Central-Difference Formulæ. By W. F. SHEPPARD, M.A., LL.M.

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PART I.—*Introductory.*

1. In the preceding paper I have found it necessary to make use of certain formulæ involving central differences. The formulæ are not new, but, in the absence of any complete text-book* on finite differences, they are not very familiar. The value of the method in practical work is well established, and it has received attention in various text-books dealing with special subjects.† But in almost all these cases the investigation of the formulæ is limited to the particular purposes for which they are required, and a discussion of their general relations would be out of place. The formulæ are therefore obtained, usually to a very few terms, from the corresponding formulæ for advancing or receding differences;‡ the notation of the latter is adopted without appreciable alteration; and very little attention is given to the meaning of the operators which appear in the formulæ. The only paper I have seen in which these operators are treated independently is one by P. A. Hansen,§ published in 1865; but even in that paper there is no clear statement as to the algebraical identities which lead up to the formulæ. In the absence of such a statement, the manipulation of symbols of operation is apt to lead to distrust of the results obtained.

The object of the present paper is to give the formulæ which are

* In Boole and Moulton's *Calculus of Finite Differences*, some central-difference formulæ are given as "examples," but the central-difference notation is condemned (see note below). In A. Murkov's *Differenzen-Rechnung* (Leipzig, 1896), which I believe to be the latest text-book, central differences are entirely ignored.

† See, e.g., *Text-Book of the Institute of Actuaries*, Part II., p. 447; Chauvenet, *Spherical and Practical Astronomy*, Vol. I., ch. ii., pp. 81-91; *Official Text-Book of Gunnery*, pp. 213 seqq. For some historical references, see C. W. Merrifield, "On Quadratures and Interpolation," *British Association Report*, 1880, pp. 352 seqq.

‡ Prof. Everett, in a recent paper, has combined advancing and receding differences in such a way as to give formulæ practically identical with the central-difference formulæ (*Quarterly Journal of Mathematics*, No. 124, 1900, p. 357). I ought to add that I am indebted to Prof. Everett for some criticisms and suggestions with regard to this and the preceding paper.

§ "Relationen zwischen Summen und Differenzen," *Abhandlungen der kön. sächs. Gesellschaft* (Leipzig), Vol. XI., 1865, pp. 505-583 (Vol. VII. of *Abhandlungen der math.-phys. Classe*). The paper is referred to in Boole and Moulton's treatise (2nd edition, 1872, p. 55), but the notation is banned as "unscientific."

of the most use for practical purposes, treating the theory of central differences as a distinct subject from the theory of advancing differences. The formulæ depend on certain algebraical expansions; for convenience of reference, these are collected in the following sections of Part I. Part II. of the paper (pp. 458-473) deals with central differences, and Part III. (pp. 473-488) with central sums, and their relation to integration.

The notation adopted* is practically identical with that of Hansen, except that, to prevent misunderstanding, I use δ and σ instead of Δ and Σ , the latter symbols being confined to the theory of advancing differences. I have also introduced a new operator, denoted by μ .

2. *Hyperbolic Functions*.—The following are familiar expansions:—

$$\begin{aligned}\cosh \phi &= \frac{1}{2} (e^{\phi} + e^{-\phi}) \\ &= 1 + \phi^2/2! + \phi^4/4! + \phi^6/6! + \dots,\end{aligned}\quad (1)$$

$$\begin{aligned}\sinh \phi &= \frac{1}{2} (e^{\phi} - e^{-\phi}) \\ &= \phi/1! + \phi^3/3! + \phi^5/5! + \phi^7/7! + \dots,\end{aligned}\quad (2)$$

$$\tanh \phi = T_1 \phi/1! - T_2 \phi^3/3! + T_3 \phi^5/5! - T_4 \phi^7/7! + \dots, \quad (3).$$

$$\phi \coth \phi = 1 + 2^2 B_1 \phi^2/2! - 2^4 B_2 \phi^4/4! + 2^6 B_3 \phi^6/6! - \dots, \quad (4)$$

$$\phi \operatorname{cosech} \phi = 1 - P_1 \phi^2/2! + P_2 \phi^4/4! - P_3 \phi^6/6! + \dots, \quad (5)$$

where $B_1, B_2, B_3, \dots$ are Bernoulli's numbers, and

$$T_r = 2^{2r} (2^{2r} - 1) B_r / 2r, \quad (6)$$

$$P_r = (2^{2r} - 2) B_r. \quad (7)$$

The series (1) and (2) are absolutely convergent. The radii of convergence of (3), (4), and (5) are $\frac{1}{2}\pi$, π , and π respectively.

Multiplying (4) and (5) by $-1/\phi$, and differentiating, we have

$$\operatorname{cosech}^2 \phi = \phi^{-2} - 2^2 B_1/2! + 2^4 B_2 \cdot 3\phi^2/4! - 2^6 B_3 \cdot 5\phi^4/6! + \dots, \quad (8)$$

$$\operatorname{cosech} \phi \coth \phi = \phi^{-2} + P_1/2! - P_2 \cdot 3\phi^2/4! + P_3 \cdot 5\phi^4/6! - \dots \quad (9)$$

The radius of convergence of each of these series is π .

* The notation has already been explained, in the previous paper (pp. 423-448); but the explanations are repeated, in order to make the present paper complete in itself.

3. *Powers of Hyperbolic Sine.*—From (2) we get $\sinh^n \phi$ in ascending powers of ϕ , n being a positive integer. If we write

$$\sinh^n \phi = {}_n A_n \phi^n + {}_n A_{n+2} \phi^{n+2} / (n+1)(n+2) + {}_n A_{n+4} \phi^{n+4} / (n+1) \dots (n+4) + \dots, \quad (10)$$

$$\text{where } {}_n A_n = 1, \quad (11)$$

we have, by differentiation,

$$\sinh^{n-1} \phi \cosh \phi = {}_n A_n \phi^{n-1} + {}_n A_{n+2} \phi^{n+1} / n(n+1) + {}_n A_{n+4} \phi^{n+3} / n \dots (n+3) + \dots \quad (12)$$

Differentiating again, substituting for $\cosh^2 \phi$ in terms of $\sinh^2 \phi$, and replacing $\sinh^{n-2} \phi$ by the series given by (10), we find

$${}_n A_{n+2r} = n^2 {}_n A_{n+2r-2} + {}_{n-2} A_{n+2r-2}, \quad (13)$$

by means of which the successive coefficients may be calculated.

The series (10) is absolutely convergent for all values of n . For we have

$$\begin{aligned} \sinh^{2m-1} \phi &= \left\{ \frac{1}{2} (e^\phi - e^{-\phi}) \right\}^{2m-1} \\ &= \left\{ \sinh (2m-1) \phi - (2m-1) \sinh (2m-3) \phi \right. \\ &\quad \left. + \frac{(2m-1)(2m-2)}{2!} \sinh (2m-5) \phi - \dots \right. \\ &\quad \left. \dots + (-)^{m-1} \frac{(2m-1)!}{(m-1)! m!} \sinh \phi \right\} / 2^{2m-2}, \quad (14) \end{aligned}$$

$$\begin{aligned} \sinh^{2m} \phi &= \left\{ \frac{1}{2} (e^\phi - e^{-\phi}) \right\}^{2m} \\ &= \left\{ \cosh 2m\phi - 2m \cosh (2m-2) \phi \right. \\ &\quad \left. + \frac{2m(2m-1)}{2!} \cosh (2m-4) \phi - \dots \right. \\ &\quad \left. \dots + (-)^{m-1} \frac{(2m)!}{(m-1)!(m+1)!} \cosh 2\phi + (-)^m \frac{1}{2} \frac{(2m)!}{m! m!} \right\} / 2^{2m-1}, \quad (15) \end{aligned}$$

so that $\sinh^n \phi$ consists of the sum of a finite number of series, each of which is absolutely convergent. Similarly, $\sinh^{n-1} \phi \cosh \phi$ might be expressed in terms of $\sinh n\phi$, $\sinh (n-2) \phi$, ..., or of $\cosh n\phi$, $\cosh (n-2) \phi$, ..., so that the series (12) is absolutely convergent.

From (14) and (15), by comparison with (10), we have

$$\begin{aligned} {}_{2m-1}A_{2p-1} = & \left\{ (2m-1)^{2p-1} - (2m-1)(2m-3)^{2p-1} \right. \\ & + \frac{(2m-1)(2m-2)}{2!} (2m-5)^{2p-1} \\ & \left. \dots + (-)^{m-1} \frac{(2m-1)!}{(m-1)! m!} 1^{2p-1} \right\} / \{2^{2m-2} (2m-1)!\}, \quad (16) \end{aligned}$$

$$\begin{aligned} {}_{2m}A_{2p} = & \left\{ (2m)^{2p} - 2m(2m-2)^{2p} + \frac{2m(2m-1)}{2!} (2m-4)^{2p} - \dots \right. \\ & \left. \dots + (-)^{m-1} \frac{(2m)!}{(m-1)!(m+1)!} 2^{2p} \right\} / \{2^{2m-1} (2m)!\}, \quad (17) \end{aligned}$$

Both of which may be included in the formula

$${}_nA_q = \left[\frac{1}{2} (E - E^{-1}) \right]^n / n! \Big|_0, \quad (18)$$

where, as usual, $E^r f(0) = f(r).$ (19)

4. *Functions of Multiple Argument.*—By expanding $\cosh n\phi$ and $\sinh n\phi$ (where n has any value, integral or fractional) in powers of $n\phi$, by (1) or (2), and then taking out the terms corresponding to the successive even or odd powers of $\sinh \phi$, as given by (10), we obtain $\cosh n\phi$ and $\sinh n\phi$ in ascending powers of $\sinh \phi$, the series commencing with 1 and $n \sinh \phi$ respectively. To find the general formula, let us write

$$\cosh n\phi = a_0 + a_2 \sinh^2 \phi / 2! + a_4 \sinh^4 \phi / 4! + \dots, \quad (20)$$

$$\sinh n\phi = n \{ {}_1B_1 \sinh \phi / 1! + {}_1B_3 \sinh^3 \phi / 3! + {}_1B_5 \sinh^5 \phi / 5! + \dots \}, \quad (21)$$

where $a_0 = 1, \quad {}_1B_1 = 1.$ (22)

Differentiating (20) twice, writing $1 + \sinh^2 \phi$ for $\cosh^2 \phi$, and comparing with (20), we find

$$a_{2r+2} = \{ n^2 - (2r)^2 \} a_{2r}, \quad (23)$$

and similarly, from (21),

$${}_1B_{2r+1} = \{ n^2 - (2r-1)^2 \} {}_1B_{2r-1}. \quad (24)$$

$$\text{Hence } \cosh n\phi = 1 + \frac{n^2}{2!} \sinh^2 \phi + \frac{n^2(n^2-2^2)}{4!} \sinh^4 \phi + \dots, \quad (25)$$

$$\begin{aligned} \sinh n\phi = & \frac{n}{1!} \sinh \phi + \frac{n(n^2-1^2)}{3!} \sinh^3 \phi \\ & + \frac{n(n^2-1^2)(n^2-3^2)}{5!} \sinh^5 \phi + \dots \end{aligned} \quad (26)$$

If n is an even integer the series (25) terminates, and if n is an odd integer the series (26) terminates. In other cases the terms in each series ultimately become alternately positive and negative, each coefficient being numerically less than the preceding. The series are therefore convergent so long as $\sinh^2 \phi \leq 1$.*

From (25) and (26), by differentiation,

$$\begin{aligned} \sinh n\phi = & \left\{ \frac{n}{1!} \sinh \phi + \frac{n(n^2-2^2)}{3!} \sinh^3 \phi \right. \\ & \left. + \frac{n(n^2-2^2)(n^2-4^2)}{5!} \sinh^5 \phi + \dots \right\} \cosh \phi, \end{aligned} \quad (27)$$

$$\cosh n\phi = \left\{ 1 + \frac{n^2-1^2}{2!} \sinh^2 \phi + \frac{(n^2-1^2)(n^2-3^2)}{4!} \sinh^4 \phi + \dots \right\} \cosh \phi, \quad (28)$$

which, like (25) and (26), are convergent so long as $\sinh^2 \phi \leq 1$.

From (26) we can obtain $\sinh^m n\phi$ in ascending powers of $\sinh \phi$, where m is an integer. Writing

$$\sinh^m n\phi/m! = n^m \{ {}_mB_m \sinh^m \phi/m! + {}_mB_{m+2} \sinh^{m+2} \phi/(m+2)! + \dots \}, \quad (29)$$

$$\text{where} \quad {}_mB_m = 1, \quad (30)$$

we have, by differentiation,

$$\begin{aligned} & \sinh^{m-1} n\phi \cosh n\phi/(m-1)! \\ &= n^{m-1} \{ {}_mB_m \sinh^{m-1} \phi/(m-1)! + {}_mB_{m+2} \sinh^{m+1} \phi/(m+1)! + \dots \} \cosh \phi. \end{aligned} \quad (31)$$

Differentiating again, substituting as before, and equating coefficients,

$${}_mB_{m+2r+2} = \{ m^2 n^2 - (m+2r)^2 \} {}_mB_{m+2r} + {}_{m-2}B_{m+2r}, \quad (32)$$

which gives the successive values of ${}_mB_{m+2r}$.

* As to the corresponding series for trigonometrical functions, see Chrystal, *Algebra*, Pt. II., pp. 304, 305; Hobson, *Trigonometry*, p. 265.

Thus we have

$$\begin{aligned}
 \sinh^2 n\phi &= \frac{n^2}{2!} 2 \sinh^2 \phi + \frac{n^2(n^2-1^2)}{4!} 2^3 \sinh^4 \phi \\
 &\quad + \frac{n^2(n^2-1^2)(n^2-2^2)}{6!} 2^5 \sinh^6 \phi + \dots \\
 \sinh^3 n\phi &= n^3 \sinh^3 \phi + \frac{n^3(n^2-1)}{2} \sinh^5 \phi + \frac{n^3(n^2-1)(13n^2-37)}{120} \sinh^7 \phi \\
 &\quad + \frac{n^3(n^2-1)(205n^4-1706n^2+3229)}{15120} \sinh^9 \phi - \dots \\
 \sinh^4 n\phi &= n^4 \sinh^4 \phi + \frac{2n^4(n^2-1)}{3} \sinh^6 \phi \\
 &\quad + \frac{n^4(n^2-1)(3n^2-7)}{15} \sinh^8 \phi + \dots \\
 \sinh^5 n\phi &= n^5 \sinh^5 \phi + \frac{5n^5(n^2-1)}{6} \sinh^7 \phi \\
 &\quad + \frac{n^5(n^2-1)(23n^2-47)}{72} \sinh^9 \phi + \dots \\
 \sinh^6 n\phi &= n^6 \sinh^6 \phi + n^6(n^2-1) \sinh^8 \phi + \dots \\
 \sinh^7 n\phi &= n^7 \sinh^7 \phi + \frac{7n^7(n^2-1)}{6} \sinh^9 \phi + \dots \\
 \sinh^8 n\phi &= n^8 \sinh^8 \phi + \dots \\
 \sinh^9 n\phi &= n^9 \sinh^9 \phi + \dots \\
 &\quad \vdots \quad \quad \quad \vdots
 \end{aligned}
 \tag{33}$$

with the corresponding series obtained by differentiation of these with regard to ϕ .

All the above series are convergent so long as $\sinh^2 \phi \leq 1$.

5. *Inverse Expansions (positive powers of argument).*—If we write $n = m, m+2, m+4, \dots$, in (10), and eliminate $\phi^{m+2}, \phi^{m+4}, \dots$, we obtain ϕ^m in ascending powers of $\sinh \phi$. Similarly, from (12), we could obtain $\phi^m/\cosh \phi$. The coefficients in these expansions are most simply found from formulæ (25)–(28), by expanding $\cosh n\phi$ and $\sinh n\phi$ in powers of $n\phi$, and equating coefficients of powers of n .

Thus we find, for the first two pairs of formulæ,

$$1 = \left\{ 1 - \frac{1^2}{1 \cdot 2} \sinh^2 \phi + \frac{1^2 \cdot 3^2}{1 \cdot 2 \cdot 3 \cdot 4} \sinh^4 \phi - \dots \right\} \cosh \phi, \quad (34)$$

$$\phi = \sinh \phi - \frac{1^2}{2 \cdot 3} \sinh^3 \phi + \frac{1^2 \cdot 3^3}{2 \cdot 3 \cdot 4 \cdot 5} \sinh^5 \phi - \dots, \quad (35)$$

$$\phi = \left\{ \sinh \phi - \frac{2^2}{2 \cdot 3} \sinh^3 \phi + \frac{2^2 \cdot 4^2}{2 \cdot 3 \cdot 4 \cdot 5} \sinh^5 \phi - \dots \right\} \cosh \phi, \quad (36)$$

$$\phi^2 = \sinh^2 \phi - \frac{2^2}{3 \cdot 4} \sinh^4 \phi + \frac{2^2 \cdot 4^2}{3 \cdot 4 \cdot 5 \cdot 6} \sinh^6 \phi - \dots \quad (37)$$

For the general formula, let us write

$$\begin{aligned} \phi^m = {}_m H_m \sinh^m \phi - {}_m H_{m+2} \sinh^{m+2} \phi / (m+1)(m+2) \\ + {}_m H_{m+4} \sinh^{m+4} \phi / (m+1) \dots (m+4) - \dots \end{aligned} \quad (38)$$

Then we have also, by differentiation,

$$\begin{aligned} \phi^{m-1} = \{ {}_m H_m \sinh^{m-1} \phi - {}_m H_{m+2} \sinh^{m+1} \phi / m(m+1) \\ + {}_m H_{m+4} \sinh^{m+3} \phi / m \dots (m+3) - \dots \} \cosh \phi. \end{aligned} \quad (39)$$

The values of the coefficients are given by

$${}_m H_m = 1, \quad (40)$$

$$\begin{aligned} {}_{2n-1} H_{2n+2r-1} = \text{sum of products of } 1^2, 3^2, \dots (2n+2r-3)^2, \\ \text{taken } r \text{ at a time,} \end{aligned} \quad (41)$$

$$\begin{aligned} {}_{2n} H_{2n+2r} = \text{sum of products of } 2^2, 4^2, \dots (2n+2r-2)^2, \\ \text{taken } r \text{ at a time,} \end{aligned} \quad (42)$$

all of which are included in the formula

$$\exp \left\{ \frac{E^{p-1} - E^{-(p-1)}}{E - E^{-1}} \log \sqrt{\theta^2 + 0^2} \right\} = \Sigma {}_m H_p \theta^{m-1}. \quad (43)$$

To obtain the relation between the successive coefficients in (38), we have

$$E^{p-1} - E^{-(p-1)} = \{ E^{p-2} + E^{-(p-2)} \} (E - E^{-1}) + E^{p-3} - E^{-(p-3)}$$

$$\text{and} \quad \{ E^{p-2} + E^{-(p-2)} \} \log \sqrt{\theta^2 + 0^2} = \log \{ \theta^2 + (p-2)^2 \}.$$

Hence, by (43),

$$\Sigma {}_m H_p \theta^{m-1} = \{ \theta^2 + (p-2)^2 \} \Sigma {}_m H_{p-2} \theta^{m-1}.$$

Equating coefficients, we find

$${}_mH_p = (p-2)^2 {}_mH_{p-2} + {}_{m-2}H_{p-2}, \quad (44)$$

which gives the successive values of ${}_mH_p$.

If all the terms in the series (25)–(28) are taken positively, the resulting series are convergent so long as $\sinh^2 \phi < 1$. The series (38) and (39) are therefore also convergent for this range of values.

6. *Inverse Expansions (negative powers of argument).*—From (38) and (39), we have

$$\sinh^m \phi = \left[\{ 1 - {}_mH_{m+2} \sinh^2 \phi / (m+1)(m+2) \right. \\ \left. + {}_mH_{m+4} \sinh^4 \phi / (m+1) \dots (m+4) - \dots \}^{-1} \right] \phi^m, \quad (45)$$

$$\sinh^{m-1} \phi = \left[(1 + \sinh^2 \phi)^{-1} \{ 1 - {}_mH_{m+2} \sinh^2 \phi / m(m+1) \right. \\ \left. + {}_mH_{m+4} \sinh^4 \phi / m \dots (m+3) - \dots \}^{-1} \right] \phi^m \cosh \phi. \quad (46)$$

The expressions in square brackets can be expanded in ascending powers of $\sinh^2 \phi$, but there is no simple relation between the successive coefficients in the resulting series. The most important cases are:—

$$\sinh \phi = \phi / \left(1 - \frac{1^2}{2 \cdot 3} \sinh^2 \phi + \frac{1^2 \cdot 3^2}{2 \cdot 3 \cdot 4 \cdot 5} \sinh^4 \phi \right. \\ \left. - \frac{1^2 \cdot 3^2 \cdot 5^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \sinh^6 \phi + \dots \right) \\ = \left(1 + \frac{1}{6} \sinh^2 \phi - \frac{17}{360} \sinh^4 \phi + \frac{367}{15120} \sinh^6 \phi \right. \\ \left. - \frac{27859}{1814400} \sinh^8 \phi + \dots \right) \phi, \quad (47)$$

$$\sinh^2 \phi = \phi^2 / \left(1 - \frac{2^2}{3 \cdot 4} \sinh^2 \phi + \frac{2^2 \cdot 4^2}{3 \cdot 4 \cdot 5 \cdot 6} \sinh^4 \phi \right. \\ \left. - \frac{2^2 \cdot 4^2 \cdot 6^2}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \sinh^6 \phi + \dots \right) \\ = \left(1 + \frac{1}{3} \sinh^2 \phi - \frac{1}{15} \sinh^4 \phi + \frac{31}{945} \sinh^6 \phi \right. \\ \left. - \frac{289}{14175} \sinh^8 \phi + \dots \right) \phi^2, \quad (48)$$

$$\begin{aligned} \sinh \phi &= \phi \cosh \phi / \left(1 + \frac{1}{3} \sinh^2 \phi - \frac{12}{3 \cdot 5} \sinh^4 \phi + \frac{2 \cdot 4}{3 \cdot 5 \cdot 7} \sinh^6 \phi - \dots \right) \\ &= \left(1 - \frac{1}{3} \sinh^2 \phi + \frac{11}{45} \sinh^4 \phi - \frac{191}{945} \sinh^6 \phi \right. \\ &\quad \left. + \frac{2497}{14175} \sinh^8 \phi - \dots \right) \cosh \phi \cdot \phi, \quad (49) \end{aligned}$$

$$\begin{aligned} \sinh^2 \phi &= \phi^2 \cosh \phi / \left\{ 1 + \frac{1}{4} \left(\frac{1}{1^2} - \frac{1}{3} \right) \sinh^2 \phi \right. \\ &\quad \left. - \frac{1 \cdot 3}{4 \cdot 6} \left(\frac{1}{1^2} + \frac{1}{3^2} - \frac{1}{5} \right) \sinh^4 \phi \right. \\ &\quad \left. + \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7} \right) \sinh^6 \phi - \dots \right\} \\ &= \left(1 - \frac{1}{6} \sinh^2 \phi + \frac{17}{120} \sinh^4 \phi - \frac{367}{3024} \sinh^6 \phi \right. \\ &\quad \left. + \frac{27859}{259200} \sinh^8 \phi - \dots \right) \cosh \phi \cdot \phi^2. \quad (50) \end{aligned}$$

I do not know of any method by which the convergence of the general series given by (45) and (46) may be established; but it is not difficult to show* that the series (47)–(50) are convergent so

* Let

$$f(x) \equiv 1 + a_1 x + a_2 x^2 + \dots$$

be a series in which every coefficient is positive, and the ratio of each coefficient to the preceding one continually increases, but is always less than unity. Then $f(x)$ is convergent so long as $1 > x > -1$. Now, let

$$1/f(x) = 1 - b_1 x - b_2 x^2 - \dots$$

Then, if we write

$$\lambda_r = a_r / a_{r-1},$$

it is easily shown that

$$b_n = (\lambda_n - \lambda_1) b_{n-1} + a_1 (\lambda_n - \lambda_2) b_{n-2} + a_2 (\lambda_n - \lambda_3) b_{n-3} + \dots + a_{n-2} (\lambda_n - \lambda_{n-1}) b_1.$$

Hence, since λ_n is greater than any of the preceding ratios $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$, and b_1 is positive, it follows, by induction, that $b_2, b_3, \dots$ are all positive. Also we find

$$\begin{aligned} a_1 x + a_2 x^2 + a_3 x^3 + \dots &= (1 + a_1 x + a_2 x^2 + \dots) (b_1 x + b_2 x^2 + b_3 x^3 + \dots) \\ &= b_1 x + (b_2 + a_1 b_1) x^2 + (b_3 + a_1 b_2 + a_2 b_1) x^3 + \dots \end{aligned}$$

The series on the left-hand side is convergent so long as $1 > x > 0$, and therefore, *a fortiori*, the series $b_1 x + b_2 x^2 + b_3 x^3 + \dots$ is convergent within the same limits. It is therefore also convergent if $0 > x > -1$.

The above method applies to formulæ (47) and (48). For (49) and (50) we are dealing with a series

$$f(x) \equiv 1 + c_1 x - c_2 x^2 + c_3 x^3 - \dots,$$

in which the coefficients after the first are alternately positive and negative. If we write

$$1/f(x) = 1 - d_1 x + d_2 x^2 - d_3 x^3 + \dots,$$

long as $\sinh^2 \phi < 1$. The question of the convergence of the series is not, however, an important one for our present purpose. They are used for determining finite-difference formulæ; and the validity of these formulæ depends on the *initial* convergence of the numerical series derived from them.

PART II.—Central-Difference Formulæ.

7. *Definitions.*—We consider that we are dealing with a function

$$u \equiv f(x),$$

whose values are tabulated by equal increments h in x . Taking x_0 to denote any value of x which appears in the table, and writing

$$u_0 = f(x_0),$$

$$x_n = x_0 + nh,$$

$$u_n = f(x_n),$$

where n is not necessarily integral, the table showing the values of u and of its differences will take this form:—

x	u	1st diff.	2nd diff.	3rd diff.	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{-2}	u_{-2}	$u_{-1} - u_{-2}$	$u_{-1} - 2u_{-2} + u_{-3}$	$u_0 - 3u_{-1} + 3u_{-2} - u_{-3}$	...
x_{-1}	u_{-1}	$u_0 - u_{-1}$	$u_0 - 2u_{-1} + u_{-2}$	$u_1 - 3u_0 + 3u_{-1} - u_{-2}$	...
x_0	u_0	$u_1 - u_0$	$u_1 - 2u_0 + u_{-1}$	$u_2 - 3u_1 + 3u_0 - u_{-1}$	...
x_1	u_1	$u_2 - u_1$	$u_2 - 2u_1 + u_0$	$u_3 - 3u_2 + 3u_1 - u_0$	...
x_2	u_2		$u_3 - 2u_2 + u_1$		...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

it may be shown that $d_1, d_2, d_3, \dots$ are all positive. The coefficients in the latter series [for (49) and (50)] are connected with those in the former series [for (47) and (48)] by a relation of the form

$$1 + d_1x + d_2x^2 + \dots = (1 - b_1x - b_2x^2 - \dots)(1 - x)^{-1}$$

$$= 1 + \left(\frac{1}{2} - b_1\right)x + \left(\frac{1 \cdot 3}{2 \cdot 4} - \frac{1}{2}b_1 - b_2\right)x^2 + \dots$$

The series $1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \dots$ is convergent so long as $1 > x > 0$; and therefore, the coefficients $d_1, d_2, \dots$ having been shown to be positive, the series $1 + d_1x + d_2x^2 + \dots$ is convergent within the same limits. It is therefore also convergent so long as $0 > x > -1$.

The quantities shown in this table are supposed to be the exact values of u and its differences, not the approximate values as they would appear in a numerical table. Each quantity in the second column is a definite function of a definite value of x , these values increasing successively by h . Each quantity in the third column is also a function of a value of x , the functions being all of the same form, and the values of x increasing by h ; and similarly for the fourth and other columns. But in any one column the form of the function will depend on the successive values of x of which the successive differences are taken to be functions; and the assignment of these is quite arbitrary. We might, for instance, regard $u_1 - u_0$ as a function of x_{-2} ; but, if we do this, $u_2 - u_1$ will be the same function of x_{-1} , and so on.

In the ordinary advancing-difference notation, $f(x+h) - f(x)$ is written $\Delta f(x)$; i.e., it is (until the Δ is separated from the $f(\quad)$, and treated as an operator) regarded as a function of x . But this is only the case if the table is read downwards; if it is read upwards, the difference $f(x+h) - f(x)$ is written $-\Delta f(x+h)$, i.e., it is regarded as a function of $x+h$. This might be considered as justified by the change of sign. But the second difference $f(x+h) - 2f(x) + f(x-h)$ has the same sign in whichever direction we read the table; it is, however, regarded as a function of $x-h$ if we read the table one way, and as a function of $x+h$ if we read it the other.

It seems more natural to regard $f(x+h) + f(x)$ as a function of the value of x about which $f(x)$ and $f(x+h)$ are symmetrically placed. We therefore write

$$f(x+h) - f(x) = \delta f(x + \tfrac{1}{2}h),$$

or, for the definition of $\delta f(x)$,

$$f(x + \tfrac{1}{2}h) - f(x - \tfrac{1}{2}h) = \delta f(x). \quad (51)$$

Similarly we have

$$\begin{aligned} \delta \delta f(x) &= \delta f(x + \tfrac{1}{2}h) - \delta f(x - \tfrac{1}{2}h) \\ &= f(x+h) - 2f(x) + f(x-h), \end{aligned} \quad (52)$$

and generally, denoting the existence of n successive δ 's by δ^n ,

$$\delta^n f(x) = f(x + \tfrac{1}{2}nh) - n f(x + \tfrac{1}{2}nh - h) + \dots + (-)^n f(x - \tfrac{1}{2}nh), \quad (53)$$

the coefficients being those of the binomial expansion. Our table of differences then becomes

x	u	1st diff.	2nd diff.	3rd diff.	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{-2}	u_{-2}		$\delta^2 u_{-2}$		...
		$\delta u_{-\frac{3}{2}}$		$\delta^3 u_{-\frac{3}{2}}$	
x_{-1}	u_{-1}		$\delta^2 u_{-1}$		...
		$\delta u_{-\frac{1}{2}}$		$\delta^3 u_{-\frac{1}{2}}$	
x_0	u_0		$\delta^2 u_0$		...
		$\delta u_{\frac{1}{2}}$		$\delta^3 u_{\frac{1}{2}}$	
x_1	u_1		$\delta^2 u_1$		...
		$\delta u_{\frac{3}{2}}$		$\delta^3 u_{\frac{3}{2}}$	
x_2	u_2		$\delta^2 u_2$		...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

where it is to be observed that $\delta^n u_r$ is merely an abbreviation for $\delta\delta\delta \dots (n \text{ times}) f(x+rh)$.

Again, let us write

$$\mu f(x) = \frac{1}{2} \{f(x + \frac{1}{2}h) + f(x - \frac{1}{2}h)\}. \quad (54)$$

Then we shall have

$$\begin{aligned} \mu\mu f(x) &= \frac{1}{2} \{\mu f(x + \frac{1}{2}h) + \mu f(x - \frac{1}{2}h)\} \\ &= \frac{1}{4} \{f(x+h) + 2f(x) + f(x-h)\}, \end{aligned} \quad (55)$$

and generally, denoting the existence of n successive μ 's by μ^n ,

$$\mu^n f(x) = \{f(x + \frac{1}{2}nh) + nf(x + \frac{1}{2}nh - h) + \dots + f(x - \frac{1}{2}nh)\}/2^n, \quad (56)$$

the coefficients inside the brackets being the same as in (53), but all positive.

If in each column of differences of u , of an odd order, we take the mean of each pair of consecutive differences, and insert it, in brackets, between the two, we get a table of this form:—

x	u	1st diff.	2nd diff.	3rd diff.	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{-1}	u_{-1}	$(\mu \delta u_{-1})$	$\delta^2 u_{-1}$	$(\mu \delta^3 u_{-1})$	...
		$\delta u_{-\frac{1}{2}}$		$\delta^3 u_{-\frac{1}{2}}$	
x_0	u_0	$(\mu \delta u_0)$	$\delta^2 u_0$	$(\mu \delta^3 u_0)$	...
		$\delta u_{\frac{1}{2}}$		$\delta^3 u_{\frac{1}{2}}$	
x_1	u_1	$(\mu \delta u_1)$	$\delta^2 u_1$	$(\mu \delta^3 u_1)$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

where $\mu \delta^n u_r$ denotes $\frac{1}{2} (\delta^n u_{r+\frac{1}{2}} + \delta^n u_{r-\frac{1}{2}})$. The quantities following any value of u , in a horizontal line, are its *central differences*.

8. μ and δ as operators.—It is easily shown that the prefixes μ and δ , both singly and in combination, follow the ordinary laws of algebra. Thus, if we have

$$f(x) = \phi(x) + \psi(x),$$

then

$$\begin{aligned} \mu f(x) &= \frac{1}{2} \{f(x + \frac{1}{2}h) + f(x - \frac{1}{2}h)\} \\ &= \frac{1}{2} \{\phi(x + \frac{1}{2}h) + \phi(x - \frac{1}{2}h)\} \\ &\quad + \frac{1}{2} \{\psi(x + \frac{1}{2}h) + \psi(x - \frac{1}{2}h)\} \\ &= \mu \phi(x) + \mu \psi(x); \end{aligned}$$

and, similarly,

$$\begin{aligned} \mu \delta f(x) &= \frac{1}{2} \{\delta f(x + \frac{1}{2}h) + \delta f(x - \frac{1}{2}h)\} \\ &= \frac{1}{2} \{f(x + h) - f(x - h)\} \end{aligned} \quad (57)$$

$$\begin{aligned} &= \frac{1}{2} \{f(x + h) + f(x)\} - \frac{1}{2} \{f(x) + f(x - h)\} \\ &= \delta \mu f(x). \end{aligned} \quad (58)$$

The prefixes may therefore be detached from the $f(x)$, and regarded as operators, following algebraical laws. Also, if

D

denotes, as usual, the process of differentiation, so that

$$Df(x) = f'(x),$$

or

$$Du = u',$$

then it may be shown that D combines with powers of μ and of δ according to these laws.

We shall have to consider, later on, the effect of the operators μ and δ when the intervals, instead of being h , are nh , n not being necessarily positive or an integer. The operators in this case will be written

$$\mu_n, \quad \delta_n,$$

so that

$$\mu_n f(x) = \frac{1}{2} \{f(x + \frac{1}{2}nh) + f(x - \frac{1}{2}nh)\}, \quad (59)$$

$$\delta_n f(x) = f(x + \frac{1}{2}nh) - f(x - \frac{1}{2}nh). \quad (60)$$

If we take $n = -1$, so that the interval is $-h$, the effect is the same as if we read the table upwards. This obviously changes the sign of δ , and leaves μ unaltered, so that

$$\left. \begin{aligned} \mu_{-1} &= \mu \\ \delta_{-1} &= -\delta \end{aligned} \right\}. \quad (61)$$

9. *Relation between μ and δ .*—From (56) and (52),

$$\mu^2 f(x) = \frac{1}{4} \{f(x+h) + 2f(x) + f(x-h)\},$$

$$\delta^2 f(x) = f(x+h) - 2f(x) + f(x-h),$$

and therefore

$$(\mu^2 - \frac{1}{4}\delta^2) f(x) = f(x),$$

or

$$\mu^2 - \frac{1}{4}\delta^2 = 1. \quad (62)$$

This gives

$$\mu^{-1} = (1 + \frac{1}{4}\delta^2)^{-1},$$

and therefore

$$\begin{aligned} u_1 &= \mu (1 + \frac{1}{4}\delta^2)^{-1} u_1 \\ &= (\mu - \frac{1}{8}\mu\delta^2 + \frac{1}{16}\mu\delta^4 - \frac{1}{160}\mu\delta^6 + \frac{1}{320}\mu\delta^8 - \dots) u_1 \quad (63) \\ &= \frac{1}{2}(u_1 + u_0) - \frac{1}{16}(\delta^2 u_1 + \delta^2 u_0) + \frac{1}{320}(\delta^4 u_1 + \delta^4 u_0) \\ &\quad - \frac{1}{2560}(\delta^6 u_1 + \delta^6 u_0) + \frac{1}{65536}(\delta^8 u_1 + \delta^8 u_0) - \dots, \quad (63A) \end{aligned}$$

which is the formula for interpolating in the middle of an interval.

10. *Relations between μ , δ , and hD .*—By Taylor's theorem,

$$\begin{aligned} f(x + \frac{1}{2}h) &= f(x) + \frac{1}{2}hf'(x) + \frac{1}{2!}(\frac{1}{2}h)^2 f''(x) + \dots \\ &= e^{\frac{1}{2}hD} f(x), \end{aligned}$$

and, similarly,

$$f(x - \frac{1}{2}h) = e^{-\frac{1}{2}hD} f(x).$$

Hence

$$\mu f(x) = \frac{1}{2}(e^{\frac{1}{2}hD} + e^{-\frac{1}{2}hD}) f(x),$$

$$\delta f(x) = (e^{\frac{1}{2}hD} - e^{-\frac{1}{2}hD}) f(x),$$

or

$$\mu = \cosh \frac{1}{2}hD, \quad (64)$$

$$\delta = 2 \sinh \frac{1}{2}hD. \quad (65)$$

These relations may be obtained directly, in the same way as Taylor's theorem was first obtained. For suppose we write

$$\theta = 1/(2m+1),$$

where m is a positive integer. Then, by (53),

$$\left. \begin{aligned} \delta_0 &= \delta_0 \\ \delta_1^3 &= \delta_{30} - 3\delta_0 \\ &\vdots \\ \delta_0^{2m+1} &= \delta_{(2m+1)0} - (2m+1)\delta_{(2m-1)0} + \frac{(2m+1)2m}{2!}\delta_{(2m-2)0} - \dots \\ &\quad \dots + (-)^m \frac{(2m+1)!}{m!(m+1)!}\delta_0 \end{aligned} \right\} \quad (66)$$

Eliminating $\delta_{30}, \delta_{50}, \dots, \delta_{(2m-1)0}$, we have*

$$\delta_{(2m+1)0} = (2m+1) \left\{ \delta_0 + \frac{4.5 \dots (m+1)}{(m-1)!} \delta_0^3 + \frac{6.7 \dots (m+2)}{(m-2)!} \delta_0^5 + \dots \right. \\ \left. \dots + \frac{2m-2}{2!} \delta_0^{2m-3} + \delta_0^{2m-1} \right\} + \delta_0^{2m+1},$$

or, if we write

$$2m+1 = n,$$

$$\text{then } \delta = n\delta_0 + \frac{n(n^2-1^2)}{2^3 \cdot 3!} \delta_0^3 + \frac{n(n^2-1^2)(n^2-3^2)}{2^5 \cdot 5!} \delta_0^5 + \dots$$

$$\dots + \frac{n(n^2-1^2)(n^2-3^2) \dots (n^2-(n-2)^2)}{2^{n-3} (n-2)!} \delta_0^{n-2} + \delta_0^n. \quad (67)$$

Now increase n indefinitely. Then the limit of

$$n^r \delta_0^r u$$

is

$$h^r D^r u,$$

$$\text{and we get } \delta = hD + \frac{1}{2^3 \cdot 3!} h^3 D^3 + \frac{1}{2^5 \cdot 5!} h^5 D^5 + \dots \\ = 2 \sinh \frac{1}{2} hD.$$

Similarly, or by means of (62), we get (64).

This latter method has some advantages, as it shows that the relations between differencing and differentiation on the one hand,

* It is not absolutely necessary to find the coefficients in this formula. We see that (66) gives δ as a linear function of $\delta_0, \delta_0^3, \dots$, so that, increasing m without limit, we have

$$f(x + \frac{1}{2}h) - f(x - \frac{1}{2}h) = \phi(h, d/dx) f(x).$$

Hence, writing $\xi = x/n$,

$$\phi(h, nd/dx) f(x) = \phi(h, d/d\xi) f(n\xi) \\ = f\left\{n\left(\xi + \frac{1}{2}h\right)\right\} - f\left\{n\left(\xi - \frac{1}{2}h\right)\right\} \\ = f\left(x + \frac{1}{2}nh\right) - f\left(x - \frac{1}{2}nh\right) \\ = \phi(nh, d/dx) f(x).$$

The powers of d/dx in $\phi(h, d/dx)$ can therefore only occur in connexion with like powers of h , and we may write

$$\delta f(x) = \psi(hD) f(x).$$

Differentiating twice with regard to h , we find successively

$$\mu f'(x) = \psi'(hD) f'(x),$$

$$\delta f''(x) = 4\psi''(hD) f''(x).$$

This gives

$$\psi(hD) = 4\psi''(hD),$$

and

$$\mu = \psi'(hD),$$

whence, since $\mu = 1$ when $hD = 0$,

$$\psi(hD) = 2 \sinh \frac{1}{2} hD.$$

and between summing and integration on the other, are really based on the relations between various sets of differences obtained by taking various intervals in x .

11. *Differences in terms of Differential Coefficients.*—From (65) and (10),

$$\begin{aligned}\delta^n &= (2 \sinh \tfrac{1}{2}hD)^n \\ &= {}_nA_n h^n D^n + \tfrac{1}{4} {}_nA_{n+2} h^{n+2} D^{n+2} / (n+1)(n+2) + \dots, \quad (68)\end{aligned}$$

and, from (64) and (12),

$$\begin{aligned}\mu\delta^{n-1} &= (2 \sinh \tfrac{1}{2}hD)^{n-1} \cosh \tfrac{1}{2}hD \\ &= {}_nA_n h^{n-1} D^{n-1} + \tfrac{1}{4} {}_nA_{n+2} h^{n+1} D^{n+1} / n(n+1) + \dots, \quad (69)\end{aligned}$$

where ${}_nA_n, {}_nA_{n+2}, \dots$ have the values given by (11) and (13). Thus we find

$$\left. \begin{aligned}u_0 &= u_0 \\ \mu\delta u_0 &= (hD + \tfrac{1}{6}h^3D^3 + \tfrac{1}{120}h^5D^5 + \tfrac{1}{5040}h^7D^7 + \dots) u_0 \\ \delta^2 u_0 &= (h^2D^2 + \tfrac{1}{12}h^4D^4 + \tfrac{1}{360}h^6D^6 + \tfrac{1}{20160}h^8D^8 + \dots) u_0 \\ \mu\delta^3 u_0 &= (h^3D^3 + \tfrac{1}{4}h^5D^5 + \tfrac{1}{45}h^7D^7 + \dots) u_0 \\ \delta^4 u_0 &= (h^4D^4 + \tfrac{1}{6}h^6D^6 + \tfrac{1}{630}h^8D^8 + \dots) u_0 \\ \mu\delta^5 u_0 &= (h^5D^5 + \tfrac{1}{3}h^7D^7 + \dots) u_0 \\ \delta^6 u_0 &= (h^6D^6 + \tfrac{1}{4}h^8D^8 + \dots) u_0 \\ \mu\delta^7 u_0 &= (h^7D^7 + \dots) u_0 \\ \delta^8 u_0 &= (h^8D^8 + \dots) u_0 \\ &\vdots\end{aligned} \right\}, \quad (70)$$

and

$$\left. \begin{aligned}\mu u_1 &= (1 + \tfrac{1}{6}h^2D^2 + \tfrac{1}{360}h^4D^4 + \tfrac{1}{40320}h^6D^6 + \tfrac{1}{100800}h^8D^8 + \dots) u_1 \\ \delta u_1 &= (hD + \tfrac{1}{24}h^3D^3 + \tfrac{1}{720}h^5D^5 + \tfrac{1}{25200}h^7D^7 + \dots) u_1 \\ \mu\delta^2 u_1 &= (h^2D^2 + \tfrac{1}{12}h^4D^4 + \tfrac{1}{360}h^6D^6 + \tfrac{1}{20160}h^8D^8 + \dots) u_1 \\ \delta^3 u_1 &= (h^3D^3 + \tfrac{1}{4}h^5D^5 + \tfrac{1}{45}h^7D^7 + \dots) u_1 \\ \mu\delta^4 u_1 &= (h^4D^4 + \tfrac{1}{6}h^6D^6 + \tfrac{1}{630}h^8D^8 + \dots) u_1 \\ \delta^5 u_1 &= (h^5D^5 + \tfrac{1}{3}h^7D^7 + \dots) u_1 \\ \mu\delta^6 u_1 &= (h^6D^6 + \tfrac{1}{4}h^8D^8 + \dots) u_1 \\ \delta^7 u_1 &= (h^7D^7 + \dots) u_1 \\ \mu\delta^8 u_1 &= (h^8D^8 + \dots) u_1\end{aligned} \right\} \quad (71)$$

12. *Differential Coefficients in terms of Differences.*—From (38) and (39),

$$h^m D^m = {}_m H_m \delta^m - \frac{1}{4} {}_m H_{m+2} \delta^{m+2} / (m+1)(m+2) + \dots, \quad (72)$$

$$h^{m-1} D^{m-1} = {}_m H_m \mu \delta^{m-1} - \frac{1}{4} {}_m H_{m+2} \mu \delta^{m+1} / m(m+1) + \dots \quad (73)$$

Hence, by (40) and (44),

$$\left. \begin{aligned} h D u_0 &= (\mu \delta - \frac{1}{6} \mu \delta^3 + \frac{1}{36} \mu \delta^5 - \frac{1}{144} \mu \delta^7 + \dots) u_0 \\ h^2 D^2 u_0 &= (\delta^2 - \frac{1}{12} \delta^4 + \frac{1}{60} \delta^6 - \frac{1}{840} \delta^8 + \dots) u_0 \\ h^3 D^3 u_0 &= (\mu \delta^3 - \frac{1}{4} \mu \delta^5 + \frac{7}{720} \mu \delta^7 - \dots) u_0 \\ h^4 D^4 u_0 &= (\delta^4 - \frac{1}{6} \delta^6 + \frac{7}{240} \delta^8 - \dots) u_0 \\ h^5 D^5 u_0 &= (\mu \delta^5 - \frac{1}{3} \mu \delta^7 + \dots) u_0 \\ h^6 D^6 u_0 &= (\delta^6 - \frac{1}{4} \delta^8 + \dots) u_0 \\ h^7 D^7 u_0 &= (\mu \delta^7 - \dots) u_0 \\ h^8 D^8 u_0 &= (\delta^8 - \dots) u_0 \\ &\vdots \end{aligned} \right\}, \quad (74)$$

$$\left. \begin{aligned} u_1 &= (\mu - \frac{1}{8} \mu \delta^2 + \frac{3}{128} \mu \delta^4 - \frac{5}{1024} \mu \delta^6 + \frac{3}{8192} \mu \delta^8 - \dots) u_1 \\ h D u_1 &= (\delta - \frac{1}{24} \delta^3 + \frac{3}{640} \delta^5 - \frac{7}{17184} \delta^7 + \dots) u_1 \\ h^2 D^2 u_1 &= (\mu \delta^2 - \frac{5}{24} \mu \delta^4 + \frac{2}{5760} \mu \delta^6 - \frac{3}{829440} \mu \delta^8 + \dots) u_1 \\ h^3 D^3 u_1 &= (\delta^3 - \frac{1}{8} \delta^5 + \frac{3}{1280} \delta^7 - \dots) u_1 \\ h^4 D^4 u_1 &= (\mu \delta^4 - \frac{7}{24} \mu \delta^6 + \frac{7}{640} \mu \delta^8 - \dots) u_1 \\ h^5 D^5 u_1 &= (\delta^5 - \frac{5}{24} \delta^7 + \dots) u_1 \\ h^6 D^6 u_1 &= (\mu \delta^6 - \frac{3}{8} \mu \delta^8 + \dots) u_1 \\ h^7 D^7 u_1 &= (\delta^7 - \dots) u_1 \\ h^8 D^8 u_1 &= (\mu \delta^8 - \dots) u_1 \\ &\vdots \end{aligned} \right\}. \quad (75)$$

13. *Accuracy of preceding Results.*—In applying the formulæ in the last section to any actual table of u , it must be borne in mind that the table does not give the exact value of u , but only its value to the nearest multiple of some particular unit. If this unit is ρ , the difference between the tabulated value of u and the true value may be any quantity between $-\frac{1}{2}\rho$ and $+\frac{1}{2}\rho$. Denote this difference, or "error," by α , so that the tabulated values of $u_0, u_1, \dots$ are $u_0 + \alpha_0, u_1 + \alpha_1, \dots$

Then, if we denote the greatest possible error in $\delta^n u$ by

$$\pm \theta_n,$$

$$\text{it is easily seen that } \theta_n = 2^{n-1} \rho. \quad (76)$$

By substitution from (53), we have

$$\begin{aligned} (p - q\delta^2 + r\delta^4 - s\delta^6 + \dots) u_0 \\ &= pu_0 - q(u_1 - 2u_0 + u_{-1}) + r(u_2 - 4u_1 + 6u_0 + 4u_{-1} - u_{-2}) \\ &\quad - s(u_3 - 6u_2 + 15u_1 - 20u_0 + 15u_{-1} - 6u_{-2} + u_{-3}) + \dots \\ &= (p + 2q + 6r + 20s + \dots) u_0 \\ &\quad - (q + 4r + 15s + \dots)(u_1 + u_{-1}) \\ &\quad + (r + 6s + \dots)(u_2 + u_{-2}) \\ &\quad - (s + \dots)(u_3 + u_{-3}) \\ &\quad + \&c. \end{aligned}$$

Hence, if $p, q, r, s, \dots$ are all of the same sign, we have

$$\begin{aligned} &\text{greatest possible error in } (p - q\delta^2 + r\delta^4 - s\delta^6 + \dots) u \\ &= (p + 2q + 6r + 20s + \dots) \frac{1}{2}\rho \\ &\quad + (q + 4r + 15s + \dots) \rho \\ &\quad + (r + 6s + \dots) \rho \\ &\quad + (s + \dots) \rho \\ &\quad + \&c. \\ &= p \frac{1}{2}\rho + q(1 + 2 + 1) \frac{1}{2}\rho + r(1 + 4 + 6 + 4 + 1) \frac{1}{2}\rho \\ &\quad + s(1 + 6 + 15 + 20 + 15 + 6 + 1) \frac{1}{2}\rho + \dots \\ &= p \frac{1}{2}\rho + q 2\rho + r 2^3\rho + s 2^5\rho + \dots \\ &= p\theta_0 + q\theta_1 + r\theta_2 + s\theta_3 + \dots \end{aligned} \quad (77)$$

Similarly, it may be shown that, on the same condition,

$$\begin{aligned} &\text{greatest possible error in } (p\delta - q\delta^3 + r\delta^5 - s\delta^7 + \dots) u \\ &= p\theta_1 + q\theta_3 + r\theta_5 + s\theta_7 + \dots \end{aligned} \quad (78)$$

If, for instance, we were finding $h^3 D^3 u$ by the second formula in (74), and differences after the fifth were negligible, the greatest possible error would be

$$(2 + \frac{1}{15} \cdot 2^5) \rho = \frac{8}{3}\rho.$$

If we were finding it by the ordinary method, which gives

$$\begin{aligned} h^2 D^2 u &= (\Delta - \tfrac{1}{2} \Delta^2 + \tfrac{1}{3} \Delta^3 - \dots)^2 u \\ &= (\Delta^2 - \Delta^3 + \tfrac{1}{3} \Delta^4 - \tfrac{8}{9} \Delta^5 + \dots) u, \end{aligned}$$

the greatest possible error, even if we did not go beyond fifth differences, would be

$$(2 + 2^2 + \tfrac{11}{3} \cdot 2^3 + \tfrac{8}{9} \cdot 2^4) \rho = \tfrac{80}{3} \rho,$$

which is ten times as great. And, as a matter of fact, we should usually have to go to differences of a higher order with this latter formula than with the central-difference formula.

Again, let us denote the greatest possible error in $\mu \delta^n u$ by

$$\pm \phi_n.$$

Then we have

$$\begin{aligned} \delta^{2m} u_0 &= u_m - 2mu_{m-1} + \dots + (-)^{m-1} \frac{(2m)!}{(m-1)! (m+1)!} u_1 \\ &\quad + (-)^m \frac{(2m)!}{m! m!} u_0 + \dots + u_{-m}, \\ \delta^{2m} u_1 &= u_{m+1} - 2mu_m + \dots + (-)^m \frac{(2m)!}{m! m!} u_1 \\ &\quad + (-)^{m+1} \frac{(2m)!}{(m+1)! (m-1)!} u_0 + \dots + u_{-m+1}, \end{aligned}$$

whence, by addition,

$$\begin{aligned} \mu \delta^{2m} u_1 &= \tfrac{1}{2} \{ u_{m+1} - A_1 u_m + A_2 u_{m-1} - \dots + (-)^m A_m u_1 \\ &\quad + (-)^m A_m u_0 + \dots + A_2 u_{-m+2} - A_1 u_{-m+1} + u_{-m} \}, \end{aligned}$$

where $A_1, A_2, \dots, A_m$ are coefficients whose values need not be written down. We have therefore

$$\phi_{2m} = \tfrac{1}{2} (1 + A_1 + A_2 + \dots + A_m + A_m + \dots + A_2 + A_1 + 1) \tfrac{1}{2} \rho.$$

But, observing how $A_1, A_2, \dots, A_m$ are obtained, we see that

$$\begin{aligned} &1 + A_1 + A_2 + \dots + A_m + A_m + \dots + A_2 + A_1 + 1 \\ &= 1 + 2m + \dots + \frac{(2m)!}{m! m!} - \frac{(2m)!}{(m+1)! (m-1)!} - \dots - 1 \\ &\quad - 1 - \dots - \frac{(2m)!}{(m-1)! (m+1)!} + \frac{(2m)!}{m! m!} + \dots + 2m + 1 \\ &= 2 \frac{(2m)!}{m! m!}, \end{aligned}$$

and therefore

$$\phi_{2m} = \frac{(2m)!}{m! m!} \tfrac{1}{2} \rho. \quad (79)$$

In the same way it may be shown that

$$\phi_{2m+1} = \frac{(2m+1)!}{m!(m+1)!} \frac{1}{2} \rho. \quad (80)$$

Now

$$(p\mu - q\mu\delta^2 + r\mu\delta^4 - \dots) u_1 = \frac{1}{2}p(u_1 + u_0) - \frac{1}{2}q(u_2 - u_1 - u_0 + u_{-1}) \\ + \frac{1}{2}r(u_3 - 3u_2 + 2u_1 + 2u_0 - 3u_{-1} + u_{-2}) - \dots,$$

and, if we collect coefficients, we shall find that, as before, the signs attached to each u are either all positive or all negative. Hence, if $p, q, r, s, \dots$ are all of the same sign,

greatest possible error in $(p\mu - q\mu\delta^2 + r\mu\delta^4 - s\mu\delta^6 + \dots) u$

$$= p\phi_0 + q\phi_2 + r\phi_4 + s\phi_6 + \dots \quad (81)$$

$$= \left(p + q \frac{2!}{1!1!} + r \frac{4!}{2!2!} + s \frac{6!}{3!3!} + \dots \right) \frac{1}{2} \rho, \quad (81A)$$

and, similarly,

greatest possible error in $(p\mu\delta - q\mu\delta^3 + r\mu\delta^5 - s\mu\delta^7 + \dots) u$

$$= p\phi_1 + q\phi_3 + r\phi_5 + s\phi_7 + \dots \quad (82)$$

$$= \left(p + q \frac{3!}{1!2!} + r \frac{5!}{2!3!} + s \frac{7!}{3!4!} + \dots \right) \frac{1}{2} \rho. \quad (82A)$$

Thus, in finding hDu_0 by the first formula in (74), the greatest possible error, if we do not go beyond sixth differences, is

$$(1 + \frac{1}{6} \cdot 3 + \frac{1}{36} \cdot 10) \frac{1}{2} \rho = \frac{1}{3} \frac{1}{2} \rho,$$

while, if we used the ordinary formula

$$hDu = (\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \frac{1}{5}\Delta^5 - \frac{1}{6}\Delta^6 + \dots) u,$$

the greatest possible error, to the same order of differences, would be

$$(1 + \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 2^2 + \frac{1}{4} \cdot 2^3 + \frac{1}{5} \cdot 2^4 + \frac{1}{6} \cdot 2^5) \rho = \frac{308}{15} \rho,$$

which is more than fifteen times as great as the former.

14. *Sub-differences in terms of Differences.*—Suppose that, starting in both directions from x_0 , we take x by intervals of

$$nh$$

instead of by intervals of h ; where n has any value, integral or not.

Let the new values of μ , δ , be denoted by

$$\mu_n, \delta_n.$$

Then we have

$$\left. \begin{aligned} \mu_n &= \cosh \frac{1}{2}nhD \\ \delta_n &= 2 \sinh \frac{1}{2}nhD \end{aligned} \right\}, \quad (83)$$

from which we notice that
$$\left. \begin{aligned} \mu_{-n} &= \mu_n \\ \delta_{-n} &= -\delta_n \end{aligned} \right\}, \quad (84)$$

$$\left. \begin{aligned} \mu_{m \pm n} &= \mu_m \mu_n \pm \frac{1}{4} \delta_m \delta_n \\ \delta_{m \pm n} &= \mu_n \delta_m \pm \mu_m \delta_n \end{aligned} \right\}. \quad (85)$$

From (83), by (25)-(28),

$$\mu_n = 1 + \frac{n^2}{2 \cdot 4} \delta^2 + \frac{n^2 (n^2 - 2^2)}{2 \cdot 4 \cdot 6 \cdot 8} \delta^4 + \frac{n^2 (n^2 - 2^2) (n^2 - 4^2)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} \delta^6 + \dots \quad (86)$$

$$\begin{aligned} \mu_n &= \mu + \frac{n^2 - 1^2}{2 \cdot 4} \mu \delta^2 + \frac{(n^2 - 1^2) (n^2 - 3^2)}{2 \cdot 4 \cdot 6 \cdot 8} \mu \delta^4 \\ &\quad + \frac{(n^2 - 1^2) (n^2 - 3^2) (n^2 - 5^2)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} \mu \delta^6 + \dots, \end{aligned} \quad (87)$$

$$\begin{aligned} \delta_n &= n\delta + \frac{n (n^2 - 1^2)}{4 \cdot 6} \delta^3 + \frac{n (n^2 - 1^2) (n^2 - 3^2)}{4 \cdot 6 \cdot 8 \cdot 10} \delta^5 \\ &\quad + \frac{n (n^2 - 1^2) (n^2 - 3^2) (n^2 - 5^2)}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14} \delta^7 + \dots \end{aligned} \quad (88)$$

$$\begin{aligned} \delta_n &= n\mu\delta + \frac{n (n^2 - 2^2)}{4 \cdot 6} \mu \delta^3 + \frac{n (n^2 - 2^2) (n^2 - 4^2)}{4 \cdot 6 \cdot 8 \cdot 10} \mu \delta^5 \\ &\quad + \frac{n (n^2 - 2^2) (n^2 - 4^2) (n^2 - 6^2)}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14} \mu \delta^7 + \dots \end{aligned} \quad (89)$$

Also, by (29) and (31),

$$\begin{aligned} \delta_n^m &= n^m \{ {}_m B_m \delta^m + {}_m B_{m+2} \delta^{m+2} / (2m+2)(2m+4) \\ &\quad + {}_m B_{m+4} \delta^{m+4} / (2m+2) \dots (2m+8) + \dots \}, \end{aligned} \quad (90)$$

$$\begin{aligned} \mu_n \delta_n^{m-1} &= n^{m-1} \{ {}_m B_m \mu \delta^{m-1} + {}_m B_{m+2} \mu \delta^{m+1} / 2m(2m+2) \\ &\quad + {}_m B_{m+4} \mu \delta^{m+3} / 2m \dots (2m+6) + \dots \}, \end{aligned} \quad (91)$$

where ${}_m B_m$, ${}_m B_{m+2}$, ${}_m B_{m+4}$, ... have the values given by (30) and (32).

The important case is that in which

$$n = 1/N,$$

where N is an integer. If, for instance,

$$N = 10, \quad n = \cdot 1,$$

we find

$$\left. \begin{aligned} \mu_n \delta_n u_0 &= \cdot 1 (\mu \delta - \cdot 165 \mu \delta^3 + \cdot 0329175 \mu \delta^5 - \cdot 0070459125 \mu \delta^7 + \dots) u_0 \\ \delta_n^2 u_0 &= \cdot 01 (\delta^2 - \cdot 0825 \delta^4 + \cdot 0109725 \delta^6 - \cdot 001761478125 \delta^8 + \dots) u_0 \\ \mu_n \delta_n^3 u_0 &= \cdot 001 (\mu \delta^3 - \cdot 2475 \mu \delta^5 + \cdot 0575025 \mu \delta^7 - \dots) u_0 \\ \delta_n^4 u_0 &= \cdot 0001 (\delta^4 - \cdot 165 \delta^6 + \cdot 02875125 \delta^8 - \dots) u_0 \\ \mu_n \delta_n^5 u_0 &= \cdot 00001 (\mu \delta^5 - \cdot 33 \mu \delta^7 + \dots) u_0 \\ \delta_n^6 u_0 &= \cdot 000001 (\delta^6 - \cdot 2475 \delta^8 + \dots) u_0 \\ \mu_n \delta_n^7 u_0 &= \cdot 0000001 (\delta^7 - \dots) u_0 \\ \delta_n^8 u_0 &= \cdot 00000001 (\delta^8 - \dots) u_0 \\ \vdots & \quad \quad \quad \vdots \end{aligned} \right\} \quad (92)$$

If $N \equiv 1/n$ is an odd integer, it is not usually necessary to make use of $\mu_n \delta_n u_0, \mu_n \delta_n^3 u_0, \dots$. For, if we suppose a table to be constructed by intervals of nh , this table will show the following differences at the middle of the interval between x_0 and x_1 :—

x	u	δ_n	δ_n^2	δ_n^3	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$x_{\frac{1}{2}(1-n)}$	$u_{\frac{1}{2}(1-n)}$	$\delta_n u_{\frac{1}{2}}$	$\delta_n^2 u_{\frac{1}{2}(1-n)}$	$\delta_n^3 u_{\frac{1}{2}}$	...
$x_{\frac{1}{2}(1+n)}$	$u_{\frac{1}{2}(1+n)}$		$\delta_n^2 u_{\frac{1}{2}(1+n)}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Any purpose, therefore, for which we might require $\mu_n \delta_n u_0, \mu_n \delta_n^3 u_0, \dots$, can be quite as well served by $\delta_n u_{\frac{1}{2}}, \delta_n^3 u_{\frac{1}{2}}, \dots$. Suppose, for instance, that

$$N = 5, \quad n = \cdot 2.$$

Then

$$\left. \begin{aligned} \delta_n^2 u_0 &= \cdot 04 (\delta^2 - \cdot 08\delta^4 + \cdot 01056\delta^6 - \cdot 0016896\delta^8 + \dots) u_0 \\ \delta_n^4 u_0 &= \cdot 0016 (\delta^4 - \cdot 16\delta^6 + \cdot 02752\delta^8 - \dots) u_0 \\ \delta_n^6 u_0 &= \cdot 000064 (\delta^6 - \cdot 24\delta^8 + \dots) u_0 \\ \delta_n^8 u_0 &= \cdot 00000256 (\delta^8 - \dots) u_0 \\ &\vdots \end{aligned} \right\}, \quad (93)$$

$$\text{and} \quad \left. \begin{aligned} \delta_n u_1 &= \cdot 2 (\delta - \cdot 04\delta^3 + \cdot 00448\delta^5 - \cdot 0006656\delta^7 + \dots) u_1 \\ \delta_n^3 u_1 &= \cdot 008 (\delta^3 - \cdot 12\delta^5 + \cdot 01824\delta^7 - \dots) u_1 \\ \delta_n^5 u_1 &= \cdot 00032 (\delta^5 - \cdot 2\delta^7 + \dots) u_1 \\ \delta_n^7 u_1 &= \cdot 0000128 (\delta^7 - \dots) u_1 \\ &\vdots \end{aligned} \right\}. \quad (94)$$

15. *Interpolation at regular intervals.*— We have, by definition.

$$\begin{aligned} \frac{1}{2} \{f(x+nh) + f(x-nh)\} &= \mu_{2n} f(x), \\ \frac{1}{2} \{f(x+nh) - f(x-nh)\} &= \frac{1}{2} \delta_{2n} f(x); \end{aligned}$$

$$\text{and therefore} \quad f(x+nh) = (\mu_{2n} + \frac{1}{2} \delta_{2n}) f(x). \quad (95)$$

Suppose now that, the values of u being tabulated by intervals h , we require to form a new table in which the intervals are

$$h/N,$$

where N is an integer. We can do this either by interpolating on each side of the given values $x_0, x_1, x_2, \dots$, or by interpolating through the intervals x_0 to x_1, x_1 to $x_2, \dots$. For the first method we have

$$u_n = \mu_{2n} u_0 + \frac{1}{2} \delta_{2n} u_0, \quad (96)$$

where

$$n = \pm p/N,$$

and p has values ranging from 0 to $\frac{1}{2}(N+1)$ or $\frac{1}{2}N$; and for the second method

$$u_{\frac{1}{2}+n} = \mu_{2n} u_{\frac{1}{2}} + \frac{1}{2} \delta_{2n} u_{\frac{1}{2}}, \quad (97)$$

where

$$n = \pm (2p+1)/2N \text{ or } \pm p/N,$$

according as N is odd or even, and p has values ranging from 0 to $\frac{1}{2}(N-1)$ or $\frac{1}{2}N$. By (86)-(89), these give respectively

$$u_n = u_0 + \frac{n}{1!} \mu \delta u_0 + \frac{n^2}{2!} \delta^2 u_0 + \frac{n(n^2-1^2)}{3!} \mu \delta^3 u_0 + \frac{n^2(n^2-1^2)}{4!} \delta^4 u_0 + \dots, \quad (98)$$

and

$$u_{i+n} = \mu u_i + \frac{2n}{2 \cdot 1!} \delta u_i + \frac{4n^2-1^2}{2^2 \cdot 2!} \mu \delta^2 u_i + \frac{2n(4n^2-1^2)}{2^3 \cdot 3!} \delta^3 u_i \\ + \frac{(4n^2-1^2)(4n^2-3^2)}{2^4 \cdot 4!} \mu \delta^4 u_i + \dots \quad (99)$$

The usual subdivision is into fifths or tenths. For subdivision into fifths, we can write $n = \mp \frac{3}{10}$ and $n = \mp \frac{1}{10}$ in the latter formula, and we have

$$\left. \begin{aligned} u_i \\ u_i \end{aligned} \right\} &= \mu u_i - \cdot 08 \mu \delta^2 u_i + \cdot 0144 \mu \delta^4 u_i - \cdot 0029568 \mu \delta^6 u_i + \cdot 000642048 \mu \delta^8 u_i - \dots \\ &\mp \{ \cdot 3 \delta u_i - \cdot 008 \delta^3 u_i + \cdot 000864 \delta^5 u_i - \cdot 00012672 \delta^7 u_i + \dots \} \\ \left. \begin{aligned} u_i \\ u_i \end{aligned} \right\} &= \mu u_i - \cdot 12 \mu \delta^2 u_i + \cdot 0224 \mu \delta^4 u_i - \cdot 0046952 \mu \delta^6 u_i + \cdot 001018368 \mu \delta^8 u_i - \dots \\ &\mp \{ \cdot 1 \delta u_i - \cdot 004 \delta^3 u_i + \cdot 000448 \delta^5 u_i - \cdot 00006656 \delta^7 u_i + \dots \}, \quad (100)$$

the upper sign being taken for u_i and u_i , and the lower for u_i and u_i . For subdivision into tenths we might obtain similar formulæ, but it is simpler to bisect the intervals by means of (63), and then apply (100).

16. *Interpolation generally.*--The preceding formulæ involve the calculation of $\mu \delta u_0$, $\mu \delta^3 u_0$, ..., or of μu_i , $\mu \delta^2 u_i$, ..., and therefore they are not convenient when we only require u for a single value of x . Let x_0 be the nearest value of x given in the table. Then we require a formula giving u in terms of

$$u_0, \delta^2 u_0, \delta^4 u_0, \dots,$$

and either

$$\delta u_{-1}, \delta^3 u_{-1}, \delta^5 u_{-1}, \dots$$

or

$$\delta u_1, \delta^3 u_1, \delta^5 u_1, \dots,$$

according as x is between $x_0 - \frac{1}{2}h$ and x_0 , or between x_0 and $x_0 + \frac{1}{2}h$.

Suppose that x is between x_0 and $x_0 + \frac{1}{2}h$, so that we may write

$$x = x_0 + nh,$$

where n is between 0 and $\frac{1}{2}$. Then we require to express

$$u_n$$

in the form

$$f_1(\delta^2) u_0 + f_2(\delta^2) \delta u_1;$$

i.e., we require to express $\mu_{2n} + \frac{1}{2}\delta_{2n}$

in the form $f_1(\delta^2) + f_2(\delta^3 \dots \mu + \frac{1}{2}\delta)\delta$.

Now

$$\begin{aligned}\mu_{2n} + \frac{1}{2}\delta_{2n} &= \cosh nhD + \sinh nhD \\ &= \left\{ \cosh\left(n - \frac{1}{2}\right)hD + \sinh nhD \left(\cosh \frac{1}{2}hD + \sinh \frac{1}{2}hD \right) \right\} / \cosh \frac{1}{2}hD,\end{aligned}$$

and therefore

$$u_n = \cosh\left(n - \frac{1}{2}\right)hD / \cosh \frac{1}{2}hD \cdot u_0 + \sinh nhD / \cosh \frac{1}{2}hD \cdot u_1.$$

Substituting from (28) and (27),

$$\begin{aligned}u_n &= \left\{ 1 + \frac{n(n-1)}{2!}\delta^2 + \frac{(n+1)n(n-1)(n-2)}{4!}\delta^4 + \dots \right\} u_0 \\ &+ \left\{ \frac{n}{1!}\delta + \frac{(n+1)n(n-1)}{3!}\delta^3 + \frac{(n+2)(n+1)n(n-1)(n-2)}{5!}\delta^5 + \dots \right\} u_1.\end{aligned}\tag{101}$$

Similarly, when x is between $x_0 - \frac{1}{2}h$ and x_0 , we have

$$\begin{aligned}u_{-n} &= \left\{ 1 + \frac{n(n-1)}{2!}\delta^2 + \frac{(n+1)n(n-1)(n-2)}{4!}\delta^4 + \dots \right\} u_0 \\ &- \left\{ \frac{n}{1!}\delta + \frac{(n+1)n(n-1)}{3!}\delta^3 + \frac{(n+2)(n+1)n(n-1)(n-2)}{5!}\delta^5 + \dots \right\} u_1.\end{aligned}\tag{102}$$

The coefficients in (101) or (102) are appreciably smaller than the coefficients in the ordinary interpolation formula.

PART III.—Central-Sum Formulæ.

Central-Sum Notation.

17. The justification of the use of δ as an operator lies in the fact that we can regard any column of differences, such as δu or $\delta^2 u$, as a series of quantities which are the same function of successive values of x , increasing by differences h , and that we can then regard the following columns as giving the first, second, ... differences of these functions. In the same way we can regard the values of u as being themselves differences of some other function, so that the

successive values of this function are found by successive additions of the values of u . Denoting the function by

$$\sigma u \equiv \sigma f(x),$$

we have, for our definition of the symbol σ ,

$$\sigma f(x + \tfrac{1}{2}h) - \sigma f(x - \tfrac{1}{2}h) = f(x), \quad (103)$$

which may also be written

$$\delta \sigma f(x) = f(x). \quad (103A)$$

Changing x in (103) into $x-h$, $x-2h$, ..., and adding the results so obtained, we have

$$\left. \begin{aligned} \sigma f(x + \tfrac{1}{2}h) &= \dots + f(x-h) + f(x) \\ \text{and, similarly,} \\ \sigma f(x + \tfrac{3}{2}h) &= \dots + f(x-h) + f(x) + f(x+h) \\ \sigma f(x + \tfrac{5}{2}h) &= \dots + f(x-h) + f(x) + f(x+h) + f(x+2h) \\ &\vdots \end{aligned} \right\} \cdot (104)$$

Since the differences of $\phi(x)$ are the same as those of $C + \phi(x)$, where C is any quantity independent of x , the solution of (103), considered as a functional equation in $\sigma f(x)$, contains an arbitrary constant. But, if the series on the right-hand side of any one of the equations (104) is made to start from any particular value of $f(x)$, and has any constant added, the other series will start from the same value, and will contain the same constant.

It is clear from (104) that, provided the constants are properly chosen,

$$\sigma \{ \phi(x) + \psi(x) \} = \sigma \phi(x) + \sigma \psi(x),$$

so that σ , regarded as an operator, follows the laws of algebra. Also, replacing $f(x)$ by u , we have, from (103A),

$$\delta \sigma u = u,$$

while, if we apply (104) to $\delta f(x)$ instead of to $f(x)$, we have, if the constants are properly chosen,

$$\begin{aligned} \sigma \delta f(x) &= \dots + \delta f(x-h) + \delta f(x) \\ &= f(x), \end{aligned}$$

or

$$\sigma \delta u = u.$$

Thus σ combines with δ according to the laws of algebra, in the same way as if

$$\sigma = \delta^{-1}. \quad (105)$$

The process represented by σ may be repeated, so that we shall have

$$\left. \begin{aligned} \sigma^2 u_0 &= \sigma \cdot \sigma u_0 \\ &= \sigma (\dots + u_{-\frac{1}{2}} + u_{-\frac{1}{2}}) \\ &= \dots + \sigma u_{-\frac{1}{2}} + \sigma u_{-\frac{1}{2}} \\ \sigma^2 u_1 &= \dots + \sigma u_{-\frac{1}{2}} + \sigma u_{-\frac{1}{2}} + \sigma u_{\frac{1}{2}} \\ &\vdots \quad \quad \quad \vdots \end{aligned} \right\}, \quad (106)$$

and so on. Each repetition of the operation involves the introduction of an arbitrary constant, so that $\sigma^m u$ will contain m of these constants.

Again, by our original definition of μ ,

$$\mu \sigma u_0 = \frac{1}{2} (\sigma u_{\frac{1}{2}} + \sigma u_{-\frac{1}{2}}).$$

Hence, if the arbitrary constant in σu is such that

$$\left. \begin{aligned} \sigma u_{-\frac{1}{2}} &= C + u_{-2} + u_{-1} \\ \sigma u_{\frac{1}{2}} &= C + u_{-2} + u_{-1} + u_0 \\ &\vdots \quad \quad \quad \vdots \\ \sigma u_{n+\frac{1}{2}} &= C + u_{-2} + u_{-1} + u_0 + \dots + u_n \end{aligned} \right\}, \quad (107)$$

$$\text{we have} \quad \mu \sigma u_0 = C + u_{-2} + u_{-1} + \frac{1}{2} u_0. \quad (108)$$

$$\begin{aligned} \text{Also} \quad \sigma \mu u_0 &= \dots + \frac{1}{2} (u_{-3} + u_{-2}) + \frac{1}{2} (u_{-2} + u_{-1}) + \frac{1}{2} (u_{-1} + u_0) \\ &= \dots + u_{-2} + u_{-1} + \frac{1}{2} u_0. \end{aligned} \quad (109)$$

If therefore the arbitrary constant is properly chosen, we have

$$\mu \sigma u = \sigma \mu u.$$

Thus the operators μ , δ , σ can be combined according to the ordinary laws of algebra, provided attention is given to the arbitrary constants introduced by σ ; and it is clear that the same applies to D , since the differential coefficient of a sum is the same as the sum of the differential coefficients.

It should be noticed that, in all cases,

$$\begin{aligned}\sigma(u_{n+1}-u_{-1}) &= \sigma u_{n+1} - \sigma u_{-1} \\ &= u_0 + u_1 + u_2 + \dots + u_{n-1} + u_n,\end{aligned}\quad (110)$$

$$\begin{aligned}\mu\sigma(u_n-u_0) &= \mu\sigma u_n - \mu\sigma u_0 \\ &= \frac{1}{2}u_0 + u_1 + u_2 + \dots + u_{n-1} + \frac{1}{2}u_n.\end{aligned}\quad (111)$$

Integration- and Summation-Formulae (Differential Coefficients).

18. By (65), and (1),

$$\begin{aligned}u_1 - u_0 &= \delta u_1 \\ &= 2 \sinh \frac{1}{2} h D \cdot u_1 \\ &= (1 + \frac{1}{24} h^2 D^2 + \frac{1}{1920} h^4 D^4 + \frac{1}{322560} h^6 D^6 \\ &\quad + \frac{1}{622080} h^8 D^8 + \dots) h D u_1,\end{aligned}\quad (112)$$

and, by (64) and (65), and (3),

$$\begin{aligned}u_1 - u_0 &= \mu \cdot \delta / \mu \cdot u_1 \\ &= \mu \cdot 2 \tanh \frac{1}{2} h D \cdot u_1 \\ &= \mu (1 - \frac{1}{24} h^2 D^2 + \frac{1}{1920} h^4 D^4 - \frac{1}{90720} h^6 D^6 \\ &\quad + \frac{1}{322560} h^8 D^8 - \dots) h D u_1.\end{aligned}\quad (113)$$

Hence, if

$$u = \int^x v dx, \quad (114)$$

$$\text{we have} \quad \int_{x_0}^{x_1} v dx = (1 + \frac{1}{24} h^2 D^2 + \frac{1}{1920} h^4 D^4 + \dots) h v_1, \quad (115)$$

$$\int_{x_0}^{x_1} v dx = \mu (1 - \frac{1}{24} h^2 D^2 + \frac{1}{1920} h^4 D^4 - \dots) h v_1. \quad (116)$$

By means of either of these formulæ we can build up a table of values of

$$\int^x v dx,$$

provided that v and its differential coefficients can be calculated. Of the two formulæ, the first requires the values of v , $D^2 v$, ... for the intermediate values of x , while the second requires them for the values ... x_0 , x_1 , ... entered in the table. The first formula converges

more rapidly than the second, but either of them is much better than the ordinary formula

$$\int_{x_0}^{x_1} v dx = \left(\frac{1}{1!} + \frac{1}{2!} hD + \frac{1}{3!} h^2 D^2 + \dots \right) h v_0.$$

19. To adapt the preceding formulæ for purposes of summation, the portion containing differential coefficients must in each case be expressed as the difference of functions of x_0 and x_1 . Thus we have

$$\begin{aligned} \delta u_1 &= hDu_1 + (\delta - hD) u_1 \\ &= hDu_1 + \delta \cdot (\delta - hD) / \delta \cdot u_1 \\ &= hDu_1 + \delta \left(1 - \frac{1}{2} hD \operatorname{cosech} \frac{1}{2} hD \right) u_1, \end{aligned}$$

and

$$\begin{aligned} \delta u_1 &= \mu hDu_1 + \delta \cdot (\delta - \mu hD) / \delta \cdot u_1 \\ &= \mu hDu_1 + \delta \left(1 - \frac{1}{2} hD \coth \frac{1}{2} hD \right) u_1. \end{aligned}$$

Hence, by (5) and (4),

$$\delta u_1 = hDu_1 + \delta U_1, \quad (117)$$

$$\delta u_1 = \mu hDu_1 + \delta U'_1, \quad (118)$$

where

$$U = \left(\frac{1}{2} hD - \frac{1}{6} \frac{1}{2} h^3 D^3 + \frac{1}{6} \frac{3}{4} \frac{1}{6} h^5 D^5 - \frac{1}{12} \frac{1}{4} \frac{3}{8} \frac{1}{6} h^7 D^7 + \dots \right) hDu, \quad (119)$$

$$U' = \left(-\frac{1}{12} hD + \frac{1}{12} \frac{1}{6} h^3 D^3 - \frac{1}{36} \frac{1}{4} \frac{1}{6} h^5 D^5 + \frac{1}{12} \frac{1}{6} \frac{1}{6} \frac{1}{6} h^7 D^7 - \dots \right) hDu. \quad (120)$$

Substituting from (114), and adding for n consecutive intervals,

$$\int_{x_0}^{x_n} v dx = h\sigma (v_n - v_0) + h (V_n - V_0), \quad (121)$$

$$\int_{x_0}^{x_n} v dx = h\mu\sigma (v_n - v_0) + h (V'_n - V'_0), \quad (122)$$

where V and V' are what U and U' become by writing v for hDu .

By transposing the terms in (121) and (122), we have

$$v_1 + v_2 + \dots + v_{n-1} = \sigma (v_n - v_0)$$

$$= h^{-1} \int_{x_0}^{x_n} v dx - \left[\left(\frac{1}{2} hD - \frac{1}{6} \frac{1}{2} h^3 D^3 + \frac{1}{6} \frac{3}{4} \frac{1}{6} h^5 D^5 - \dots \right) v \right]_{x=x_0}^{x=x_n}, \quad (123)$$

$$\frac{1}{2} v_0 + v_1 + v_2 + \dots + v_{n-1} + \frac{1}{2} v_n = \mu\sigma (v_n - v_0)$$

$$= h^{-1} \int_{x_0}^{x_n} v dx + \left[\left(\frac{1}{2} hD - \frac{1}{12} h^3 D^3 + \frac{1}{36} \frac{1}{4} \frac{1}{6} h^5 D^5 - \dots \right) v \right]_{x=x_0}^{x=x_n}. \quad (124)$$

The second of these is the Euler-Maclaurin formula for the sum of n consecutive values of a function, and first is an alternative form which is sometimes useful.

20. The preceding formulæ may be expressed in other ways. If, for instance, it is more convenient to use the even than the odd differential coefficients of v , we have

$$\begin{aligned}\delta u_1 &= hDu_1 + \delta^2 \cdot (\delta - hD)/\delta^2 \cdot u_1 \\ &= hDu_1 + \delta^2 \left(\frac{1}{2} \operatorname{cosech} \frac{1}{2}hD - \frac{1}{4}hD \operatorname{cosech}^3 \frac{1}{2}hD \right) u_1,\end{aligned}$$

and, similarly,

$$\delta u_1 = \mu hDu_1 + \delta^2 \left(\frac{1}{2} \operatorname{cosech} \frac{1}{2}hD - \frac{1}{4}hD \operatorname{cosech} \frac{1}{2}hD \coth \frac{1}{2}hD \right) u_1.$$

Substituting from (5), (8), and (9),

$$\delta u_1 = hDu_1 + \delta^2 W_1, \quad (125)$$

$$\delta u_1 = \mu hDu_1 + \delta^2 W'_1, \quad (126)$$

where

$$W = \left(\frac{1}{24} - \frac{1}{840}h^2D^2 + \frac{1}{32400}h^4D^4 - \frac{1}{184800}h^6D^6 + \dots \right) hDu, \quad (127)$$

$$W' = \left(-\frac{1}{12} + \frac{1}{1440}h^2D^2 - \frac{1}{16320}h^4D^4 + \frac{1}{163800}h^6D^6 - \dots \right) hDu; \quad (128)$$

and therefore

$$\int_{x_0}^{x_n} v dx = h\sigma(v_n - v_0) + h(\delta X_n - \delta X_0), \quad (129)$$

$$\int_{x_0}^{x_n} v dx = h\mu\sigma(v_n - v_0) + h(\delta X'_n - \delta X'_0), \quad (130)$$

where X and X' are what W and W' become by writing v for hDu in (127) and (128).

21. Again, by (65) and (15),

$$\begin{aligned}\delta^2 u &= 4 \sinh^2 \frac{1}{2}hD \cdot u \\ &= \left(1 + \frac{1}{12}h^2D^2 + \frac{1}{840}h^4D^4 + \frac{1}{32400}h^6D^6 + \frac{1}{184800}h^8D^8 + \dots \right) h^2D^2 u, \quad (131)\end{aligned}$$

and therefore, if $\frac{d^2 u}{dx^2} = w,$ (132)

so that $u = \int \int^x w(dx)^2,$ (132A)

we have $u = \sigma^2 \delta^2 u$

$$= \sigma^2 h^2 (w + \frac{1}{12} h^2 D^2 w + \frac{1}{240} h^4 D^4 w + \dots). \quad (133)$$

The arbitrary constants in σ^2 must be adjusted according to the circumstances of the case. Thus, if we know the values of u_0 and u_1 , the expression following σ^2 in (133) has to be calculated for $x = x_1, x_2, \dots$, and the values of $u_2, u_3, \dots$ are found by successive additions.

To adapt the formula for summation, we have

$$\delta^2 u = h^2 D^2 u + \delta^2 (1 - \frac{1}{2} h^2 D^2 \operatorname{cosech}^2 \frac{1}{2} h D) u,$$

and therefore, by (8),

$$\delta^2 u = h^2 D^2 u + \delta^2 (\frac{1}{12} - \frac{1}{240} h^2 D^2 + \frac{1}{8064} h^4 D^4 - \frac{1}{1774560} h^6 D^6 + \dots) h^2 D^2 u. \quad (134)$$

Substituting from (132A), and performing the successive additions up to $x = x_p$, we find

$$\int_{x_0}^{x_p} \int_{x_0}^x w (dx)^2 = A + pB + h^2 \{ p w_0 + (p-1) w_1 + (p-2) w_2 + \dots + w_{p-1} \} + h^2 \{ Z_p - Z_0 \},$$

where A and B are constants, and

$$Z = (\frac{1}{12} - \frac{1}{240} h^2 D^2 + \frac{1}{8064} h^4 D^4 - \frac{1}{1774560} h^6 D^6 + \dots) w. \quad (135)$$

Taking $p = 0$ and $p = 1$ in the above relation, we find, O being an arbitrary value of x ,

$$\int_{x_0}^{x_p} \int_c^x w (dx)^2 = p \int_{x_0}^{x_1} \int_c^x w (dx)^2 + h^2 \{ (p-1) w_1 + (p-2) w_2 + \dots + w_{p-1} \} + h^2 \{ Z_p - p Z_1 + (p-1) Z_0 \}. \quad (136)$$

The above methods might be extended to any number of successive summations. But the resulting formulæ are not of practical value.

Integration-Formulæ (Differences).

22. In order to apply (115) or (116) to the tabulation of

$$\int v dx$$

by intervals of h in x , we have to calculate the successive values not only of $h v$, but also of $h^3 D^2 v, h^5 D^4 v, \dots$. The calculation is avoided by

expressing the differential coefficients in terms of the differences. We have then only to calculate the values of $h\nu$.

By (47),

$$\begin{aligned}\delta u_1 &= 2 \sinh \frac{1}{2} h D . u_1 \\ &= (1 + \frac{1}{2} \delta^2 - \frac{1}{8} \frac{1}{7} \delta^4 + \frac{3}{8} \frac{1}{7} \frac{1}{8} \delta^6 - \frac{3}{4} \frac{1}{7} \frac{1}{8} \frac{1}{8} \delta^8 + \dots) h D u_1 \quad (137) \\ &= \{1 + \frac{1}{2} \delta^2 - (\frac{1}{8} \frac{1}{7} + \frac{1}{16} \cdot \frac{1}{8} \frac{1}{8}) \delta^4 + \cdot 000 \ 379 \ 258 \dots \delta^6 \\ &\quad - \cdot 000 \ 059 \ 978 \dots \delta^8 + \dots\} h D u_1,\end{aligned}$$

and similarly, by (49),

$$\begin{aligned}\delta u_1 &= \mu (1 - \frac{1}{2} \delta^2 + \frac{1}{2} \frac{1}{7} \delta^4 - \frac{1}{6} \frac{1}{7} \frac{1}{8} \delta^6 + \frac{2}{3} \frac{1}{7} \frac{1}{8} \frac{1}{8} \delta^8 - \dots) h D u_1 \quad (138) \\ &= \mu \{1 - \frac{1}{2} \delta^2 + (\frac{1}{6} - \frac{1}{2} \cdot \frac{1}{8} \frac{1}{8}) \delta^4 - \cdot 003 \ 158 \ 069 \dots \delta^6 \\ &\quad + \cdot 000 \ 688 \ 106 \dots \delta^8 - \dots\} h D u_1.\end{aligned}$$

$$\text{Hence} \quad \int_{x_0}^{x_1} v dx = (1 + \frac{1}{2} \delta^2 - \frac{1}{8} \frac{1}{7} \delta^4 + \dots) h v_1, \quad \text{---} \quad (139)$$

$$\int_{x_0}^{x_1} v dx = \mu (1 - \frac{1}{2} \delta^2 + \frac{1}{2} \frac{1}{7} \delta^4 - \dots) h v_1. \quad \text{---} \quad (140)$$

By taking the successive intervals x_0 to x_1 , x_1 to x_2 , ..., we get the differences in the table of

$$u \equiv \int^x v dx,$$

and by successive additions of these differences we get the values of u .

If the values of $h\nu$ are accurate within $\pm \frac{1}{2} \rho$, and if the expression given in (139) is taken to the same number of places as $h\nu$, a further error of $\pm \frac{1}{2} \rho$ may be introduced by the portion

$$(\frac{1}{2} \delta^2 - \frac{1}{8} \frac{1}{7} \delta^4 + \dots) h v,$$

so that after p additions, if the initial value of u is correct within $\frac{1}{2} \rho$, there may be a total error of $\pm (p + \frac{1}{2}) \rho$. On account of the accumulation of errors, a table formed by the above method must always be checked at intervals by direct calculation of the integral. The amount of the accumulated error may be kept down by retaining one or two extra figures in the differences of the integral, so that the error in u after p additions will lie within the limits $\pm (p + 2) \frac{1}{2} \rho$. But the retention of these extra figures is avoided, and the values of u at intervals are more easily checked, by expressing the sums of the

differences $\delta^2, \delta^4, \dots$ in (139) in terms of $\delta, \delta^3, \dots$. Thus by n successive additions of (139) we find*

$$u_n = \int_{x_0}^{x_n} v dx = (\sigma + \frac{1}{2}\delta - \frac{1}{6}\frac{7}{60}\delta^3 + \frac{1}{6}\frac{9}{7}\frac{7}{60}\delta^5 - \frac{1}{6}\frac{2}{6}\frac{4}{3}\frac{9}{6}\frac{5}{60}\delta^7 + \dots) hv_n, \quad (141)$$

and similarly, from (140),

$$u_n = \int_{x_n}^{x_0} v dx = \mu (\sigma - \frac{1}{2}\delta + \frac{1}{6}\frac{7}{60}\delta^3 - \frac{1}{6}\frac{9}{7}\frac{7}{60}\delta^5 + \frac{1}{6}\frac{2}{6}\frac{4}{3}\frac{9}{6}\frac{5}{60}\delta^7 - \dots) hv_n. \quad (142)$$

The arbitrary constant in σ or $\mu\sigma$ in either formula depends on the lower limit of integration; but, if we take σhv to be such that

$$u_0 = (\sigma + \frac{1}{2}\delta - \frac{1}{6}\frac{7}{60}\delta^3 + \dots) hv_0, \quad (143)$$

i.e., if we tabulate the values of hv , and then take σhv so that

$$\sigma hv_0 = u_0 - \frac{1}{2}\delta hv_0 + \frac{1}{6}\frac{7}{60}\delta^3 hv_0 - \dots, \quad (143A)$$

then we shall have always

$$u_n = (\sigma + \frac{1}{2}\delta - \frac{1}{6}\frac{7}{60}\delta^3 + \dots) hv_n. \quad (143B)$$

Similarly, if we take $\mu\sigma hv$ so that

$$u_0 = \mu (\sigma - \frac{1}{2}\delta + \frac{1}{6}\frac{7}{60}\delta^3 - \dots) hv_0, \quad (144)$$

i.e., if we take

$$\sigma hv_1 = u_0 + \frac{1}{2}\delta hv_0 + \frac{1}{6}\mu\delta hv_0 - \frac{1}{6}\frac{1}{60}\mu\delta^3 hv_0 + \dots, \quad (144A)$$

then we shall have always

$$u_n = \mu (\sigma - \frac{1}{2}\delta + \frac{1}{6}\frac{7}{60}\delta^3 - \dots) hv_n. \quad (144B)$$

Suppose that σhv_0 is fixed by (143A), its value being taken to the number of decimal places to which we require u_n . Then for checking the table after an interval ph , we ought to have

$$\sigma hv_p = u_p - \frac{1}{2}\delta hv_p + \frac{1}{6}\frac{7}{60}\delta^3 hv_p - \dots$$

If $\sigma hv_0 + hv_1 + hv_2 + \dots + hv_{p-1}$ does not agree with this value of σhv_p , one or more of the values of hv must be altered, by inspection

* This modification of the formula is taken from Hansen. I do not know to whom the original formula (137) is due, but it was used by Legendre in calculating his table of elliptic functions.

of the sequence of differences. But the coefficients of δhv , $\delta^3 hv$, ... in (143) are so small that it is not usually necessary to go through the labour of making the corresponding alterations in the differences of hv . The same remark applies to the use of formula (144A).

23. In the last section we have only considered the tabulation of u at intervals of h in x , by means either of (141) or of (142); the values of hv being calculated for the tabulated values of x in the case of the latter, and for the intermediate values of x in the case of the former. From the table so formed we could construct a table at intervals of $\frac{1}{2}h$ by taking the differences of u , and applying (63). But it should be noticed that the arbitrary constants in (141) and (142) are the same, so that by taking hv at intervals of h we get u at intervals of $\frac{1}{2}h$ with very little trouble. For (141) may be written

$$u_n = (1 + \frac{1}{2}\delta^2 - \frac{1}{24}\delta^4 + \dots) \sigma hv_n.$$

Taking this with

$$u_{n+1} = (1 + \frac{1}{2}\delta^2 - \frac{1}{24}\delta^4 + \dots) \sigma hv_{n+1},$$

we have

$$\begin{aligned} u_{n+1} &= \mu^{-1} (1 + \frac{1}{2}\delta^2 - \frac{1}{24}\delta^4 + \dots) \mu \sigma hv_{n+1} \\ &= (1 - \frac{1}{24}\delta^2 + \frac{1}{720}\delta^4 - \dots) \mu \sigma hv_{n+1} \\ &= (\mu \sigma - \frac{1}{24}\mu \delta^2 + \frac{1}{720}\mu \delta^4 - \dots) hv_{n+1} \\ &= \mu (\sigma - \frac{1}{24}\delta^2 + \frac{1}{720}\delta^4 - \dots) hv_{n+1}, \end{aligned}$$

the constant in σ being unaltered, by (108), and the coefficients in the series just obtained being the same as the coefficients in (142), since they are obtained by the development of

$$\mu^{-1} \cdot 2 \sinh \frac{1}{2}hD = 2 \tanh \frac{1}{2}hD.$$

In applying this method, it is simplest to calculate first the values of

$$\frac{1}{2}\delta hv, \quad \frac{1}{24}\delta^3 hv, \quad \dots,$$

for the tabulated values of x , and thence write down the values of

$$\alpha \equiv (\sigma - \frac{1}{24}\delta^2 + \frac{1}{720}\delta^4 - \dots) hv,$$

keeping in one or two extra figures. The intermediate values of u

are then written down first, by taking the arithmetic means of successive values of a ; and the terms in

$$(\sigma + \frac{1}{2}\delta - \frac{1}{8}\frac{1}{2}\delta^3 + \dots) h\nu$$

are found from those already used, by the relations

$$\begin{aligned} \frac{1}{2}\delta &= \frac{1}{2} \text{ of } \frac{1}{2}, \\ \frac{1}{8}\frac{1}{2}\delta &= \frac{1}{8}\frac{1}{2} \text{ of } \frac{1}{2}\frac{1}{2} = \cdot 19318 \text{ of } \frac{1}{2}\frac{1}{2}, \\ &\&c. \end{aligned}$$

24. Again, by (48),

$$\begin{aligned} \delta^2 u &= 4 \sinh^2 \frac{1}{2} hD \cdot u \\ &= (1 + \frac{1}{12}\delta^2 - \frac{1}{24}\delta^4 + \frac{1}{60}\frac{1}{4}\delta^6 - \frac{1}{360}\frac{1}{8}\frac{1}{2}\delta^8 + \dots) h^2 D^2 u \quad (145) \\ &= (1 + \frac{1}{12}\delta^2 - \frac{1}{24}\delta^4 + \cdot 000\ 512\ 566 \dots \delta^6 \\ &\quad - \cdot 000\ 079\ 641 \dots \delta^8 + \dots) h^2 D^2 u. \end{aligned}$$

Hence, if the second differential coefficients of u are known, we can tabulate the values of u by a double summation. Thus, taking

$$u = \int^x \int^x w (dx)^2,$$

we find

$$u_n = \int^x \int^x w (dx)^2 = (\sigma^2 + \frac{1}{12}\delta^2 - \frac{1}{24}\delta^4 + \frac{1}{60}\frac{1}{4}\delta^6 - \frac{1}{360}\frac{1}{8}\frac{1}{2}\delta^8 + \dots) h^2 w_n. \quad (146)$$

To determine the two arbitrary constants in σ^2 , we usually know the initial values u_0 and u_1 . We must then fix the value of $\sigma h^2 w_1$ so that

$$u_1 - u_0 = \delta u_{\frac{1}{2}} = (\sigma + \frac{1}{12}\delta - \frac{1}{24}\delta^3 + \frac{1}{60}\frac{1}{4}\delta^5 - \dots) h^2 w_{\frac{1}{2}}, \quad (147)$$

and we shall have always

$$\delta u_{n+\frac{1}{2}} = (\sigma + \frac{1}{12}\delta - \frac{1}{24}\delta^3 + \frac{1}{60}\frac{1}{4}\delta^5 - \dots) h^2 w_{n+\frac{1}{2}}. \quad (147A)$$

Then, choosing $\sigma^2 h^2 w_0$ so that

$$u_0 = (\sigma^2 + \frac{1}{12}\delta^2 - \frac{1}{24}\delta^4 + \frac{1}{60}\frac{1}{4}\delta^6 - \dots) h^2 w_0, \quad (148)$$

we shall have always

$$u_n = (\sigma^2 + \frac{1}{12}\delta^2 - \frac{1}{24}\delta^4 + \frac{1}{60}\frac{1}{4}\delta^6 - \dots) h^2 w_n. \quad (148A)$$

If, on the other hand, we know u_0 and Du_0 , the value of $\sigma h^2 w_1$ must be chosen so as to give, as in (142),

$$hDu_0 = \mu \left(\sigma - \frac{1}{12}\delta + \frac{1}{720}\delta^3 - \frac{1}{60480}\delta^5 + \frac{1}{362880}\delta^7 - \dots \right) h^2 w_0, \quad (149)$$

$$i.e., \quad \sigma h^2 w_1 = hDu_0 + \frac{1}{12}h^2 w_0 + \frac{1}{720}\mu\delta h^2 w_0 - \frac{1}{720}\mu\delta^3 h^2 w_0 + \dots \quad (150)$$

If, however, we do not know either u_1 or Du_0 , we must choose $\sigma h^2 w_1$ so as to agree with u_0 and with the next value of u by which the table is to be tested. Calling this u_p , we have, as in § 21,

$$u_p = u_0 + p\delta u_1 + (p-1)h^2 w_1 + (p-2)h^2 w_2 + \dots + h^2 w_{p-1}, \quad (151)$$

whence

$$\delta u_1 = \{u_p - u_0 - (p-1)h^2 w_1 - (p-2)h^2 w_2 - \dots - h^2 w_{p-1}\} / p. \quad (151A)$$

To check the table at intervals ph , when $u_0, u_p, \dots$ are known, and also either $u_1, u_{p+1}, \dots$ or $Du_0, Du_p, \dots$, we can proceed in either of two ways. We may first check the table of values of $\sigma h^2 w$ independently, altering values of $h^2 w$ where necessary, and correcting their differences; and then check the table of $\sigma^2 h^2 w$ by altering values of $\sigma h^2 w$. Or we may perform the two processes simultaneously, by altering values of $h^2 w$; the effect on u_p of alteration of values at different points being given by (151). If we only know $u_0, u_p, u_{2p}, \dots$, we can of course only use (151).

If we wish to use the table of $\sigma^2 h^2 w$ to construct a table of values of u by intervals of $\frac{1}{2}h$ in x , we have, by (50),

$$\begin{aligned} u_{n+\frac{1}{2}} &= \mu \left(\sigma^2 - \frac{1}{24}\delta + \frac{1}{1680}\delta^2 - \frac{1}{16800}\delta^3 + \frac{1}{604800}\delta^4 - \dots \right) h^2 w_{n+\frac{1}{2}} \quad (152) \\ &= \mu \left\{ \sigma^2 - \frac{1}{24}\delta + \left(\frac{1}{1680} + \frac{1}{1680} \cdot \frac{1}{12}\delta \right) \delta^2 - \cdot 001\,896\,288 \dots \delta^3 \right. \\ &\quad \left. + \cdot 000\,419\,847 \dots \delta^4 - \dots \right\} h^2 w_{n+\frac{1}{2}}. \end{aligned}$$

25. The above methods can be extended to any number of summations. In practice, if we wish to make more than two summations, we can generally arrange to make an even number. The formulæ required are found from (45) and (46) by taking $m = 2, 4, 6, \dots$. Going up to $m = 8$, we get the following, which will be found sufficient for most purposes:—

$$\left. \begin{aligned} u &= \left(\sigma^2 + \frac{1}{12}\delta - \frac{1}{720}\delta^3 + \frac{1}{60480}\delta^5 - \frac{1}{362880}\delta^7 + \dots \right) h^2 D^2 u \\ hDu &= \mu \left(\sigma - \frac{1}{12}\delta + \frac{1}{720}\delta^3 - \frac{1}{60480}\delta^5 + \frac{1}{362880}\delta^7 - \dots \right) h^2 D^2 u \end{aligned} \right\}, \quad (153)$$

$$\left. \begin{aligned} n^4 &= (\sigma^4 + \frac{1}{6}\sigma^2 - \frac{1}{24}\sigma + \frac{1}{720}\delta^2 - \frac{1}{2880}\delta^4 + \dots) h^4 D^4 u \\ hDu &= \mu (\sigma^3 + 0.\sigma + \frac{1}{24}\delta - \frac{1}{240}\delta^3 + \frac{1}{1200}\delta^5 - \dots) h^4 D^4 u \end{aligned} \right\}, \quad (154)$$

$$\left. \begin{aligned} n &= (\sigma^6 + \frac{1}{4}\sigma^4 + \frac{1}{24}\sigma^2 + \frac{1}{720}\delta - \frac{1}{5760}\delta^3 + \dots) h^6 D^6 u \\ hDu &= \mu (\sigma^5 + \frac{1}{12}\sigma^3 + 0.\sigma - \frac{1}{864}\delta + \frac{1}{1728}\delta^3 - \dots) h^6 D^6 u \end{aligned} \right\}, \quad (155)$$

$$\left. \begin{aligned} n &= (\sigma^8 + \frac{1}{3}\sigma^6 + \frac{1}{40}\sigma^4 + \frac{1}{560}\sigma^2 - \frac{1}{1280}\delta + \dots) h^8 D^8 u \\ hDu &= \mu (\sigma^7 + \frac{1}{6}\sigma^5 + \frac{1}{240}\sigma^3 + 0.\sigma + \frac{1}{1728}\delta - \dots) h^8 D^8 u \end{aligned} \right\}. \quad (156)$$

In using these formulæ, it is assumed that each column representing summation can be checked by its own guiding values; otherwise the small errors in the values of $h^{2n}D^{2n}u$ might accumulate very rapidly. The formulæ to be used for checking the several columns are different according as the initial sums are determined by the accurate calculation of a few successive values of u , or by calculation of u_0 and its differential coefficients. For the former case it is easily shown that the effect of successive summations is to multiply the initial sums, and also the added values of $h^{2n}D^{2n}u$, by the coefficients appearing in the expansion of different powers of

$$(1-x)^{-1}.$$

Thus, denoting by U the quantity $h^{2n}D^{2n}u$, whose values for successive values of x are calculated and tabulated, and supposing that we perform m summations, we have

$$\begin{aligned} \sigma^m U_{i+m,p} &= \left\{ \sigma^m U_{i+m} + \frac{p}{1!} \sigma^{m-1} U_{i+m-1} + \frac{p(p+1)}{2!} \sigma^{m-2} U_{i+m-2} + \dots \right. \\ &\quad \left. \dots + \frac{p(p+1) \dots (p+m-2)}{(m-1)!} \sigma U_i \right\} \\ &+ \left\{ \frac{p(p+1) \dots (p+m-2)}{(m-1)!} U_1 + \frac{(p-1)p \dots (p+m-3)}{(m-1)!} U_2 + \dots \right. \\ &\quad \left. \dots + \frac{1.2 \dots (m-1)}{(m-1)!} U_p \right\}, \quad (157) \end{aligned}$$

the second term in which may also be written

$$\left\{ U_p + \frac{m}{1!} U_{p-1} + \frac{m(m+1)}{2!} U_{p-2} + \dots + \frac{m(m+1) \dots (m+p-2)}{(p-1)!} U_1 \right\}. \quad (157A)$$

The coefficients in (157) for values of m from 1 to 8, and for values of p from 1 to 20, are given by the following table, the coefficients

being read horizontally from the value of p , and then upwards to the value of m :—

p	m							
	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8
3	1	3	6	10	15	21	28	36
4	1	4	10	20	35	56	84	120
5	1	5	15	35	70	126	210	330
6	1	6	21	56	126	252	462	792
7	1	7	28	84	210	462	924	1716
8	1	8	36	120	330	792	1716	3432
9	1	9	45	165	495	1287	3003	6435
10	1	10	55	220	715	2002	5005	11440
11	1	11	66	286	1001	3003	8008	19448
12	1	12	78	364	1365	4368	12376	31824
13	1	13	91	455	1820	6188	18564	50388
14	1	14	105	560	2380	8568	27132	77520
15	1	15	120	680	3060	11628	38760	116280
16	1	16	136	816	3876	15504	54264	170544
17	1	17	153	969	4845	20349	74613	245157
18	1	18	171	1140	5985	26334	100947	346104
19	1	19	190	1330	7315	33649	134596	480700
20	1	20	210	1540	8855	42504	177100	657800

(158)

If the initial sums, instead of being $\sigma U_1, \sigma^2 U_1, \sigma^3 U_1, \dots$, are $\sigma U_1, \sigma^2 U_0, \sigma^3 U_0, \dots$, these being determined by the formulæ (153)–(156), the coefficients are practically given by (101), and are got from the table (158) by going alternately horizontally and diagonally upwards until we arrive at the coefficient of σU_1 , and then going vertically upwards.

26. The various formulæ given in the preceding sections may be applied to the calculation of integrals involving two or more variables. The only important cases, in practice, are those of

$$u = \int^x \int^y f(x, y) \, dx \, dy,$$

and

$$u = \int^x \int^y \int^z f(x, y, z) \, dx \, dy \, dz.$$

The method is the same in both cases, and it will be sufficient to

consider the former. Suppose that we require our double table, giving u for different values of x and y , to proceed by intervals of h in x , and of k in y , the values of x being

$$\dots, x_0, x_1, \dots,$$

and those of y being $\dots, y_0, y_1, \dots$

In order to use (141) we should calculate the values of

$$hkw_{p,q} = hkf(x_p, y_q)$$

for each of the intermediate values

$$\dots, x_{-\frac{1}{2}}, x_{\frac{1}{2}}, x_{\frac{3}{2}}, \dots$$

of x , taken with each of the intermediate values

$$\dots, y_{-\frac{1}{2}}, y_{\frac{1}{2}}, y_{\frac{3}{2}}, \dots$$

of y , and arrange them in a double table, thus :—

y	x				
	$\dots$	$x_{-\frac{1}{2}}$	$x_{\frac{1}{2}}$	$x_{\frac{3}{2}}$	$\dots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	
$y_{-\frac{1}{2}}$	$\dots$	$hkw_{-\frac{1}{2}, -\frac{1}{2}}$	$hkw_{\frac{1}{2}, -\frac{1}{2}}$	$hkw_{\frac{3}{2}, -\frac{1}{2}}$	$\dots$
$y_{\frac{1}{2}}$	$\dots$	$hkw_{-\frac{1}{2}, \frac{1}{2}}$	$hkw_{\frac{1}{2}, \frac{1}{2}}$	$hkw_{\frac{3}{2}, \frac{1}{2}}$	$\dots$
$y_{\frac{3}{2}}$	$\dots$	$hkw_{-\frac{1}{2}, \frac{3}{2}}$	$hkw_{\frac{1}{2}, \frac{3}{2}}$	$hkw_{\frac{3}{2}, \frac{3}{2}}$	$\dots$
$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	

Applying (141) to the column marked

$$x_{\frac{1}{2}},$$

we get the successive values of

$$h \int^y f(x_{\frac{1}{2}}, y) dx dy.$$

This must be done for each column, and the results arranged in a double table in which the values of x are, as before,

$$\dots, x_{-\frac{1}{2}}, x_{\frac{1}{2}}, x_{\frac{3}{2}}, \dots,$$

those of y being

$$\dots, y_0, y_1, \dots$$

Then, repeating the process on each row, we finally get the value of

$$\int^x \int^y f(x, y) dx dy$$

for each of the values $\dots, x_0, x_1, \dots$

of x , taken with each of the values

$$\dots, y_0, y_1, \dots$$

of y . Or we can use (142) in both cases, or (141) in one case and (142) in the other, with the necessary modifications.

APPENDIX (a).

(SESSION 1899–1900.)

It should have been stated on p. 283 that the date of Mr. S. O. Roberts' death was May 31st, 1899. A fuller account of his life-work is given in *The Taylorian*, Vol. XXI., No. 6, July, 1899.

Major-General Frederick Close, whose death took place on November 23rd, 1899; at his residence, Cranleigh, Shanklin, Isle of Wight, was son of the late Captain Close, of the Royal Artillery. He was born April 15th, 1830, and received his first commission in the Royal Artillery in December, 1847. During the earlier portion of his career he was employed in purely military work. About the year 1865 he passed the "advance class" (afterwards called the "Artillery College"), and subsequently received an appointment as Assistant Superintendent of the Royal Carriage Department of the Woolwich Arsenal. He then held the post of Professor of Artillery, from November, 1873, to March, 1875, at the Royal Military Academy; from that date until the end of March, 1880, he was Superintendent of the Royal Small Arms Factory at Enfield. Finally, from 1881 to 1886, he was Superintendent of the Royal Carriage Department at Woolwich.