the point x. Conversely, assume this condition to be satisfied for every value of n sufficiently great; then it will be shown that x is a point of continuity of $s(x)$. A neighbourhood of x can be so chosen that for any point p in it both $|s_n(p)-s_n(P)| < \eta$, where η is an arbitrarily chosen positive number, and also

$$|R_n(p)-R_n(P)| \leqslant |R_n(p)|+|R_n(P)| < \tfrac{4}{3}\epsilon;$$

in this neighbourhood of P, $|s(p)-s(P)| < \frac{4}{3}\epsilon+\eta$, and thus, since ϵ and η are both at choice, the function $s(x)$ is continuous at x. It has now been shown that the necessary and sufficient conditions that $s(x)$ may be continuous at the point P are that corresponding to every fixed positive number ϵ chosen arbitrarily, and as small as we please, and for any value whatever of n which is greater than or equal to a fixed number dependent on ϵ, a neighbourhood of P can be found such that at every point in it $|R_n(x)| < \epsilon$.

4. Let the function $R(x, y)$ be represented on the plane of (x, y), the function being defined for all values of x in the interval (a, b) and for all values of y in the interval $(0, 1)$; the values of the function for those values of y which are not the reciprocal of an integer may be defined* so that $R(x, y)$ is a continuous function of y; moreover, $R(x, 0) = 0$. If a positive number ϵ is prescribed, then for a certain range of values of y from zero upwards the condition $|R(x, y)| < \epsilon$ is satisfied; the upper limit of these values of y may be denoted by $\phi_\epsilon(x)$; there may be other greater values of y not continuous with the interval $(0, \phi_\epsilon(x))$ for which the condition $|R(x, y)| < \epsilon$ is also satisfied. At a point P of non-uniform convergence of the series, the lower limit of $\phi_\epsilon(x)$ for the values of x in any neighbourhood of P is zero, whereas for a point of uniform convergence a neighbourhood of P can be found for which the lower limit of $\phi_\epsilon(x)$ is greater than zero.

The distinction between the three classes of points in the interval (a, b), viz., (i.) those at which the series is uniformly convergent, (ii.) those at which the series is non-uniformly convergent, but at which the sum-function is continuous, and (iii.) those points at which the function is discontinuous, may be illustrated by means of figures which indicate the regions of the plane (x, y) in the neighbourhood of $(x, 0)$, at which $|R(x, y)|$ is less than an arbitrarily chosen ϵ.

* See the paper already referred to, *Proc. London Math. Soc.*, Vol. XXXIV.

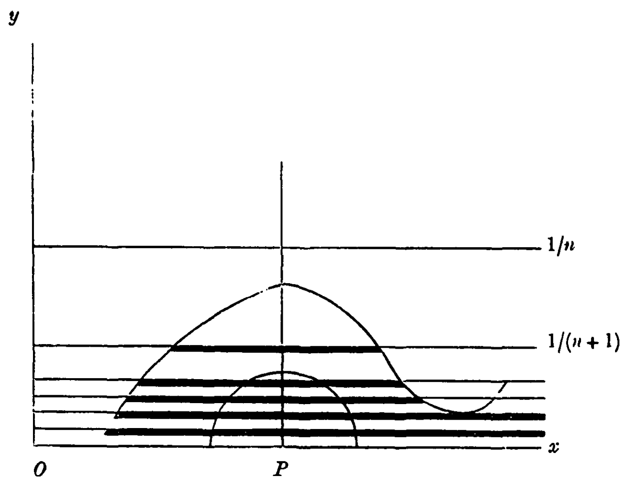

FIG. 1.

Fig. 1 represents the neighbourhood of a point P at which the convergence of the series is uniform; the blackened lines represent those portions of the lines whose ordinates are $1/n$, $1/(n+1)$, $1/(n+2)$, ..., at which $|R_n(x)|$, $|R_{n+1}(x)|$, ... are $\leqslant \epsilon$. These portions consist of all those parts of the lines which are bounded by the curve $y = \phi_\epsilon(x)$, and also possibly of other pieces outside this curve. An area, for example, semi-circular, can be drawn which is bounded by a portion of the x axis which contains P, and is such that for every point within it $|R(x, y)| < \epsilon$; that this should be possible for every value of ϵ is the condition that $R(x, y)$ is continuous at the point P, with regard to the two-dimensional continuum (x, y).

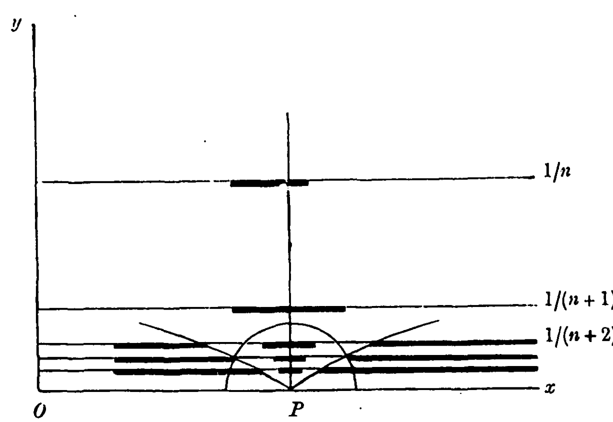

FIG. 2.

Fig. 2 represents the neighbourhood of a point P at which the function

$s(x)$ is continuous but at which the series is non-uniformly convergent. In this case the function $\phi_\epsilon(x)$ is for a sufficiently small value of ϵ, discontinuous at P; the value of $\phi_\epsilon(x)$ at P is finite, but the functional limits $\phi_\epsilon(x+0)$, $\phi_\epsilon(x-0)$ at P are both zero. The breadth of the blackened portions of the straight lines parallel to the x-axis, which represent the portions of those lines at which $|R_n(x)| \leqslant \epsilon$, diminishes indefinitely as y approaches the value zero at P. In this case no semicircle can be drawn with P as centre, for all internal points of which $|R(x, y)| < \epsilon$: thus the point P is one of non-uniform continuity, the measure of non-uniform convergence being greater than ϵ. In the figure the convergence is non-uniform on both sides of P; it is clear, however, how the figure must be modified for the case in which the convergence is uniform on one side of P. In case the measure of non-uniform convergence is indefinitely great the figure will be essentially similar whatever value of ϵ is chosen; otherwise the figure applies to an ϵ which is less than the measure of non-uniform convergence, viz., the saltus at P of $|R(x, y)|$ in the two-dimensional continuum.

Fig. 3 represents the neighbourhood of a point P at which $s(x)$ is discontinuous, for a value of ϵ less than the measure of non-uniform convergence of the series at P. In this case, as before, $\phi_\epsilon(x)$ is finite at P, and

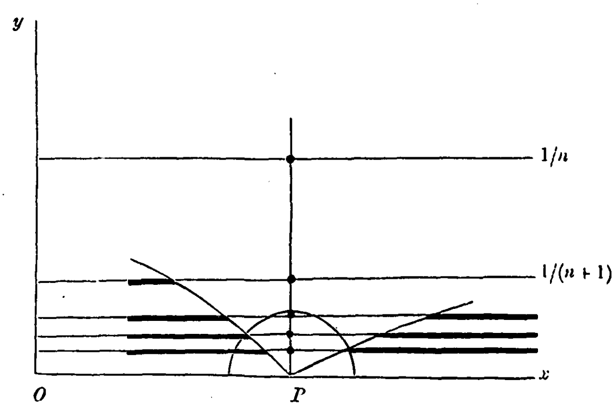

FIG. 3.

$\phi_\epsilon(x+0)$, $\phi_\epsilon(x-0)$ are zero, but there are no intervals near P on the ordinate at which $|R(x, y)| < \epsilon$, but only points on the ordinate through P itself.

As an example we may take

$$R_n(x) = \frac{nx}{1+n^2x^2}, \quad \text{and thus} \quad R(x, y) = \frac{xy}{x^2+y^2},$$

and we may suppose the domain of x to be the interval (0, 1). In this case the end-point $x = 0$ is a point of discontinuity of $R(x, y)$, and we find that, if $\epsilon < \frac{1}{2}$, the condition $|R_n(x)| < \epsilon$ is satisfied for the space bounded by the x-axis and by the straight line

$$y = \phi_\epsilon(x) = x\left[\frac{1}{2\epsilon} - \left(\frac{1}{4\epsilon^2} - 1\right)^{\frac{1}{2}}\right].$$

The same condition is satisfied for the space between the y-axis and the straight line $y = x\left[\frac{1}{2\epsilon} + \left(\frac{1}{4\epsilon^2} - 1\right)^{\frac{1}{2}}\right]$, and thus the point $x = 0$ is a point of continuity of the function $s(x)$, although the convergence at that point is non-uniform. If $\epsilon > \frac{1}{2}$, $|R(x, y)| < \epsilon$, for the whole space between the axes; thus the measure of non-uniform convergence is $\frac{1}{2}$.

5. *Conditions for the Continuity of the Sum-Function in its whole Domain.*

The necessary and sufficient conditions can now be determined that $s(x)$ may be continuous for the whole interval (a, b). First suppose that $s(x)$ is everywhere continuous, and choose an arbitrarily small number ϵ; then, if n is sufficiently large, there are points in (a, b) at which $|R_n(x)| < \epsilon$. If P be such a point, a neighbourhood of P can be found such that at every point within it the condition $|R_n(x)| < \epsilon$ is satisfied; the size of this neighbourhood may be extended in both directions until points p, q are reached, at which $|R_n(p)| = |R_n(q)| = \epsilon$; this follows from the fact that $R_n(x)$ is a continuous function. Every such point P in (a, b) at which $|R_n(x)| < \epsilon$ may, in a similar manner, be enclosed in an interval of finite length, and in all internal points of such intervals the condition $|R_n(x)| < \epsilon$ is satisfied. Thus for any fixed value of n which is sufficiently large there exists a finite or infinite system D_n of intervals in (a, b) such that at every point which is interior to one of the intervals of D_n the condition $|R_n(x)| < \epsilon$ is satisfied; moreover, the intervals of D_n contain in their interiors all points except a, b at which the condition is fulfilled. Two intervals of the set may have a common end-point in case $|R_n(x)|$ has a maximum equal to ϵ; otherwise the intervals will be separated from one another.

Consider the systems of intervals $D_n, D_{n+1}, D_{n+2}, \ldots$, every value of n being taken from a fixed value onwards: the whole set thus formed is such that every point in (a, b), except the end-points, is interior to an infinite number of intervals of the compound set; this follows from the fact that, for any point x, a value of n, say n_1, can be found such that $|R_{n_1}(x)| < \epsilon$, $|R_{n_1+1}(x)| < \epsilon$, ... Moreover, intervals can be found with

a and b as end-points which for a sufficiently large value of m belong to D_{n+m}. The Heine-Borel theorem states that, if in (a, b) any infinite set of intervals is taken such that every point within (a, b) is an internal point of one interval at least of the set, and such that a, b are end-points of two intervals of the set, then a finite number of the intervals can be selected which is such that every internal point of (a, b) is in the interior of one at least of the finite set. If we apply this theorem to the set of intervals D_n, $D_{n+1}, \ldots$, we see that a finite number of these intervals can be chosen which contains every point within (a, b) as an internal point of one at least of the intervals. The number n can be chosen so great that $|R_n(a)|$ and $|R_n(b)|$ are both less than ϵ, and thus that the points a, b are end-points of two of the finite set of intervals. It thus appears that on the supposition that $s(x)$ is continuous at every point of (a, b), and n is a chosen number, then a finite series of numbers $n+\iota_1, n+\iota_2, \ldots, n+\iota_r$, all greater than or equal to n, can be chosen such that, for every value of x, $|R_m(x)| < \epsilon$ where m has one of the values $n+\iota_1, n+\iota_2, \ldots, n+\iota_r$; the particular value of m varies with x, but the same value of m is applicable to the whole of one of a finite number of continuous intervals. The intervals of the set will overlap, but an overlapping portion may be considered as belonging to one of the intervals only, so that (a, b) may be divided into a finite number of parts, in each of which for some value of m constant for that part, the condition $|R_m(x)| \leqslant \epsilon$ is satisfied.

It should be remarked that the set D_n for a fixed n is not necessarily a finite set, for, besides those intervals for which $|R_n(x)| < \epsilon$ and those for which $|R_n(x)| > \epsilon$, there may be points of (a, b) which are not in either set of intervals, but are limiting points of end-points of the intervals D_n; at such points $|R_n(x)| = \epsilon$. For example, if

$$|R_n(x)| = \frac{1}{n^2} + \frac{x-c}{n^3} \sin\left(\frac{1}{x-c}\right),$$

where $a < c < b$, and, if $\epsilon = 1/n^2$, the point c is a point of continuity of $R_n(x)$, and is a limiting point of end-points of those intervals for which $R_n(x)| < \epsilon$. Conversely, if, for every value of ϵ, a finite set of intervals exists, having the property described, then $s(x)$ is continuous throughout (a, b). Consider a point P of non-uniform convergence of the series: then, for a given ϵ, P is inside an interval for every point of which, for a fixed value of m, $|R_m(x)| < \epsilon$, and we have shown that this is a sufficient condition that $s(x)$ is continuous at the point P; hence every point of non-uniform convergence of the series is a point of continuity.

It has now been established that *the necessary and sufficient condition* *that a series* $u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots$, *each term of which is a continuous function of* x *throughout* (a, b) *and converges at every point of this domain to a definite value* $s(x)$, *is that, corresponding to any arbitrarily chosen number* ϵ *and to an arbitrarily chosen integer* n, *we have* $|R_m(x)| < \epsilon$, *for every value of* x *in* (a, b), *where* m *has one of a finite number of values greater than or equal to* n, *and that the value of* m *depends in general on that of* x, *but is constant for points* x *which lie in one of a number of finite portions of the interval* (a, b).

This theorem, which has been established otherwise by Prof. Arzelà, shows that a certain mode of convergence of the series is the necessary and sufficient condition for the continuity of the sum; this mode has been termed by Arzelà *convergenza uniforme a tratti* ("uniform convergence by intervals"); this term is, perhaps, not altogether appropriate, because the intervals are dependent in number and length upon the arbitrarily chosen ϵ. Uniform convergence and simple uniform convergence are special cases of uniform convergence by intervals; they correspond to the case in which the finite set of intervals which correspond to an ϵ reduce to one interval, namely, the whole interval (a, b).

*Conditions of Integrability of the Sum of a Series of Integrable Functions.**

6. Let the functions $u_1(x), u_2(x), \ldots, u_n(x), \ldots$ be no longer necessarily continuous functions, but let us suppose that each one has a proper integral in every interval contained in (a, b); that this may be the case it is necessary that the upper limits of each of $|s_1(x)|$, $|s_2(x)|$, ..., $|s_n(x)|$, ... in the interval (a, b) is finite. Denoting these upper limits by $l_1, l_2, \ldots, l_n, \ldots$, then, provided $l_1, l_2, \ldots l_n, \ldots$ have a finite upper limit L, it can be shown that $|s(x)|$ has a finite upper limit in (a, b); for, if the upper limit of $|s(x)|$ were indefinitely great, we could find a value of x such that $|s(x)| = L+a$, where a is some positive number. Now n can be taken so great that $|s(x)-s_n(x)| < \epsilon$, where ϵ is arbitrarily small; hence $|s(x)| < |s_n(x)| + \epsilon < L+\epsilon$, and ϵ can be chosen to be less than a; thus it is impossible that the upper limit of $|s(x)|$ in (a, b) is not finite. The condition just stated that $|s(x)|$ may have a finite upper limit is a sufficient one but not a necessary one; in fact we know that at the point $(x, 0)$ the function $s(x, y)$ may have an infinite discontinuity whilst $s(x)$ has only a finite discontinuity or is continuous at the point x.

* §§ 6 and 7 have been rewritten December 30th, 1903.

We shall suppose, in what follows, that $|s(x)|$ has a finite upper limit in (a, b), so that, in case $s(x)$ is integrable, the integral is a proper one.

To find the conditions that $s(x)$ may be an integrable function, we shall first suppose it to be so; then those points of (a, b) at which $s(x)$ has a saltus which is $\geqslant \epsilon$ form a closed set of points with zero content. These points may be included in a finite set of intervals whose sum is arbitrarily small, say η, and the end-points of these intervals $\delta_1, \delta_2, \ldots, \delta_r$ may be so chosen that in each the saltus of $s(x)$ is $< \epsilon$. At every point of (a, b) which is not in the interior of one of the intervals $\{\delta\}$, the saltus of $s(x)$ is less than ϵ. In a similar manner the points of (a, b) at which the saltus of $s_n(x)$ is $\geqslant \epsilon$ may be included in a finite set of intervals $\delta_1^{(n)}, \delta_2^{(n)}, \ldots, \delta_p^{(n)}$, whose sum η_n is arbitrarily small, and such that at the end-points the saltus of $s_n(x)$ is less than ϵ. If both sets $\{\delta\}$, $\{\delta^{(n)}\}$ be excluded from (a, b), the remainder consists of a finite number of closed intervals, such that at every point in them both $s(x)$ and $s_n(x)$ have a saltus which is $< \epsilon$; these intervals we may denote by $\Delta_1^{(n)}, \Delta_2^{(2)}, \ldots, \Delta_s^{(n)}$, forming a finite set $\{\Delta^{(n)}\}$.

Let us consider the set $G^{(n)}$ which consists of those points of the intervals $\{\Delta^{(n)}\}$ at which $|R_n(x)| < 3\epsilon$; the set $G^{(n)}$ consists in general of a set $\{D^{(n)}\}$ of intervals, and of a set $H^{(n)}$ of points which are not in intervals for which $|R_n(x)|$ is everywhere $< 3\epsilon$; the intervals of the set $\{D^{(n)}\}$ may be finite or enumerably infinite in number, and each interval of the set may be closed, or open, at either end, according as the end-point is, or is not, a point at which $|R_n(x)| < 3\epsilon$; two intervals of the system may abut on one another, in case the common end-point does not belong to $G^{(n)}$. As regards the set of points $H^{(n)}$ and the set of intervals $\{D^{(n)}\}$, we observe that, if P be any point belonging to $G^{(n)}$, then, if no neighbourhood of P either on the right or on the left can be found, such that every point in the neighbourhood belongs to $G^{(n)}$, P is a point of the set $H^{(n)}$; if, on the other hand, a neighbourhood $(x-\alpha, x+\beta)$ of $P(x)$ exists, every point of which belongs to $G^{(n)}$, the number β is capable of having a set of values which have an upper limit β_1, and the number α is capable of having a set of values which have a lower limit α_1: in that case $(x-\alpha_1\, x+\beta_1)$ is an interval belonging to the set $\{D^{(n)}\}$; it is a closed interval in case both $x-\alpha_1$ and $x+\beta_1$ are points of $G^{(n)}$, and it is open at one or both ends in case one or both of the points $x-\alpha_1, x+\beta_1$ does not belong to the set $G^{(n)}$. In case x has only a neighbourhood $(x, x+\beta)$ on the right, or only a neighbourhood $(x-\alpha, x)$ on the left, every point of which belongs to $G^{(n)}$, we see, in a similar manner to the above, that a definite interval $(x, x+\beta_1)$ or $(x-\alpha_1, x)$ exists, every point of which, with the possible exception of the end-point, belongs to $G^{(n)}$, and thus that the interval belongs to $\{D^{(n)}\}$. Since P is any point whatever of $G^{(n)}$, it is now clear that the set of points $H^{(n)}$ and the set of intervals $\{D^{(n)}\}$ which together constitute $G^{(n)}$ are determinate; either $\{D^{(n)}\}$, or $\{G^{(n)}\}$, or both, may be absent for the n considered; it will, however, appear that, if n is sufficiently great, the set $\{D^{(n)}\}$ must exist.

We now see that every point x, of (a, b), which is such that $|R_n(x)| < 3\epsilon$, either lies in the interior of one of the intervals of $\{\delta\}$, or of $\{\delta^{(n)}\}$, or else belongs to one of the intervals of $\{D^{(n)}\}$, or to the set of points $H^{(n)}$.

Similar reasoning applies if, instead of n, we take $n+1, n+2, \ldots, n+m, \ldots$; if we take $\eta_n = \zeta$, $\eta_{n+1} = \frac{1}{2}\zeta$, $\eta_{n+2} = \frac{1}{2^2}\zeta$, $\ldots$, $\eta_{n+m} = \frac{1}{2^m}\zeta$, $\ldots$, where ζ is an arbitrarily small fixed number, we see that the sum of all the intervals $\{\delta^{(n)}\}$, $\{\delta^{(n+1)}\}$, $\ldots$, $\{\delta^{(n+m)}\}$, $\ldots$ is 2ζ.

It will now be shown that there is no point x which belongs to all the sets of points $H^{(n)}, H^{(n+1)}, \ldots, H^{(n+m)}, \ldots$, or to an infinite number of them; to prove this, we observe that, for any fixed x, a value of m can be found such that $|R_{n+m}(x)| < \epsilon$, for this value of m and for all greater values; taking m to have such a value, unless x lies in the interior of one of the intervals of $\{\delta\}$, $\{\delta^{(n+m)}\}$, a neighbourhood of x can be found such that, for every point $x+h$ in it, $|s(x+h)-s(x)| < \epsilon$, and also

$$|s_{n+m}(x+h)-s_{n+m}(x)| < \epsilon;$$

in this neighbourhood of x,

$$\begin{aligned} &|R_{n+m}(x+h)| \\ &\quad \leqslant |s(x+h)-s(x)| + |s(x)-s_{n+m}(x)| + |s_{n+m}(x)-s_{n+m}(x+h)| \\ &\quad < 3\epsilon. \end{aligned}$$

This holds for definite values of m; the neighbourhood in which it holds depends upon the values of m.

It has now been shown that, if x is any fixed point not in the interior of $\{\delta\}$, nor an end-point of $\{\delta\}$, a value of m can be found so great that x is either (1) interior to one of the intervals $\{\delta^{(n+m)}\}$, or (2) interior to one of the intervals of $\{D^{(n+m)}\}$, or (3) an end-point of an interval of $\{\delta^{(n+m)}\}$. In the last case, since every end-point of the $\{\delta^{(n+m)}\}$ is a point at which the saltus of $s_{n+m}(x)$ is $< 3\epsilon$, we can find a neighbourhood of x, which encroaches on the interval of which x is an end-point, but is such that the above reasoning applies to it. If x is an end-point of an interval of $\{\delta\}$, we can, in a similar manner, find an interval encroaching on the interval of $\{\delta\}$ of which it is an end-point, but such that throughout it $|R_{n+m}(x)| < 3\epsilon$. For the end-points a, b, values of m can be found such that these points are end-points of intervals through which $|R_{n+m}(x)| < 3\epsilon$, unless a or b is already an end-point of the set $\{\delta\}$. If we take the whole set of intervals which consists of (1) the intervals $\{\delta\}$ whose sum is η, of (2) all the sets $\{\delta^{(n+m)}\}$ whose sum is 2ζ, and (3) of all the sets $\{D^{(n+m)}\}$, where m has all values 0, 1, 2, 3, ..., with (4) the addition of intervals introduced as above whenever an end-point of one of the above intervals becomes a point of a $G^{(n+m)}$, we have, when these are taken altogether, an enumerable set of intervals which contains every point of (a, b) in the interior of at least one interval, except that a and b are end-points of intervals of the set. To this complex set of intervals the Heine-Borel theorem is applicable, and we deduce that a finite number of intervals of the set can be chosen which contain every point of (a, b) except a and b, in the interior of at least one interval, and such that a, b are themselves end-points of the intervals of the set. Those intervals of the finite set which belong to $\{\delta\}$ or $\{\delta^{(n)}\}$, $\{\delta^{(n+1)}\}$, ... have a sum less than $\eta+2\zeta$, and the rest of the intervals are such that for each of them there is a value of m such that $|R_{n+m}(x)| < 3\epsilon$, for every point of that one interval. Since $\eta+2\zeta$ is arbitrarily small, we see that, by excluding from (a, b) a finite set of intervals whose sum is arbitrarily small, the remainder of (a, b) is such that, for every point in it, $|R_{n+p}(x)| < 3\epsilon$, where p has one of a finite number of values. That this should hold for every ϵ is a necessary condition for the integrability of $s(x)$.

It can now be shown, conversely, that, provided this condition is satisfied for every ϵ, the function $s(x)$ must be integrable. To show this, we apply Riemann's test of integrability, according to which the whole interval $(a\ b)$, is divided into a finite number of parts $h_1, h_2, \ldots, h_s$, and the value of the sum of the products of each h into the fluctuation of the function in that part h is estimated; the condition of integrability is that this sum should have a zero limit when the number s is increased indefinitely, and at the same time the greatest of the intervals h has a zero limit. It is well known that it is sufficient to make the sub-division of (a, b) according to any law we please, provided the condition as to the greatest of the h is satisfied. Taking a given ϵ, let the intervals $(\delta_1, \delta_2, \ldots, \delta_r)$ be marked out, and also the finite set of intervals into which the rest of (a, b) can be divided, so that in each one of them, for some particular value of p, $|R_{n+p}(x)| < 3\epsilon$; let the end-points of $h_1, h_2, \ldots, h_s$ contain among them all the end-points of both these finite sets of intervals. In any h which lies within one of the δ, the product of the h into the fluctuation of $s(x)$ is less than or equal to $h(M-m)$, where M, m are the upper and lower limits of $s(x)$ in (a, b); in any h which lies within an interval for which $|R_{n+p}(x)| < 3\epsilon$, the fluctuation of $s(x)$ cannot exceed that of $s_{n+p}(x)$ by more than 6ϵ. It follows that the sum of the products required for Riemann's test is less than $(M-m)\Sigma\delta + \sum_p \Sigma h \{\text{fluctuation of } s_{n+p}(x) + 6\epsilon\}$, where in the double summation the first summation refers to all those of the h which are within an interval for which p is constant, and the second sum is taken for all the finite number of values of p. Now $s_{n+p}(x)$ is integrable through any interval; hence, through the interval for which p has a fixed value $\{\Sigma h \text{ fluctuation of } s_{n+p}(x)\}$ has a zero limit; also $(M-m)\eta$ is arbitrarily small; hence the product in Riemann's test is arbitrarily small as η becomes indefinitely small and s is indefinitely increased; therefore $s(x)$ is integrable in (a, b).

It has now been shown that, *if* $u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$ *converges to a definite value* $s(x)$ *at every point in* (a, b), *and if all the functions* $u_1(x), u_2(x), \ldots, u_n(x), \ldots$ *have proper integrals in* (a, b), *the necessary and sufficient conditions that* $s(x)$ *may have a proper integral are* (1) *that the upper limit of* $s(x)$ *in* (a, b) *is finite, and* (2) *that corresponding to any arbitrarily small positive number* σ, *and to any positive integer* n, *a finite number of intervals whose sum is arbitrarily small can be excluded from* (a, b) *so that, in the remainder of* (a, b), $|R_{n+p}(x)| < \sigma$, *for every* x, *where* p *has one of a finite number of values which depend on* x, *but are such that the same* p *is applicable to all points* x *in a certain continuous interval.*

7. The conditions which have been found that $s(x)$ may be an integrable function are expressed by saying that $|s(x)|$ must have a finite upper limit, and that there must be a certain mode of convergence of the series, which Arzelà has denominated *convergenza uniforme a tratti in generale*, "uniform convergence by intervals in general." This mode of convergence is distinguished from uniform convergence by intervals, in that a finite number of intervals of arbitrarily small sum are excluded from the domain. As regards the proof which Arzelà has given of the theorem which has been just obtained, it may be remarked that it appears to depend upon the assumption that not only the points of (a, b) at which $s(x)$ has a saltus $\geqslant \epsilon$ form a set of zero content, but also that the set of points at which any of the functions $s_n(x), s_{n+1}(x), s_{n+m}(x), \ldots$ have a saltus $\geqslant \epsilon$ can be enclosed in a finite number of intervals whose sum is arbitrarily small; this appears from the statement on p. 708 and in the first paragraph of p. 709 (*loc. cit.*). This is, however, not necessarily the case, and has not been assumed to be true in the present paper. The set

of points for which any one $s_{n+m}(x)$ has a saltus $\geqq \epsilon$ is a closed set, but it does not follow that the set of points at which any of the $s_{n+m}(x)$ has a saltus $\geqq \epsilon$ can be enclosed in a finite number of intervals of arbitrarily small sum. The sets G_0, G_1, ..., G_m, ..., which correspond to $s_n(x)$, $s_{n+1}(x)$, ..., $s_{n+m}(x)$, ..., being each closed, we may consider the sequence of sets G_0, G_0+G_1, $G_0+G_1+G_2$, ..., $G_0+G_1+\dots+G_m$, ...; each of these sets is closed and is contained in the succeeding set; hence, by a theorem given by Prof. Osgood,* the limiting set, *provided it is closed*, has for content the limit of the content of $G_0+G_1+\dots+G_m$; in our case this is zero. In the general case it is quite true that all the points at which any of the $s_{n+m}(x)$ has a saltus $\geqq \epsilon$ can be enclosed in finite intervals whose sum is arbitrarily small, but not in a finite number of such intervals. It would thus appear that the proof of the theorem given by Arzelà cannot be accepted as valid, at all events in the form in which it has been given by him.

* *American Journal*, Vol. XIX., p. 178; see also Schönflies, *Bericht über die Mengenlehre*, p. 91; the proof given by Schönflies is, however, invalid.