Thus if α be a single root of the equation $\phi_0\left(\frac{1}{x}\right) = 0$, the corresponding solution of (15) is

$$y_m = \Gamma(m+b) A \alpha^m,$$

and the corresponding expansion of y from (14)

$$y = A \left\{ \Gamma(b) + 1' (b+1) \alpha x + \Gamma(b+2) \frac{\alpha^2 x^2}{2} + \dots \right\} \dots \dots (16).$$

The n arbitrary constants $A, \dots$ are not $y_0, y_1, \dots$, but are connected with them by linear equations.

When $b=1$, equation (14) reduces to

$$\left\{ \phi_0(x) D^n + n \phi_0'(x) D^{n-1} + \frac{n(n-1)}{2} \phi_0''(x) D^{n-2} + \dots \right\} y = 0,$$

or

$$D^n \{ \phi_0(x) y \} = 0.$$

This gives the expansion for rational fractions; and in the same manner, when b is put $=0$, or any positive or negative integer, the expansion obtained is that of $D^{b-1}(f)$, where f stands for the rational fraction, and $D^{b-1}(f)$ for its $(b-1)^{\text{th}}$ differential coefficient; negative indices of course meaning integrations.

*The Diagonal Scale Principle applied to Angular Measurement
in the Circular Slide Rule.* By JOHN R. CAMPBELL.

[Abstract of Paper, read January 14th, 1875.]

Before entering upon the construction of diagonal scales, having, in place of the usual equidistant parallel lines crossed by a straight diagonal, as many equidistant concentric arcs of circles crossed by a curved diagonal, it will be necessary for me briefly to describe the instrument, or rather the somewhat rough home-made model of one, in which I have introduced them.

It is simply a form of circular slide rule, combining in one arrangement both the ordinary principle of two logometric scales for multiplication and division, and that introduced by the late Dr. Roget (*vide* Phil. Trans., Nov. 17, 1814) for the finding of powers and roots.

Fig. 1 is a plan of the face; fig. 2, a section of the instrument by a vertical plane through the centre.

The face is a circular cardboard surface AA, 12 or 14 inches in diameter, but which might well be made of smaller dimensions. It consists of two parts,—an annular rim AB for the *fixed* scale, and a circular disc BCB corresponding to the *slide*, turning on an axis C in

the centre of the rim. The rim is mounted on a wooden backing DD, through which the axis passes. For an axis I have used a common pin, but a small bolt, capped by a milled-edged nut, would be better. *aa*, *a'a'*, *bb*, *cc*, *dd*, &c., are scales contained between equidistant circles described from the common centre C. *aa* on the rim is logometrically graduated, or, in other words, a scale of $\log x$. Of those on the disc used in conjunction with it, *a'a'* is also a scale of $\log x$; *bb*, *cc*, and *dd* portions of a scale of $\log \log x$. (In Dr. Roget's instrument the scale of $\log \log x$ is continuous in the form of a spiral—a neater construction than mine, but involving much more labour, and therefore possible inaccuracy in its execution). Other scales, *e.g.*, of $\log \sin \theta$, $\log \tan \theta$, may be added in the spaces *ee*, *ff*, &c.; but those already named appear to me of most general importance, and a greater number is liable to produce confusion; to prevent which as much as possible, the rings *aa*, *a'a'*, *bb*, &c., are tinted of various contrasting colours,—red, violet, orange, &c.; or, if left white, they have their numbers marked in distinctive colours, red for one scale, blue for the next, and so on. *h, h, h* ... are holes in the disc, to afford a finger purchase in turning it. ECE is a cardboard marker, turning on the axis C, and having its edges radii. Either of these, placed over any point on a scale, indicates the point on any other scale which lies on the same radius.

Without going fully into the theory of the slide rule (for a concise account of which see De Morgan's article in the "Penny Cyclopaedia"), it may be as well to remark, that in all circular varieties of that instrument the measurements are angular—the value of an angle, however, being represented by the ratio of its arc to the circumference. The entire circle is taken as unity, but, being visibly identical with *n* coinciding circles, it also represents any integer *n*, positive or negative. Hence it represents the characteristic of a logarithm, whatever that may be, $>$ or $<$ 0. In the scale *bb* of $\log \log x$ the characteristic is 0, in *cc* it is -1 , and in *dd*, -2 . The mantissa, or decimal portion, of a logarithm, being in the tables always a positive quantity < 1 , alone requires measurement. The measurements are taken from a scale of equal parts constructed round the margin of the rim, but which, in the finished instrument, need scarcely be retained.* They are in the direction of the arrow, those of the disc scales commencing from a common radius *oo'*. The graduation up to at least two places of figures is by radial divisions (omitted in fig. 1). Every tenth division, only, has a number attached—that number being a quantity of which the arc is a certain function.

I now come to the diagonal scales, the object of which is to afford

* It might be used in conjunction with *aa* for finding the logarithm of a number, but this can be done by *a'a'* in conjunction with the scales of $\log \log x$.

an extra figure in the reading. Where the ordinary slide rule reads only to two figures, by the insertion of these scales between each two consecutive radial divisions, we obtain a third.

Fig. 3 is an enlarged view of two contiguous scales, showing the radial divisions with their diagonal scales and the system of numbering. Each of the rings aa , $a'a'$, bb , &c. (fig. 1) is portioned into ten smaller ones, of uniform breadth, by circles,—the middle circle (corresponding to the figure 5) being described in red. The arcs of these *elementary* circles, as I may call them, lying between consecutive radial divisions, are crossed by a curved diagonal whose equation, of course, depends on the law of graduation of the scale to which it pertains, the number of its intersections with an arc, reckoned from the circles MN , mn , determining the value of the right-hand figure.

Suppose $MmnN$ (fig. 4) to be the space between two consecutive radial divisions; Mn , the diagonal cutting two consecutive elementary arcs in P and Q ; and C the common centre. Draw the radii CM , CP , CQ , and CN , and take C for the pole.

Let $r = CP$, the radius vector;
 $\theta = \text{actual angle from a fixed line};$
 $\phi = \text{angle of reading on the scale};$

then we have $\phi = F(\theta) \dots \dots \dots (\alpha),$

where F represents the function which the measurement is of the reading on the scale.

Now, although conformable to decimal notation, we insert only nine equidistant arcs between MN and mn , the curve is evidently the same, whatever be their number. Suppose it therefore n , and consider the difference $CQ - CP$ of two consecutive radii, Δr , corresponding to the angle PCQ or $\Delta\phi$.

Then $\frac{\Delta\phi}{\Delta r} = k$, a constant;

hence, making n infinite, $\frac{d\phi}{dr} = k \dots \dots \dots (\beta).$

Combining (α) and (β) , we obtain, on integration,

$$F(\theta) = k(r+c),$$

which is the general equation to diagonal curves.

In the scale of equal parts used as a measure in the construction of the other scales,

$$F(\theta) = a\theta;$$

hence

$$r+c = A\theta$$

is the equation to its diagonal curve,—the Archimedeian spiral.

In the scale of $\log x$,

$$\theta = a \log_{10} \phi, \quad \text{or} \quad \phi = A10^\theta,$$

and $r+c = B10^\theta$ is the equation to a diagonal.

In the scale of $\log \log x$, it may be shown in a similar manner that

$$r+c = B 10^{10}$$

is the equation.

Of course, in the $\log x$ and $\log \log x$ scales, the constants vary with the position of each diagonal scale in the circumference; and, were we to limit the radial graduation to a two-figure reading, the diagonals at the commencement of each scale would be very long, compared with those near the end, cutting the middle elementary arcs at an angle too acute to afford satisfactory reading, especially when, as might easily occur, there was an appreciable inaccuracy in the tracing of the curve. In order to avoid these long diagonals, the arc MN, when it is more than, say, twice the length of the breadth Mm of the scale, ought to be split into ten portions by the insertion of additional radii, and a diagonal scale constructed on each—the reading, of course, extending to four places of figures.

Instead of all the elementary circles being described on the face of the instrument, short arcs of them might be marked on the margins of the radial marker, the fifth only being shown on the scale.

In my models I have generally substituted, for the true curve of the diagonal, the arc of a circle which would cut it in three points, viz., M, n, and an intermediate point on the fifth elementary circle. Where, however, the diagonals are long, it would be a better plan to determine from tables a greater number of points, and trace the curves in the ordinary way.*

Although I have nowhere seen any description of the diagonal scale as applied to angular measurement, the following statement of Sir D. Brewster, in his "Martyrs of Science," p. 123, which I only came across a few days ago—months after the idea had suggested itself to me—renders it probable that something of the kind was in use in 1576, or even earlier. After giving a list of the astronomical instruments in Tycho Brahe's observatory at Uraniberg, the author remarks:—

"In almost all the instruments now enumerated, the limb was subdivided by diagonal lines,—a method which Tycho first brought into use, but which, in modern times, has been superseded by the inventions of Nonius and Vernier."

Diagonal curves might be advantageously applied to the cylindrical form of slide rule, where the scales are on zones capable of being turned independently of one another round a cylindrical core. A

* Should the instrument ever become an article of sale, much expense might be avoided by engraving the whole face from one carefully executed copper-plate, and afterwards cutting out the portion which forms the disc. The disc ought to have a metal backing. Probably a more generally useful arrangement than the one described might consist of two $\log x$ scales alone, having a face 6 inches in diameter.

marker, having its edge parallel to the axis, must in this case be attached to the instrument.

Suppose OM (fig. 5) to be a portion of one of these scales developed into a plane; O, the point from which the measurements are taken. $ON = x$, $NP = y$, coordinates of a point P on the diagonal Cd.

Then we have $\frac{dF(x)}{dy} = k$;

and hence $F(x) = k(y+c)$ is the general equation to these diagonals.

It is easily seen that the diagonals Ab, Bc, Cd, &c., are identical with the arcs ab' , $b'c'$, $c'd'$, &c., of the continuous curve $Oab'c'.....$, the extension of the first diagonal Oa, and which may be formed by placing the rectangles containing them in the positions shown by dotted lines.

In the scale of log x , $F(x) = A10^x$,
and therefore $y+c = A10^x$ is the equation to Cd. And since, when $x = 0$, $y = 0$, and $c = A$, $y+A = A10^x$ is the equation to $Oab'c'$.

If $A = 1$, this assumes the simpler form,

$$x = \log(1+y).$$

Similarly it may be shown that the equation to a diagonal in the scale of log log x , is of the form

$$y+c = B10^{10^x}.$$

February 11th, 1875.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Mr. C. E. Bickmore was elected a Member; and Messrs. A. B. Kempe, B.A., Scholar of Trinity College, Cambridge, and S. A. Renshaw, were proposed for election.

Prof. Cayley, F.R.S., read two short papers: "On a Point in the Theory of Attractions," and on the question of "The Mechanical Description of a Quartic Curve." Prof. Sylvester, F.R.S., exhibited a new sort of Lady's Fan, and briefly indicated its mode of construction and properties. He then spoke in detail on the expression of the curves generated by any given system whatever of link-work under the form of an irreducible determinant. The Chairman made a few remarks on the paper. Mr. Tucker read portions of papers by the Rev. W. H. Lavery, on Peaucellier's Problem; by Mr. Routh, on Laplace's

Three Particles; and by Mr. J. Griffiths, "Note on some relations between certain Elliptic and Hyperbolic Functions."

The following presents were received:—

"Monatsbericht," Sept. and Oct., 1874.

Carte-de-visite of Col. A. R. Clarke, F.R.S., who also gives a photographic likeness of the late Dr. Rutherford, F.R.A.S. (taken by himself), and a photographic copy of a likeness of Prof. Schumacher.

Note on a Point in the Theory of Attraction. By Prof. CAYLEY.

[Read February 11th, 1875.]

Consider a mass of matter distributed in any manner on a surface, and attracting points P, Q not on the surface. Consider a point Q accessible from P , viz., such that we can pass continuously from P to Q without passing through the surface. [It is hardly necessary to remark that, if for example the matter is distributed over a hemisphere or segment of a closed surface, then by the surface we mean the hemisphere or segment, not the whole closed surface.] The potential and its differential coefficients *ad infinitum*, in regard to the coordinates of the attracted point, all vary continuously as we pass from P to Q ; and it follows that the potential is one and the same analytical function of (a, b, c) , the coordinates of the attracted point, for the whole series of points accessible from the original point P ; in particular, if the surface be an unclosed surface, for instance a hemisphere or segment of a sphere, then every point Q whatever not on the surface is accessible from P , and the theorem is that the potential is one and the same analytical function of (a, b, c) , the coordinates of the attracted point, for any position whatever of this point (not being a point on the surface). But this seems to give rise to a difficulty. Consider the matter as uniformly distributed over a closed surface, and divide the closed surface into two segments: the potential of the whole shell is the sum of the potentials of the two segments; and the potential of the first segment being always one and the same function of (a, b, c) , whatever may be the position of the attracted point, and similarly the potential of the second segment being always one and the same function of (a, b, c) , whatever may be the position of the attracted point; then the potential of the whole shell is one and the same function of (a, b, c) , whatever may be the position of the attracted point. This we know is not the case for a uniform spherical shell; for the potential is a different function for external and interior points, viz., for internal points it is