where, in the bracket, there are $n(2n-1)$ terms, and $\epsilon_1, \dots, \epsilon_{2n}$ are imaginary associative symbols satisfying the equations

$$\epsilon_i^2 = 0, \quad \epsilon_i \epsilon_j = -\epsilon_j \epsilon_i, \quad \epsilon_1 \epsilon_2 \dots \epsilon_{2n} = 1.$$

Also, the expression

$$\epsilon_1 \epsilon_2 (12) + \epsilon_1 \epsilon_3 (13) + \dots + \epsilon_r \epsilon_s (rs) + \dots$$

can be written $\frac{1}{2} [\epsilon_1 (1) + \epsilon_2 (2) + \dots + \epsilon_{2n} (2n)]^2$,

where (1), ..., (2n) are imaginary symbols such that

$$\epsilon_r (r) \epsilon_s (s) = \epsilon_r \epsilon_s (rs) = \epsilon_s \epsilon_r (sr).$$

In the equations (I.), (II.), the left sides may be described as Pfaffians whose elements are Pfaffians of (two or) four indices. It would seem to be worth investigation what analogous expansions exist for the general Pfaffian in terms of Pfaffians of any assigned number of indices.

On Discontinuous Fluid Motions involving Sources and Vortices.

By B. HOPKINSON. Received December 23rd, 1897. Read January 13th, 1898.

The theory of fluid motion in two dimensions where the boundary consists partly of straight walls and partly of free stream lines over which the pressure is constant has been investigated in a general manner by Mr. Love.* His method, however, only applies to cases in which there are no sources or other singularities in the moving fluid, the motion being produced entirely by streams proceeding to or from infinity. This paper contains an extension of the theory to cases in which there are singularities present anywhere in the fluid.

In order the more clearly to show the modifications of Mr. Love's method necessitated by this extension, I will quote the preliminary

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account of the problem and of the method of solving it which appears at the head of his paper :—

“The motion of the fluid is supposed to take place in the plane of a complex variable z , and to be given by means of a velocity-potential ϕ and a stream-function ψ , so that $\phi + i\psi$, or w , is a function of z . The object of the theory is to show how in any given problem the functional relation between w and z can be discovered. For this purpose we consider two functions ζ and Ω such that

$$\zeta = \frac{dz}{dw} \quad \text{and} \quad \Omega = \log \zeta,$$

and it is well known that Ω is a complex quantity whose real part is the logarithm of the reciprocal of the velocity at the point z , and whose imaginary part is the angle which the direction of this velocity makes with the real axis in the z -plane. In the kind of motion to which this method applies, the region of the z -plane within which the motion takes place is bounded partly by fixed straight lines and partly by free stream-lines. Along the fixed boundaries the direction of the velocity is given, so that the corresponding parts of the boundary in the Ω -plane are lines parallel to the real axis. Along the free stream-lines the velocity is a given constant, so that the corresponding parts of the boundary in the Ω -plane are parts of a straight line parallel to the imaginary axis. Hence the boundary in the Ω -plane is a polygon which we know how to draw. In like manner, the boundary in the w -plane, consisting of parts of straight lines parallel to the real axis ($\psi = \text{constant}$), is a polygon which we know how to draw. If, now, we take an auxiliary complex variable u , the polygons in the Ω - and w -planes can be conformally represented on the half-plane for which the imaginary part of u is positive. The required transformations are given by the theory of Schwarz and Christoffel, and we are thus in a position to write down two relations,

$$\frac{dz}{du} = f_1(u),$$

$$\frac{d\Omega}{du} = f_2(u),$$

from which $\frac{dz}{du} = Cf_1(u) \exp \left\{ \int f_2(u) du \right\}.$

In this way a relation $z = f(u)$ is obtained which is such that the area of motion is conformally represented on that half of the plane of u for which the imaginary part of u is positive.

In cases where sources, sinks, or vortices are present in the moving fluid, it is still possible to obtain the relation $z = f(u)$ by finding Ω and w in terms of u . Mr. Love's method of arriving at these functions here fails, however, because they become infinite at the singularities, and the half-plane of u can no longer be conformally represented on areas in the planes of Ω and of w . We cannot, therefore, have recourse to Schwarz's expression for $\frac{d\Omega}{du}$ and $\frac{dw}{du}$, but must obtain these functions direct from their boundary and infinity conditions. I proceed to show how to do this.

Suppose that the motion goes on in a simply-connected area in the z -plane, and that the boundary consists of straight walls and free stream-lines alternately. There may be streams proceeding to or from infinity, and these streams may be bounded either by parallel straight walls or by free surfaces ultimately straight and parallel. ψ is constant over the boundary, save for discontinuities at those points at infinity whence the streams come or whither they go. The velocity is supposed to be finite at all points of the boundary. At isolated points within the area the velocity becomes infinite. The type of these points is $z = \zeta$, where $\frac{dw}{dz}$ becomes finite in the form

$$\frac{A_1}{z-\zeta} + \frac{A_2}{(z-\zeta)^2} + \dots + \frac{A_m}{(z-\zeta)^m}.$$

Here $A_1, A_2, \dots, A_m$ may be any complex quantities. The index m is, of course, an integer, and we may call it the order of the singularity at ζ .

The area of motion can always be conformally represented on that half of the plane of a complex variable u for which the imaginary part of u is positive. The problem is to find the relation between z and u in any given case, so that this representation can be effected.

Now, if w be expressed as a function of u , it will represent a fluid motion in the half-plane of u . In this motion the real axis of u is a rigid boundary over which ψ is constant save for discontinuities of amounts $\pi m_1, \pi m_2, \dots$ at points $t_1, t_2, \dots$ (which correspond to the discontinuities in ψ on the boundary in the z -plane). $\pi m_1, \pi m_2, \dots$ are the rates of discharge of fluid respectively by the several streams in the z -plane to which the points $t_1, t_2, \dots$ correspond. There will be singularities at the points corresponding to the singularities in the z -plane; and the order of each singularity in the u -plane will be equal to the order of the corresponding singularity in the z -plane. Thus, at the point

$u = r$ corresponding to $z = \zeta \frac{dw}{du}$ becomes infinite, like

$$\frac{B_1}{u-r} + \frac{B_2}{(u-r)^2} + \dots + \frac{B_m}{(u-r)^m},$$

and we note that $B_1 = A_1$.

The relation between w and u must, therefore, be

$$\begin{aligned} \frac{dw}{du} = \frac{m_1}{u-t_1} + \frac{m_2}{u-t_2} + \dots + \Sigma \left[\frac{B_1}{u-r} + \frac{B_2}{(u-r)^2} + \dots + \frac{B_m}{(u-r)^m} \right] \\ + \Sigma \left[\frac{B'_1}{u-r'} + \frac{B'_2}{(u-r')^2} + \dots + \frac{B'_m}{(u-r')^m} \right], \end{aligned}$$

where $B'_1, B'_2, \dots, r'$ are the complex quantities respectively conjugate to $B_1, B_2, \dots, r$, and the summations extend to all singularities, such as $u = r$.

This equation reduces to the form

$$\frac{dw}{du} = \frac{F(u)}{\phi(u)},$$

where $F(u)$ and $\phi(u)$ are rational integral functions of u with real coefficients, and $F(u)$ is of less degree than $\phi(u)$. The real roots of $\phi(u)$ are $t_1, t_2, \dots$, and no two of them are equal. The complex roots occur in conjugate pairs, and those of them whose imaginary parts are positive are at the points r which correspond to the singularities. Each complex root is repeated a number of times equal to the order of the singularity to which it corresponds.

The method of procedure which I adopt is as follows:—Beginning with the motion in the half-plane of u given by the general formula

$$\frac{dw}{du} = \frac{F(u)}{\phi(u)},$$

I shall seek the most general relation $z = f(u)$ which is such that—

(1) The half-plane of u is conformally represented on a simply-connected area in the z -plane.

(2) In the motion in that area given by w (expressed as a function of z) the velocity of q is constant over some parts of the boundary, while the direction of motion of the fluid is constant over the other parts.

It will then be possible, by proper choice of the functions $F(u)$

and $\phi(u)$ and of the constants which arise in the general form of $z = f(u)$, to arrive at any possible case of motion of the kind described in the z -plane. The proper choice of these functions and constants to suit any particular case forms the second part of the problem.

The first step is to determine the infinity and boundary conditions to be satisfied by Ω . Suppose that there are n free pieces in the boundary in the z -plane. Corresponding to these there will be n segments of the real axis of u —say from u_1 to u_2 , u_3 to u_4 , ..., u_{2n-1} to u_{2n} . The remaining n segments, occurring alternately with these, correspond to the straight parts of the boundary. Ω is a function of u , the real part of which does not vary over any one of the first-named segments. On the other segments the imaginary part does not vary, except discontinuously at isolated points which correspond to the points on the straight walls at which the fluid is at rest.

Infinities of Ω .—Consider, first, points within the region of motion and not on the boundary. Since the representation of the region of motion on the half-plane of u is conformal, $\log \frac{dz}{du}$ does not become infinite at any such point. Now

$$\Omega = \log \frac{dz}{du} - \log \frac{dw}{du},$$

whence it follows that Ω becomes infinite at the same points, and in the same manner as $-\log \frac{dw}{du}$. That is, Ω becomes infinite at every complex root of $F(u)$ and of $\phi(u)$, the imaginary part of which is positive, and at no other points. Let σ be such a root of $F(u)$, and let σ' be its conjugate root, and let it be repeated k times, so that $F(u)$ contains the factor $(u-\sigma)^k(u-\sigma')^k$. Then Ω becomes infinite when $u = \sigma$, like $-k \log(u-\sigma)$. Similarly, at the point τ , where τ , as before, is a complex root of $\phi(u)$ within the half-plane, and repeated m times, Ω becomes infinite, like $m \log(u-\tau)$.

Turning now to points on the boundary, we remember that one of the postulates as to the nature of the motion was that the velocity should not become infinite at any point on the boundary. Hence Ω can only become infinite on the boundary, if at all, at those points at which the fluid is at rest. These points of rest can clearly only occur on the straight walls, and not on the free stream lines. Now we have

$$\frac{dw}{dz} = \frac{dw}{du} \cdot \frac{du}{dz}.$$

But, if we confine ourselves to points at which both u and z are finite $\frac{dz}{du}$ cannot be zero or infinite on a straight wall. Hence the fluid is at rest, and Ω is infinite at such roots of $F(u)$ as fall on segments of the real axis which correspond to straight parts of the boundary. If s be such a root of $F(u)$, and if it be repeated p times, then Ω becomes infinite when $u = s$, like $-p \log(u-s)$. Ω does not become infinite at any other points on the real axis of u , except in certain cases when $u = \infty$.

The motion is supposed to go on in a finite area, except for the streams proceeding to or from infinity. Hence the fluid is not at rest, and Ω is not infinite for infinite values of z .

If the point in the z -plane corresponding to $u = \infty$ be either at an infinite distance from the origin, or on a free stream-line, Ω cannot be infinite there. If, however, it be on a straight wall, Ω may be infinite. But we have

$$\frac{d\Omega}{du} = -\frac{d}{du} \log \left(\frac{dw}{du} \right) + \frac{d}{du} \log \left(\frac{dz}{du} \right).$$

Now, at the point on the straight wall corresponding to $u = \infty$, $\frac{dz}{du}$ vanishes, like $\frac{\text{const.}}{u^3}$. Also $\frac{dw}{du}$ vanishes when $u = \infty$, like $\frac{\text{const.}}{u}$.

Hence $\frac{d\Omega}{du}$ vanishes when u is infinite.

We may now sum up the conditions to be satisfied by the function Ω :—

(1) Its real part is constant over each of the n segments of the real axis (u_1 to u_2 , u_3 to u_4 , ..., u_{2n-1} to u_{2n}) which correspond to the free parts of the boundary in the z -plane.

(2) Its imaginary part is constant over each of the remaining n segments, save for certain discontinuous changes.

(3) It becomes infinite at every complex root of $F(u)$ or of $\phi(u)$ which falls within the half-plane. At a root σ of $F(u)$, repeated k times, Ω is infinite, like $-k \log(u-\sigma)$; and at a root τ of $\phi(u)$, repeated m times, it is infinite, like $+m \log(u-\tau)$.

(4) It becomes infinite at those real roots of $F(u)$, of which the type is s , which are such that $u_{2r+1} > s > u_{2r}$. The infinity is of the type $-p \log(u-s)$, where the root is repeated p times.

(5) For infinite values of u , Ω may be infinite, but $\frac{d\Omega}{du}$ vanishes.

From these boundary and infinity conditions we have now to determine Ω .

Consider the equation

$$\frac{d\Omega}{du} = \frac{\chi}{\sqrt{(u-u_1)(u-u_2)\dots(u-u_{2n})}}, \quad (1)$$

where χ is a function of u which is real for real values of u . It is clear that, as u increases along the real axis from $-\infty$ to $+\infty$, the growth of Ω is alternately entirely in its real and in its imaginary part. On those segments of the real axis for which

$$u_{2r+1} > u > u_{2r},$$

and which correspond to the straight parts of the boundary in the fluid motion, the real part of Ω alone varies, and its imaginary part is constant. On the other, and alternate, segments the real part of Ω is constant. Thus, Ω , if determined from this equation, will satisfy boundary conditions of the right character; and by means of the function χ we can introduce infinities and discontinuities into Ω as we please, the only restriction on that function being that it must be real for real values of u .

Suppose that we take

$$\chi = C_0 + C_1 u + C_2 u^2 + \dots + C_{n-1} u^{n-1} - \sum \frac{p\alpha}{u-s} - \sum k \left(\frac{\beta}{u-\sigma} + \frac{\beta'}{u-\sigma'} \right) + \sum m \left(\frac{\gamma}{u-\tau} - \frac{\gamma'}{u-\tau'} \right), \quad (2)$$

where $C_0, C_1, \dots, C_{n-1}$ are real. The symbols $p, s, \sigma, \sigma', k, \tau, \tau', m$ bear the same meanings as before, and the summations extend to all the infinities of Ω . The quantities α are real; and the quantities β', γ are respectively the complex quantities conjugate to β, γ .

Then χ clearly satisfies the conditions of being real for real values of u . Also, at the point $u = s$, $\frac{d\Omega}{du}$ becomes infinite, like

$$- \frac{p\alpha}{\sqrt{(s-u_1)(s-u_2)\dots(s-u_{2n})}} \frac{1}{u-s},$$

and Ω becomes infinite, like

$$- \frac{p\alpha}{\sqrt{(s-u_1)(s-u_2)\dots(s-u_{2n})}} \log(u-s).$$

Hence Ω will become infinite at $u = s$ in the right way if

$$\alpha = \sqrt{(s-u_1)(s-u_2) \dots (s-u_{2n})}.$$

Since the root s is so situated that

$$u_{2r+1} > s > u_{2r},$$

it follows that α is real for every such root, as was postulated above.

Similarly, in order that Ω may become infinite in the right way when $u = \sigma$, and when $u = \tau$, we must take

$$\beta = \sqrt{(\sigma-u_1)(\sigma-u_2) \dots (\sigma-u_{2n})}$$

and

$$\gamma = \sqrt{(\tau-u_1)(\tau-u_2) \dots (\tau-u_{2n})},$$

and we shall then have

$$\beta' = \sqrt{(\sigma'-u_1)(\sigma'-u_2) \dots (\sigma'-u_{2n})},$$

$$\gamma' = \sqrt{(\tau'-u_1)(\tau'-u_2) \dots (\tau'-u_{2n})},$$

the sign affixed to the square roots being such that β' is the complex quantity conjugate to β , and γ' that conjugate to γ .

The function Ω determined in this way only becomes infinite at the points whose types are s , σ , τ ; and $\frac{d\Omega}{du}$ vanishes for infinite values of u . Thus Ω satisfies the right conditions.

The introduction of the n arbitrary constants $C_0, C_1, \dots, C_{n-1}$ insures that Ω shall be the most general possible function that satisfies the conditions. This is most readily seen by considering a simple electrical analogy. Suppose that the half-plane of u is a uniformly conducting sheet, the real axis being its edge. Let the segments of the edge from u_1 to u_2 , u_2 to u_3 , ..., u_{2n-1} to u_{2n} be made perfectly conducting and kept at arbitrary potentials, the remaining segments being left free. Let there be electrodes in the sheet and on the edges thereof, at the points whose types are σ , τ , s . Let the electrode at s discharge an amount $-p\pi$ of electricity per second, that at σ an amount $-k\pi$, and that at τ an amount $+m\pi$. Then the electrical potential is clearly the real part of Ω , where Ω is the function that we have been seeking in the fluid motion problem. The arbitrary potentials at which the n segments of the edge are maintained are respectively the logarithms of the reciprocals of the velocities over the n free pieces of the boundary of the fluid motion. Now the real part of Ω , as determined from equation (1) above, contains $n+1$ arbitrary constants — i.e., the n constants $C_0, C_1, \dots, C_{n-1}$, and one

constant of integration. These can be determined in one way, and in one way only, for any given set of values of the potentials of the n segments of the edge, and of the potential at infinity.

This electrical problem has some little interest of its own. It has not, so far as I am aware, been solved hitherto, except for cases where all the electrodes are on the edge of the sheet. In such cases the ordinary equation of Schwarz and Christoffel effects an immediate solution. In the particular form of Ω found above, which corresponds to the fluid motion problem, the strengths of the electrodes are integers. The solution is, however, clearly not confined to cases where the electrodes are of integral strength.

It is worth noting that the above equation for $\frac{d\Omega}{du}$ is a particular case of a more general equation. The ordinary transformation of Schwarz and Christoffel,

$$\frac{dz}{du} = (u-u_1)^{-\alpha_1/\pi} (u-u_2)^{-\alpha_2/\pi} \dots,$$

gives z as a function of u which is finite and continuous for all values of u , and which is such that the argument of $\frac{dz}{du}$ is constant over the real axis save for discontinuous changes at the points $u_1, u_2, \dots$. If we take, instead of the equation last written, the modified equation

$$\frac{dz}{du} = \chi (u-u_1)^{-\alpha_1/\pi} (u-u_2)^{-\alpha_2/\pi} \dots,$$

where χ is a function of u which is real for real values of u , we shall get a function z which is such that the argument of $\frac{dz}{du}$ is constant on the real axis save for discontinuous changes at the same points and of the same amounts as before. The function z will, however, no longer be finite and continuous for all values of z ; but we can introduce into it any infinities or discontinuities that we please by proper choice of χ .

The expression for χ in equation (2) may be put into a more compact form. The complex roots of $F(u)$ and of $\phi(u)$ occur in conjugate pairs, since the coefficients in these functions are all real. Hence the quantities of which the types are σ' and τ' are roots of $F(u)$ and of $\phi(u)$ respectively. We also have

$$\begin{aligned} \beta' &= \sqrt{(\sigma' - u_1)(\sigma' - u_2) \dots (\sigma' - u_{2n})}, \\ \gamma' &= \sqrt{(\tau' - u_1)(\tau' - u_2) \dots (\tau' - u_{2n})}. \end{aligned}$$

We may therefore write

$$\chi = -\sum \frac{\lambda}{u-\sigma} + \sum \frac{\mu}{u-\tau}, \quad (3)$$

where σ now stands for any root of $F(u)$, real or complex, and in either half-plane, which does not lie on a segment of the real axis of u corresponding to a free part of the boundary. Thus σ now includes all the roots of $F(u)$ previously denoted by the symbols s , σ , and σ' . τ stands for any complex root of $\phi(u)$ in either half-plane. We also have

$$\lambda = \sqrt{(\sigma-u_1)(\sigma-u_2) \dots (\sigma-u_{2n})},$$

$$\mu = \sqrt{(\tau-u_1)(\tau-u_2) \dots (\tau-u_{2n})}.$$

If a root be repeated any number of times, the term corresponding to it in the expression for χ is repeated the same number of times.

We have now, therefore, determined Ω in terms of u . The relation between z and u which we seek is given by

$$\frac{dz}{du} = e^n \frac{dw}{du}.$$

This relation will involve a number of constants, such as the roots of $F(u)$ and $\phi(u)$, and the quantities $C_0, C_1, \dots, C_{n-1}, u_1, u_2, \dots, u_{2n}$. If we assign particular values to these constants, and if, furthermore, we make one particular point in the z -plane correspond to one in the half-plane of u , so as to fix z as a one-valued function of u in the half-plane, then the half-plane of u will be conformally represented on a determinate region in the plane of z . The motion in this region given by w will be such that over n pieces of the boundary the velocity is constant, while over the remaining n pieces, occurring alternately with these, the direction of motion of the fluid is constant. By properly choosing the constants we can arrive at any possible motion whatever which satisfies these conditions. The singularities in the motion will be the same in number and order as those in the corresponding motion in the half-plane of u .

In the examples which follow, I investigate in each case the general expressions for a particular class of motion in which there are singularities of a given number and order, and in which the boundary consists of a given number of straight and free pieces. The number and order of the singularities in the motion in the half-plane of u are then known, but their positions and strengths are not known. Three of the constants which occur in the general relation connecting z and u

can be arbitrarily assigned without affecting the generality. For, suppose that we take a new complex variable v , given by

$$v = \frac{au + b}{a'u + b'},$$

where a, b, a', b' are real quantities. Then the region in the z -plane can be conformally represented on the half-plane of v , and the functional relation between z and v will involve the three ratios of a, b, a', b' ; to each of which we can give any value we please.

By proper choice of the remaining constants in the general relation between z and u , we can arrive at any particular case of motion which comes within the class which is being investigated.

Example I.—The boundary consists of a free stream-line and a straight wall. ψ is constant all round the boundary; and the only singularity is a vortex, the circulation round which is $2m\pi$. The velocity on the free stream-line is unity.

If the area of motion be conformally represented on the half-plane of u , the motion in that half-plane represented by w will be that due to a single vortex, the circulation round which is $2m\pi$. The representation can always be effected so that this vortex is on the imaginary axis at the point $u=i\gamma$; and, at the same time, the straight wall may be made to correspond to that segment of the real axis of u which lies between -1 and $+1$.

The relation between w and u with which we start is, therefore,

$$\begin{aligned} \frac{dw}{du} &= \frac{mi}{u-i\gamma} - \frac{mi}{u+i\gamma} \\ &= -\frac{2m\gamma}{u^2+\gamma^2}, \end{aligned}$$

and from this it must be possible, by a proper transformation, to arrive at any possible motion which satisfies the conditions enumerated above. We note that Ω is not infinite when u is infinite, since the point corresponding to $u = \infty$ is on the free stream-line.

We shall take the axis of z parallel to the straight wall. Ω is then a function of u , the imaginary part of which vanishes on the real axis between -1 and $+1$, while the real part vanishes for real values of u outside these limits. Also, Ω becomes infinite when $u = i\gamma$, like $\log(u-i\gamma)$, and does not become infinite for any other value of u . We have, therefore,

$$\frac{d\Omega}{du} = \frac{1}{\sqrt{1-u^2}} \left[\frac{\sqrt{1+\gamma^2}}{u-i\gamma} + \frac{\sqrt{1+\gamma^2}}{u+i\gamma} \right].$$

Take a new variable θ , given by $u = \sin \theta$, and let $\cos \theta = +\sqrt{1-u^2}$. The effect of this is to conformally represent the area of motion on an area on the plane of θ bounded by the lines parallel to the imaginary axis and by a segment of the real axis between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$; the latter segment corresponding to the straight wall (see Fig. 1). Let $\gamma = \sinh \alpha$, so that the point $u = \gamma$ corresponds to the point $\theta = i\alpha$.

FIG. 1.

The equation for Ω now integrates in the form

$$\Omega = \log \frac{\cos \theta - \cosh \alpha}{\cos \theta + \cosh \alpha},$$

the constants being so chosen that the imaginary part vanishes if θ lie on the segment AA' in Fig. 1, while the real part vanishes if it be on AB or $A'B'$.

Thus we find $\frac{dz}{du} = e^{\Omega} \frac{dw}{du}$

$$= -2m\gamma \frac{1}{(\cos \theta + \cosh \alpha)^2}.$$

This is the relation between z and u , and it contains one arbitrary constant, i.e., γ . By choice of γ it must be possible to cover every possible case of motion which satisfies the conditions with which we started.

We have
$$\frac{dz}{d\theta} = \frac{-2m \sinh \alpha \cos \theta}{(\cos \theta + \cosh \alpha)^2},$$

and, if the origin be taken in the middle of the straight wall, the co-ordinates of the ends of the wall will be

$$\begin{aligned} y &= 0, \\ x &= \pm \int_0^{\pi} \frac{2m \sinh \alpha \cos \theta}{(\cos \theta + \cosh \alpha)^2} \\ &= \pm 2m \left(\frac{1}{\gamma} - \frac{1}{\gamma^2} \tan^{-1} \gamma \right). \end{aligned}$$

The expression last written is equal to half the length of the wall, so that we have an equation for determining γ .

The vortex is at the point corresponding to $\theta = \alpha$, and its co-ordinates are

$$\begin{aligned} x &= 0, \\ y &= - \int_0^{\alpha} \frac{2m\gamma \cosh \phi \, d\phi}{(\cosh \phi + \cosh \alpha)^2} \\ &= -2m\gamma \left\{ 1 - \frac{1}{\gamma^2} \log(1 + \gamma^2) \right\}. \end{aligned}$$

Thus the vortex must be on the line of symmetry of the wall, and it must, furthermore, be at a determinate distance from the wall, in order that it may be possible to complete the boundary with a free stream-line.

One half of the free stream-line corresponds to AB in Fig. 1, and the other half to $A'B'$. At points on the first half θ is equal to $\frac{\pi}{2} + \chi$, and on the other half to $-\frac{\pi}{2} + \chi$, where χ is real and positive, and ranges from 0 to α .

In Fig. 2 I have drawn the case where $m = 1$, and $\gamma = 1$. In this case points on the free stream-line are given by

$$\begin{aligned} \frac{dz}{d\chi} &= - \frac{2i \sinh \chi}{(\cosh \alpha - i \sinh \chi)^2} \\ \text{or} \quad \frac{dx}{d\chi} &= - \frac{2 \sinh \chi (3 - \cosh^2 \chi)}{(1 + \cosh^2 \chi)^2}, \\ \text{and} \quad \frac{dy}{d\chi} &= - \frac{4\sqrt{2} \sinh^2 \chi}{(1 + \cosh^2 \chi)^2}. \end{aligned}$$

The vortex is at P . The figures affixed to the various points are

the values of u corresponding to these points. The length of the wall is $4-\pi$, and the distance of the vortex from it is $4-\log_2$.

FIG. 2.

Example II.—The motion is produced by a vortex and doublet at the same point. The boundary consists wholly of a free stream-line, over which the velocity is unity. The circulation round the vortex is $-2\pi m$, where m is positive.

If the area of motion be conformally represented on the half-plane of u , the motion represented by w will be that excited by a vortex and doublet combined. The representation can, in all cases, be effected (by choice of the three constants at our disposal) in such wise that the vortex and doublet are at the point $u = i$, while the axis of the doublet is parallel to the real axis. The form of the boundary in the z -plane will manifestly depend solely on the ratio of the strengths of the doublet and vortex; its dimensions only being affected by the absolute magnitudes of these quantities. We therefore assume that the strength of the doublet in the u -plane is unity. The strength of the doublet in the z -plane will then depend upon the relation between z and u .

$$\begin{aligned} \text{We have } \frac{dw}{du} &= \frac{1}{(u-i)^2} + \frac{1}{(u+i)^2} - m \left(\frac{1}{u-i} - \frac{1}{u+i} \right) \\ &= \frac{2(1+m)(u^2+\beta^2)}{(u^2+1)^2}, \end{aligned}$$

where

$$\beta^2 = \frac{m-1}{m+1}.$$

The motion in the half-plane of u given by this equation is to be transformed into a motion in the z -plane, over the boundary of which the velocity is unity. I propose to investigate the nature of this motion for various values of m .

The real part of Ω is zero for all real values of u ; and Ω becomes infinite at those roots and poles of $\frac{dw}{du}$ which are not on the real axis. Two classes of cases, therefore, arise, according as β is real or imaginary.

(a) $m > 1$ and β real.—In this case Ω becomes infinite when $u = i\beta$, like $-\log(u - i\beta)$; and when $u = i$, like $2 \log(u - i)$. We have, therefore,

$$\Omega = \log \frac{u + i\beta}{u - i\beta} + 2 \log \frac{u - i}{u + i},$$

whence

$$\begin{aligned} \frac{dz}{du} &= e^{\Omega} \frac{dw}{du} \\ &= \frac{2(1+m)(u + i\beta)^3}{(u + i)^4}. \end{aligned}$$

This gives, on integration,

$$\frac{z}{2(1+m)} = -\frac{1}{u+i} + \frac{i(1-\beta)}{(u+i)^2} + \frac{(1-\beta)^2}{(u+i)^3},$$

the origin being taken at $u = \infty$.

The vortex is at $u = i$, and the fluid is at rest at $u = i\beta$. The equation to the boundary is formed by giving u real values in the equation for z . Thus we find for points on the boundary—

$$\begin{aligned} \frac{x}{2(1+m)} &= \frac{u}{3(1+u^2)^3} [-3u^4 + (1-8\beta+\beta^2)u^3 - 3\beta^2], \\ \frac{y}{2(1+m)} &= \frac{1}{3(1+u^2)^3} [3(2-\beta)u^4 + 3(1+2\beta-\beta^2)u^3 + 1+\beta+\beta^2]. \end{aligned}$$

Fig. 3 shows the case where $m = \frac{2}{3}$ and $\beta = \frac{1}{2}$. The vortex and doublet are at P , and the point of rest ($u = i\beta$) is at Q .

As m diminishes a dimple forms in the boundary. This dimple deepens, and its sides approach until at last they overlap, as shown in Figs. 4 and 5. In these figures the part of the area of motion in the neighbourhood of the point of rest and vortex is shown on an enlarged scale.

FIG. 3.

FIG. 4.

FIG. 5.

Such overlapping will often take place in the general case. By making one particular point in the z -plane correspond to a particular point in the u -plane, we can always get z as a one-valued function of u , but we cannot insure that u shall be a one-valued function of z .

A case of motion with overlap is, of course, only of analytical interest; though it is possible to conceive it as going on in a very thin layer.

As m further diminishes the overlap increases, until we reach the limiting case shown in Fig. 6, in which $m = 1$ and $\beta = 0$.

In this case
$$\frac{dz}{du} = \frac{2u^3}{(u+i)^4},$$

and
$$\Omega = 2 \log \frac{u-i}{u+i}.$$

The point $u = \beta$, which in Figs. 3, 4, and 5 was the point of rest, is now on the boundary, and the fluid is no longer at rest there.

FIG. 6.

In Figs. 3, 4, 5, and 6, the boundary stream-line is drawn thick.

The lighter lines show the general disposition of stream-lines. In Figs. 3 and 6 the boundary is drawn accurately. Figs. 4 and 5 are mere sketches to show how the motion develops as m diminishes. In these figures, as in Fig. 6, the complete drawing would be of unmanageable dimensions; so part only of some of the stream-lines is shown.

In each case P is the vortex and doublet ($u = i$), and Q is the point $u = \beta$, which, in Figs. 3, 4, and 5, is the point of rest.

When there is overlap, the stream-lines of the underlying layer are shown dotted.

(b) If m still further diminishes, β becomes imaginary.—We have

$$\frac{dw}{du} = \frac{2(1+m)(u^2 - \gamma^2)}{(u^2 + 1)^2},$$

where γ is real and equal to $\sqrt{\frac{1-m}{1+m}}$.

Ω is now infinite at $u = i$, like $2 \log(u - i)$, and at no other point. Its real part vanishes for real values of u , as before. We have, therefore,

$$\Omega = 2 \log \frac{u-i}{u+i},$$

and now

$$\frac{dz}{du} = \frac{2(1+m)(u^2 - \gamma^2)}{(u+i)^4}.$$

This integrates in the form

$$\frac{z}{2(1+m)} = -\frac{1}{u+i} + \frac{1}{(u+i)^2} + \frac{1+\gamma^2}{3(u+i)^3},$$

the origin being taken at $u = \infty$.

The coordinates of any point on the free stream-line are given by

$$\begin{aligned} \frac{3x}{2(1+m)} &= -\frac{u\{3u^4 - (1+\gamma^2)u^2 + 3\gamma^2\}}{(u^2+1)^3}, \\ \frac{3y}{2(1+m)} &= \frac{6u^4 + 3(1-\gamma^2)u^2 + 1 + \gamma^2}{(u^2+1)^3}. \end{aligned}$$

There is a cusp with the point turned inwards at the points $u = \pm\gamma$. Fig. 7 shows the case where $\gamma^2 = \frac{1}{3}$, $m = \frac{1}{3}$. Fig. 8 is a sketch similar to Figs. 4, 5, and 6, showing an intermediate stage where there is overlap.

FIG. 7.

FIG. 8.

It will have been noted that, in this example, the point of rest develops quite continuously with change of m into two cusps on the boundary. The same thing will happen in the same sort of way in the general case of discontinuous fluid motion considered at the beginning of this paper. Suppose that there is an unpeated complex root σ in $\frac{dw}{du}$. Then $\frac{dw}{du}$ contains a factor $(u-\sigma)(u-\sigma')$ in the numerator, and the fluid is at rest at the point corresponding to $u = \sigma$. It will also be remembered that, if

$$\frac{d\Omega}{du} = \frac{\chi}{\sqrt{(u-u_1)(u-u_2) \dots (u-u_{2n})}},$$

then χ contains a term $\frac{\alpha}{u-\sigma} + \frac{\alpha'}{u-\sigma'}$, where

$$\alpha = \sqrt{(\sigma - u_1)(\sigma - u_2) \dots (\sigma - u_{2n})}.$$

Now let the strengths and positions of the sources or other singularities in the z -plane be varied, so that the root σ tends to approach a segment of the real axis of u which corresponds to a free part of the boundary. A change in the form of that part of the boundary will then take place, somewhat as shown in Figs. 3, 4, and 5. After a certain amount of this change, σ will become real and equal to σ' . At the same time α becomes a pure imaginary and equal to $-u'$; so that the term in χ corresponding to σ disappears. Thus, at this stage, $\frac{dz}{du}$ vanishes, like $\text{const.} \times (u-\sigma)^2$, when $u = \sigma$; for Ω and $\frac{dz}{dw}$ —hitherto infinite at $u = \sigma$ —are now finite at that point. It follows that, when σ has just become real, there must be overlap in the neighbourhood of the point corresponding to $u = \sigma$, somewhat like that shown in Fig. 6.

If the change be carried further, the two roots σ, σ' —now both real—separate. There is no term in χ corresponding to either, and so, at each of the points $u = \sigma, u = \sigma', \frac{dz}{du}$ vanishes, like $(u-\sigma)$ and $(u-\sigma')$ respectively. Thus, at the point corresponding to each on the bounding curve in the z -plane, there is a cusp with its point turned inwards. In this way we get a succession of shapes such as those shown in Figs. 7 and 8.

The overlap will, in general, begin before σ has actually become real; and will not disappear until after σ and σ' have separated some distance.

Example III.—A stream escapes from between parallel walls and proceeds to infinity with free surfaces, which are ultimately straight and parallel. The stream is disturbed by a doublet at any point in its path.

FIG. 9.

Let the area of motion be conformally represented on the half-plane of u . The representation can always be effected so that the straight walls correspond to that segment of the real axis of u which lies between $u = -1$ and $u = +1$, while the point at infinity to which the stream proceeds corresponds to $u = \infty$. Let the point at infinity whence the stream comes correspond to $u = \gamma$ (see Fig. 9).

Suppose that the stream discharges an amount π of fluid per second. In the motion in the u -plane represented by w , there is a source at the point $u = \gamma$ discharging at the same rate as the stream. There is no other singularity on the real axis of u . There is also a doublet in the u -plane corresponding to that in the z -plane, say at the point $u = \tau$. We have, therefore,

$$\frac{dw}{du} = \frac{1}{u-\gamma} + \frac{m}{(u-\tau)^2} + \frac{m'}{(u-\tau')^2},$$

which may be written in the form

$$\frac{dw}{du} = \frac{(u-\sigma_1)(u-\sigma_2)(u-\sigma_3)(u-\sigma_4)}{(u-\gamma)(u-\tau)^2(u-\tau')^2}.$$

Now Ω is a function of u which becomes infinite when $u = \tau$ and when $u = \tau'$, and also at such of the roots $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ as do not lie on the real axis of u outside the limits ± 1 . Also, assuming that the velocity on the free surfaces is unity, Ω is a pure imaginary for real values of u outside the limits ± 1 , and is real if u be within those limits. There are, therefore, five classes of cases, distinguished from one another by the number of cusps appearing on the free part of the boundary, such cusps corresponding in each case to the excepted roots of $\frac{dw}{du}$. The form of the relation between z and u will be different for these several classes; but any one of the classes may be continuously developed into any other in the manner illustrated in the last example.

These classes are as follows:—

I. No excepted roots of $\frac{dw}{du}$.—In this case we have

$$\frac{d\Omega}{du} = \frac{1}{\sqrt{1-u^2}} \left[-\frac{\sqrt{1-\sigma_1^2}}{u-\sigma_1} - \frac{\sqrt{1-\sigma_2^2}}{u-\sigma_2} - \frac{\sqrt{1-\sigma_3^2}}{u-\sigma_3} - \frac{\sqrt{1-\sigma_4^2}}{u-\sigma_4} \right. \\ \left. + 2 \frac{\sqrt{1-\tau^2}}{u-\tau} + 2 \frac{\sqrt{1-\tau'^2}}{u-\tau'} \right],$$

where it is to be noted that the roots $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ may consist either of two conjugate complex pairs, one complex pair and two real roots, or four real roots. Thus, in this case there may be four points of rest on the boundary, or two on the boundary and one in the fluid, or two in the fluid. The form of Ω will be the same in all cases of the class.

II. One excepted root. Here

$$\frac{d\Omega}{du} = \frac{1}{\sqrt{1-u^2}} \left[-\frac{\sqrt{1-\sigma_1^2}}{u-\sigma_1} - \frac{\sqrt{1-\sigma_2^2}}{u-\sigma_2} - \frac{\sqrt{1-\sigma_3^2}}{u-\sigma_3} + 2\frac{\sqrt{1-\tau^2}}{u-\tau} + 2\frac{\sqrt{1-\tau'^2}}{u-\tau'} \right].$$

In this case there is one cusp on the free stream-line, corresponding to the excepted root. There are either two or three points of rest.

III. Two excepted roots, and two cusps on the boundary.—Either one or two points of rest.

IV. Three excepted roots and three cusps on the boundary. One point of rest, which must be on the boundary.

V. Four excepted roots.—In this case there are four cusps on the boundary, and we have

$$\frac{d\Omega}{du} = \frac{1}{\sqrt{1-u^2}} \left[2\frac{\sqrt{1-\tau^2}}{u-\tau} + 2\frac{\sqrt{1-\tau'^2}}{u-\tau'} \right].$$

The above equations are readily integrated by the substitution $u = \sin \theta$. If we put $\sigma_1 = \sin \theta_1$, &c., and $\tau = \sin \theta_0$, we shall get for the first class of cases

$$\frac{dz}{d\theta} = \frac{\cos \theta}{\sin \theta - \gamma} \frac{\cos^{\frac{1}{2}}(\theta + \theta_1) \cos^{\frac{1}{2}}(\theta + \theta_2) \cos^{\frac{1}{2}}(\theta + \theta_3) \cos^{\frac{1}{2}}(\theta + \theta_4)}{\cos^{\frac{1}{2}}(\theta + \theta_0) \cos^{\frac{1}{2}}(\theta - \theta'_0)},$$

and for the second class

$$\frac{dz}{d\theta} = \frac{\cos \theta}{\sin \theta - \gamma} \frac{\cos^{\frac{1}{2}}(\theta + \theta_1) \cos^{\frac{1}{2}}(\theta + \theta_2) \cos^{\frac{1}{2}}(\theta + \theta_3)(\sin \theta - \sin \theta_4)}{\cos^{\frac{1}{2}}(\theta + \theta_0) \cos^{\frac{1}{2}}(\theta + \theta'_0)},$$

and so for the other classes, until for the fifth class we have

$$\frac{dz}{d\theta} = \frac{\cos \theta}{\sin \theta - \gamma} \frac{(\sin \theta - \sin \theta_1)(\sin \theta - \sin \theta_2)(\sin \theta - \sin \theta_3)(\sin \theta - \sin \theta_4)}{\cos^{\frac{1}{2}}(\theta + \theta_0) \cos^{\frac{1}{2}}(\theta + \theta'_0)}.$$

These five classes will exhaustively cover every possible case.

The quantities $\sin \theta_1, \sin \theta_2, \sin \theta_3$, and $\sin \theta_4$ are the roots of the equation

$$(u-\tau)^2(u-\tau')^2+(u-\gamma) \left[m(u-\tau')^2+m'(u-\tau)^2 \right] = 0,$$

M 2

and are immediately expressible in terms of m, m', τ, τ' , and γ . In any given case the data are the coordinates and direction of the doublet with reference to the straight walls, the magnitude of the doublet, and the amount PQ by which one wall projects beyond the other. In working out such a case, it is necessary to proceed successively on the five assumptions that the case belongs to each of the classes enumerated above. For each such assumption values will be obtained for m, τ , and γ in terms of the data. Thus we have—

$$\text{strength of doublet} = \text{mod} \left\{ m \left(\frac{dz}{d\theta} \right)_{\theta=\theta_0} \right\},$$

$$\text{inclination of doublet} = \text{argument of } m \left(\frac{dz}{d\theta} \right)_{\theta=\theta_0},$$

$$\text{doublet is at point corresponding to } \theta = \theta_0,$$

$$PQ = \text{real part of } \int_{-\tau}^{+\tau} \left(\frac{dz}{d\theta} \right) d\theta,$$

which are just sufficient equations for the determination of m, τ , and γ ; and, consequently, of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$.

If the motion is a possible one, it will be found that on one of these assumptions, and one only, the distribution of the roots $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ is appropriate to that assumption. We then know that this is the correct assumption, and can proceed to work out the motion completely.

The effect of the transformation is to conformally represent the area of motion on that portion of the plane of θ which is shown in Fig. 1. The form of the free stream-lines is given by ascribing to θ values of the form $\pm \frac{\pi}{2} + i\chi$, where χ is real and positive.