SECT. II.—OTHER SELECTED PAPERS.

No. 1,455.—“On the Conditions of Resistance of Swing Bridges.”

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THE Author intends specially to investigate the conditions of resistance of swing bridges, but considers it advisable, in the first instance, to make some general observations on the types commonly employed, illustrating them by examples.

There are several varieties of movable bridges. For instance, there are bridges of boats, and platforms supported by floating timbers, thus avoiding the difficulties of foundations. Sometimes a single bay is made movable so as to open a passage for vessels navigating the river; in some rare instances, the whole of the construction can be removed and placed in safety in a dock during the breaking up of ice, or in charge of the states bordering on the river in the event of war. Again, dock caissons serve as ordinary fixed bridges when once in place, and they are readily removed each time that the entrances which they close have to be opened. There are also ferry bridges, which, shorter than the space to be traversed, have to convey the load across; train ferries, steam ferries, and floating railways. In a few instances the ferry bridge rests on the bed of the stream instead of floating, and moves on rollers; of this the roller bridge of St. Malo¹ is an example. Another type of bridge has a platform resting on fixed supports, which slides out of place to allow vessels to pass, or to stop the way; but this form is rarely adopted now, on account of the difficulty of working, and of the inconvenience of the shifting platform, which is usually required for covering over the gap left in the bridge recess, sliding backwards or forwards like a drawer, and guided by rollers. At the Leven bridge,² the movable bay is lowered and drawn under the fixed platform. In other cases the

¹ “*Annales des Ponts et Chaussées*,” July 1874.

² *Vide Minutes of Proceedings Inst. C.E.*, vol. xvii., p. 442.

movable bay is withdrawn by raising it vertically, and is balanced by weights descending into wells, so that a considerable mechanical work for lifting it is dispensed with, of which the vertical lift bridge over the Grand Surrey Canal¹ is an instance. Again, movable bridges rise vertically in turning on a horizontal axis, such as the wooden drawbridges of ancient fortresses, old bridges on canals, and counterpoised bridges, such as the one at Hull, with a tail-end, serving as a balance-weight, which sinks into an underground chamber whilst the drawbridge rises; the underground recesses, however, require to be kept dry by frequent pumping, and the drawbridges sometimes, when only partially raised, catch the yards of passing vessels. Lastly, other bridges turn round a quarter of a circle on a vertical axis, so as to come parallel to and within the line of the side-walls of the canal each time that a vessel requires to pass. This is the type of bridge generally adopted on canals and in docks, as, if properly constructed, only a small amount of labour is required for turning them; and the beams of a drawbridge and the well of a counterpoised bridge are dispensed with. It is of this latter class of bridges alone that the Author proposes to treat in the present communication; and even amongst these there is a certain diversity of type. On rivers whose breadth considerably exceeds the requirements of navigation, and where a central pier does not present too great an obstacle, swing bridges with double spans are naturally erected, affording two equal openings to vessels passing either way (Plate 5, Figs. 1, 2, and 3). On the other hand, in harbour entrances or canals the width is too nearly adapted to the requirements of the case to allow of this double opening, entailing as it does a central pier, and the entire space has to be set free for vessels. But this can be accomplished in two ways: either by adopting a bridge in one span, turning on one of the abutments by means of a single gearing, or by making the bridge in two half-bays joining in the centre of the span, each portion being turned on either bank by an independent gearing. This last plan is the most complicated, but it may be necessary to keep down the weight of the girders if the span is very great, as in the case of the bridge at Brest (Plate 5, Fig. 7), or for shortening the girders if the available side quay space is too small to receive a long bridge when open; or for the rotation in a quadrant of a proportionately long tail-end.

The material employed in the construction will, moreover, somewhat influence the choice of design. Wrought iron, from its com-

¹ *Vide* Minutes of Proceedings Inst. C.E., vol. ix, p. 303.

parative lightness in proportion to its strength, admits of spans which, with cast iron or wood, would be difficult of attainment. The Americans, indeed, do not hesitate to construct bridges of large span with wood;¹ but wrought iron is taking the place of cast iron and wood, especially for the structures under consideration, which, more than any others, require strength combined with lightness. Largeness of span is generally now considered desirable, and with this object in view, it has been suggested that the substitution of steel for wrought iron would constitute a fresh step in advance. On the canals in Holland, in olden times, numerous wooden drawbridges existed, but the injuries caused by shocks, and the wear and tear of working, rendered frequent repairs necessary, and this, added to the increasing difficulty of procuring timber large enough for the principal beams, has led to the repairs being now made with iron.

Whilst the swing bridges with two spans (Plate 5, Figs. 1, 2, and 3) balance themselves when open on account of the symmetry of their construction on each side of the central point of support; swing bridges over a single opening, whether the space is crossed by a single span, or by two half-spans from opposite sides, must be balanced on the other side of the axis of rotation by a prolongation called a tail-end. Sometimes the tail-end is as long as the portion over the opening (Plate 5, Fig. 4); but generally it is shorter, to economise materials and space, and then it must be loaded with kentledge. A weight of kentledge just sufficient to counterbalance the excess of weight of the opening span suffices where the opening span is in one piece, so that when the bridge is closed it rests on the farther abutment, and is therefore properly supported when subjected to the rolling load, and the bridge, whilst being swung, is secured from rocking by a ring of rollers. If, however, each girder spans only half the opening, or if, on account of the nearness of the pivot to the face of the wall, or for any other reason, it is not possible to guide the bridge by a ring of rollers, it is advisable to put on an excess of counterpoise in order to secure the bridge against overbalancing whilst being swung, or when subjected to the maximum load, and to insure a certain amount of pressure on the rollers, at the extremity of the tail-end of the bridge, guiding the motion. The pivot and the rollers are merely employed as means of turning the bridge, and consequently, when this function has been discharged, the tail-end is lifted by wedges or screws, so that the bridge may be raised off the

¹ *Vide* Minutes of Proceedings Inst. C.E., vol. xxii., p. 305.

rollers, and to a sufficient extent to insure a perfect bearing on the farther abutment. The pivot may be used as a support, but generally it is better to make the bridge rest on bearings nearer the edge of the abutment, so as to diminish the length of the span (Plate 5, Fig. 6).

Many bridges are worked solely by manual labour, as there is only the friction of the machinery, the sliding friction on the pivot, and the rolling friction of the supporting and guiding rollers to overcome. When, however, the weight of the bridge is considerable, and the roller ring large, the friction of the rollers and of the complicated rack movements is so much increased, that the rate of rotation has to be reduced to render the working possible. The sliding friction is small, on account of the very small space through which the pivot moves, and by dispensing with rollers, and concentrating all the weight of the carefully balanced bridge on the pivot alone, which rests on a socket, A, and passes through a cast-iron bearing girder, B (Plate 5, Fig. 4), the bridge can be worked, without gearing, by one man, a great advantage being gained by the large amount of leverage, and the friction being considerably reduced by being confined to the central pivot. Large bridges require rollers to maintain them in position; but as these bear very little of the weight, they interfere but slightly with the movement of the bridge. In harbours with wide entrances, and where the constant traffic renders rapidity of motion indispensable, swing bridges, besides being supported on the pivot, are moved by hydraulic machinery devised by Sir William Armstrong, Vice-President Inst. C.E. This system of working has been adopted in England for the swing bridges at the Regent's Canal, the Millwall Docks, and Woolwich Arsenal, weighing respectively 450, 500, and 410 tons; and in France for the swing bridge at Marseilles,¹ weighing 700 tons, which can be opened in three minutes.

The bridge at Brest,² which cost £120,000, is worked solely by manual labour, turning on a circular bearing which rests on a solid support (Plate 5, Fig. 7). The span of the bridge is 385 feet 6 inches from centre to centre of the piers; it is divided into two half-spans, each balanced by a tail-end, 93 feet 10 inches long, weighted with kentledge of masonry. The cast-iron drum rotates on conical cast-iron rollers, and the bridge is turned by a rack

¹ *Vide* "Annales des Ponts et Chaussées," May 1875, p. 413; also Minutes of Proceedings Inst. C.E., vol. xlii., p. 305.

² *Vide* "Annales des Ponts et Chaussées," November 1867.

arrangement working on the drum, and is moved by a capstan, of which there are two for each portion of the bridge, one only being used except in windy weather. The process of unwedging, lowering the screws, and releasing the bridge from its fastenings, and, lastly, turning it round, occupies nine men for a quarter of an hour: the mere operation of turning it round is performed by two men at each bay in eight minutes, but in stormy weather a double gang is required. The weight of the movable portion of each span being 350 tons, the friction of the rollers being reckoned at 4.4 lbs. per ton, and the distance traversed (which is a quarter of the circumference of a circle of $29\frac{1}{2}$ feet diameter) being 23 feet, the work performed is $350 \times 4.4 \times 23 = 35,420$ foot-lbs.; doubling this quantity, to allow for the friction of the machinery, and dividing by 8, the duration of the work of rotation in minutes, the result is 8,855 foot-lbs. of work done by two men in one minute. In comparing this with the work a man can perform for a short period, it must be remembered that the resistance of the wind has to be taken into account. The weight of the bridge is removed from the rollers by means of four screws acting under each drum. The top of the lever for turning the bridge bends down with a hinge, so that it can be lowered under the platform (Plate 5, Fig. 7, cross section x). The cylinders of the hydraulic press are so arranged as to allow of the removal and renewal of the rollers when worn down; and at the extremity of the tail-ends of the bridge there is a contrivance for firmly grasping the masonry of the abutments.

The bridge cannot be regarded as an arch subjected to longitudinal pressure, as the contact of the extremities of the girders in the centre of the span is imperfect, and as the centre is too depressed, and there is no construction at the springing of the arch calculated to resist the longitudinal strain. The girders must therefore be regarded as long brackets, having to support the rolling load in addition to the dead weight; and the fastening of the extremities of the tail-ends to the masonry of the abutments is, accordingly, of great service in bearing a portion of the strains which tend to lift up the shore ends of the bridge.

The bridge at Brest was opened in 1861; it carries a roadway 16 feet 5 inches wide, and footpaths 3 feet $3\frac{1}{2}$ inches in width on each side.

The bridge over the great lock (100 feet wide) at Havre (Plate 5, Fig. 6) may better be regarded as an arch than the bridge at Brest, but the difficulty of placing an abutment with sufficient accuracy for the three girders to butt against simultaneously, prevented this

method of resisting the strains being taken into account, and the tail-ends are merely raised sufficiently to remove the weight of the bridge from the pivots, and to cause the bridge to rest on the edges of the abutments. The traffic on the bridge in each direction is separated by a central division along the roadway, by which all chance of collision is removed.

The bridge at the Marseilles repairing docks¹ (Plate 5, Fig. 8) has a total length of 209½ feet, and is 46 feet wide: three trellis-work girders, with curved top flanges, carry the bridge; and on one side of the central girder there is a single line of rails, and on the other side a roadway, with a footway supported on brackets outside the outer girder: the length of the portion of the bridge crossing the opening (91 feet 10 inches wide) in a single span is 126 feet. The bridge is raised and turned on an hydraulic pivot, the apparatus of which can be removed, repaired, and replaced in a very short time without disturbing the bridge, so that all chance of stoppage in moving the bridge is avoided. The three main girders are supported on a large transverse bearing girder, resting on the Bramah press, which serves as the axis for lifting and rotation. Each girder carries a roller under it near the extremity of its shore end: the three rollers are in a line parallel to the bearing girder, so that the bridge may rest equally on all three when slightly raised. This arrangement necessitates two circular roller paths, whose radii are 64 feet and 67 feet 7 inches respectively, measured from the pivot as the centre. The tail-end is secured by wedges, moved by a small special hydraulic cylinder. A circular cast-iron ring, 46 feet in diameter, with a groove in its outside face, is fixed under the bridge concentric with the pivot. In the groove a chain is placed, worked in opposite directions by two hydraulic cylinders, one serving for opening and the other for closing the bridge.² All the hydraulic engines are supplied with water under pressure from the engine and accumulator of the docks, 1,530 yards distant.

In the operation of turning the bridge, the tail-end is first released sufficiently from the wedges for the rollers to rest on the roller paths; the press then lifts the bridge, so that the far end of

¹ *Vide* "Annales des Ponts et Chaussées," May 1875, p. 413; also Minutes of Proceedings Inst. C.E., vol. xlii., p. 305.

² There is a single cylinder at the bridge over the Ourcq canal at Villette (Paris), in which a piston works with a rod on each side, to which is fastened an end of the chain for turning the bridge, the ends of the chain cross and work on return pulleys. The water for the cylinder is obtained from the reservoirs of Ménilmontant at a pressure of about 4 atmospheres.

the bridge is raised from its supports, and the cylinders attached to the chain which works in the groove of the circular ring are set in motion and turn the bridge. This is all done by one man in three minutes, however hard the wind may be blowing; whereas for the Havre bridge, weighing only 120 tons, four men are required to push the tail-end.

The kentledge of the Marseilles bridge is so placed as to throw 15 tons, out of the total weight of 700 tons, on the guiding rollers, to insure the stability of the bridge whilst being swung. The press, which is of wrought iron, 1 foot 11 inches internal diameter, and 6·3 inches thick, is fastened by keys into a cast-iron case, from which it is readily withdrawn for repairs as soon as the pivot is released from the water-pressure.

The prismatic top of the piston of the press is inserted in a bearing plate fixed to the bottom of the bearing girder, which has three web plates: its surface of contact is made slightly convex, or with a very obtuse bevel, so that the bridge may always have a good bearing on the axis, notwithstanding slight displacements in the process of raising. A horizontal sector is fastened to the head of the piston, which rests against two rollers with light movable axles, supported by a cast-iron bracket to counteract the lateral strain caused by the chains in turning the bridge. It was impossible to place rollers all round the piston, as it requires a certain amount of play during the wedging up of the bridge.

The ordinary water-pressure at the Marseilles docks, amounting to 52 atmospheres, is amply sufficient for working the cylinder, 13·8 inches in diameter, for wedging up the bridge; and also the two conjugate pistons, for swinging the bridge, whose diameters are 11·8 inches, and their length of stroke 9 feet 2½ inches, which is supplemented by a system of multiplying pulleys. In the case however, of the press for lifting the pivot, which, with a diameter of 1 foot 11 inches, has to support a weight of 700 tons, the water-pressure required is equal to 250 atmospheres; this increase of pressure is obtained by a double-acting force-pump which, receiving a supply of water at the ordinary pressure of 52 atmospheres, raises it to five times that pressure. M. Barret, the engineer who designed the bridge, considers that, owing to the great advance in metallurgy and casting, it would be feasible to make the press with an internal diameter of 4 feet 11 inches, and thus dispense entirely with auxiliary apparatus for increasing the normal water-pressure.

The leather lining makes a water-tight joint with the piston; but in order to prevent its tendency to tear from the turning

strain imparted by the adherence of the rotating pivot, the interstice between the edges has been fitted with a band of india-rubber, which, by the interior adhesion it gives to the opposite edges of the leather, causes its exterior surface to slide on the metal. When in the act of turning, the motion of the bridge becomes a little too accelerated; the moment of inertia of the large mass is only partially restrained, owing to the imperfect elasticity of the hydraulic cushion, and consequently it was found necessary to modify the slide-valves regulating the water supply, so that, whilst not liable to be opened by each stroke of the ram, a regulating counter-pressure might obviate the jerks that would otherwise occur.

The Author is inclined to think that the employment of the hydraulic machines for wedging up the bridge is superfluous, and that the fall of the pivot, after it has been raised for lifting the bridge from its supports at its extreme end, might be utilised to insure the proper pressure of the bridge on three points of support, fixed once for all at suitable levels. This is, indeed, the opinion of M. Barret, for he says that if he had to construct another similar bridge he would dispense with the apparatus and cylinder for wedging up the bridge; that he would retain the three rollers, weighted with the slight load of 5 tons apiece; that he would make the roller supports movable in a vertical line, and so balanced with counterpoises that the rollers should always be kept in contact with the roller path, whilst the bearing girder lifts both extremities of the bridge off their supports. The top of the pivot, instead of being rounded off, could be made quite flat; and, in place of only two horizontal rollers to steady the pivot on the side where the pull of the turning chains takes place, there could be a set of rollers encircling the pivot. M. Barret considers that by introducing the improvements suggested, a bridge over 2,000 tons in weight might be turned with perfect safety in three minutes.¹

The examples referred to suffice to illustrate, in a general way, the types of swing bridges usually constructed at the present day.

The Author now proposes to investigate the conditions of resist-

¹ The Marseilles bridge rests on its pivot by superposition, a simpler system than that followed by the Dutch engineers, who hang their bridges by the cast-iron pin, placed over the pivot, by means of a circular row of large and strong bolts. At the Dordrecht bridge, there are only three bolts, of steel, being not less than 7 inches in diameter.—*Vide* "Notice sur les Travaux publics en Hollande," par Ph. Croizette Desnoyers. Paris, 1874.—J. G.

ance of swing bridges. In the first place, it is necessary to draw a distinction between the swing bridges having two half-spans, and those formed in one piece. Whereas the latter when closed have a firm bearing on the farther abutment, the two spans of the others only reach the centre of the opening, and even when firmly butting against each other can only afford a horizontal reaction, which is injurious in a straight beam, and could only be of service if the girders, made in the form of an arch, rested at their springing upon very solid abutments.

The bridge formed with two spans shall first be considered.

In the absence of a mutual support at A (Fig. 1), each span,

FIG. 1.

B A, must be regarded as a simple bracket, fixed to the side-wall by reason of being counterbalanced by the tail-end, B C, carrying the counterpoise. The counterpoise at C, having to keep in equilibrium the bridge when weighted with the maximum load, as well as the bridge itself, generally throws a considerable excess of weight on the rollers on which the bridge turns, and consequently increases the work of swinging the bridge. At Brest the case is different, for the movable load is counterpoised by fastening the tail-end to the solid masonry when the bridge is closed, and in turning the bridge the balancing of the dead weight alone has to be dealt with.

The theory of the resistance of two half-spans not butting against each other presents no difficulty. If the pivot, P, is one of the points of support of the bridge when closed, it is evident that the maxima moments of flexion are necessarily produced by the action of the full load; but as it is generally advisable to remove the weight off the pivot and rollers by lifting the extremity, C, of the tail-end with wedges or screws, to bring the bearing on to B, the edge of the side-walls, the length of the bracket is reduced to B A when loaded, and is P A when unloaded and being swung round; and it accordingly may happen that the reduction in length may more than compensate for the increase in weight.

Let q be the load for a unit of length on the portion B A of the bridge whose length is l , and q' on the portion B C whose length is l' . The maximum bending moment at B will be $\frac{q l^2}{2}$ with the

bridge closed, and the variable moment μ for a section M, with abscissa x and B the origin of co-ordinates, will be $\mu = \frac{q}{2} (l - x)^2$.

The moment of the portion B C supported only at B and C must, in order to maintain equilibrium, be equal to $\frac{q l^2}{2}$, and for this purpose a counterpoise is usually added. This condition being fulfilled, the equation for the moment μ' of the portion B C, the abscissa x' having its origin at B and being measured along B C, will be

$$\mu' = \left(q \frac{l^2}{2} - q' x' \right) \frac{l' - x'}{2}.$$

The reaction at C is $\frac{q' l'}{2} - \frac{q l^2}{2 l'}$, neglecting the counterpoise, the object of which is to prevent the reaction ever becoming negative. The force required for wedging up the bridge will be this reaction, with the dead loads substituted for q and q' , increased by the weight of the counterpoise. Similar formulæ would give the resistance of the bridge when open, the origin of co-ordinates being transferred from B to P. The greatest strains being produced at B and P, the girder must be strengthened at these points and lightened at the ends, especially as this form of equal resistance will not affect the distribution of the forces directly the bridge rests on only two supports.

Considering briefly the case where the horizontal thrust at the centre of the span when the bridge is shut is utilised by treating the bridge as an arch, it must be assumed that the under side of the bracket, B A (Fig. 1), is formed in a half arc of sufficient rise, and that the lifting at C not only releases the pivot and rollers, but turns over the part B A on the support B, at the edge of the side-wall, till the requisite thrust is produced at the extremity, A, of the half-span. The bridge will thus be placed in the condition of an iron arch, provided the side-walls resist the thrust; and the portion B C must be regarded as a girder fixed to the masonry at B and merely supported at C. The object of raising the bridge at C is to develop the thrust due to the dead weight. Let w be the weight for a unit of length, then the thrust is given by the simple approximate formula $\frac{w l^2}{2 r}$, r being the rise of the arch:

the force required at C to produce it is $\frac{q l^2}{2 l'}$ in addition to the force which sufficed in the preceding case. In fact, the counter-

poise and half the weight of the portion B C, turning on the support B, has to be lifted just as if the portion B A was not there, for the moment tending to overturn this latter portion is supported by the counter-thrust of the other half-span. To obtain a thrust against the side of the abutment hinged struts have been sometimes employed (Plate 5, Fig. 5), which are let down by chains when the bridge is closed, so as to make them rest against their bearings, but this system appears to have been abandoned as too complicated an operation for frequent adjustment.

RESISTANCE OF A SWING BRIDGE OF SINGLE SPAN.

By the term bridge of single span is intended a bridge crossing the entire opening in one piece, and it accordingly includes the case of symmetrical bridges having two spans balancing each other and crossing two openings, as each of these spans may be considered as the tail-end of the other. In these bridges the central support is exactly in the centre of the bridge, and coincides with the pivot on which the bridge turns, and evidently is a particular instance of the more general case in which the bridge does not rotate on its central point and requires a counterpoise to adjust it, and where the intermediate support on which the bridge rests when closed is at a certain distance, δ (Fig. 2), from the pivot.

FIG. 2.

When the bridge is open and placed on the side-wall it can be supported by a slight wedging at any point, as the strains are at their minimum. During the rotation, however, the bridge is entirely supported on the pivot, assisted in some cases by the ring of rollers. The bridge having the span $l + \delta$, and weighing w per unit of length, is subjected, at the point whose abscissa is $\delta + x$ (taking the pivot as the origin, and x the distance from the point of support on the side-wall), to the bending moment

$\mu = \frac{w}{2} (l - x)^2$, the equation to a parabola whose axis is vertical,

whose parameter is $\frac{2}{w}$, and its vertex at the extremity of the span.

The maximum moment at the pivot, where $x = -\delta$, is $\frac{w}{2}(l+\delta)^2$.

With regard to the tail-end, it is of primary importance to weight it sufficiently to prevent the bridge upsetting whilst being swung. If the length of this portion was precisely $l' - \delta$, and the kentledge could be concentrated in one point at the extreme end, and assuming the tail-end to weigh the same per unit of length as the rest of the bridge, the weight of this kentledge would be $\frac{w}{2} \frac{(l+\delta)^2 - (l' - \delta)^2}{l' - \delta}$, and consequently the equation of the variable bending moment, with abscissa x' measured from the side-walls, is $\mu' = w \frac{(l+\delta)^2 - (l' - \delta)(x' - \delta)}{2(l' - \delta)}(l' - x')$, which is a parabola of less curvature than the preceding one, but with the same parameter $\frac{2}{w}$, and a vertical axis.

In practice the kentledge is generally spread over a certain length $a + \beta$ (Fig. 2) on each side of the supporting rollers. Let g be the weight of the kentledge per unit of length, then, the length of the tail-end being $l' - \delta + \beta$, the weight is w per unit for a length $l' - \delta - a$, and $w + g$ for the remaining length $a + \beta$. The minimum value of one of the quantities g, a, β , is given by the equation

$$\frac{w}{2}(l+\delta)^2 = \frac{w}{2}(l' + \beta - \delta)^2 + g(a + \beta)\left(l' - \delta + \frac{\beta - a}{2}\right)$$

the condition of equilibrium for rotation on the pivot, the value of the other two being determined by the requirements of the design. The value of one of the three is always somewhat increased to insure a pressure on the rollers in the process of swinging. The bending moment of the tail-end of the bridge is represented by two parabolic arcs (Fig. 3).

FIG. 3.

From the extreme end where $x' = l' + \beta$ to the point where $x' = l' - a$ the equation is

$$\mu' = \frac{w+g}{2} (l' + \beta - x')^2,$$

and from this latter point to the pivot, where $x' = \delta$, the equation is

$$\mu' = \frac{w}{2} (l' + \beta - x')^2 + g(a + \beta) \left(l' - x' + \frac{\beta - a}{2} \right).$$

The greatest strains on the portion of the bridge near the pivot are generally produced under the conditions just considered when the bridge is being opened and supports merely its dead weight; but other parts of the girders are subjected to greater strains when the bridge is closed, wedged up and loaded.

The bridge when closed forms a continuous beam over two openings, l and l' , with an overhanging piece of length β attached to the tail-end. The strain on this piece is very slight, on account of its shortness, but it produces a moment at the point of support which tends to press down the tail-end as if it were fastened to the bearing.

The Author proposes to analyse successively the strains produced by the various forces acting on the bridge; and, assuming the three points of support of the mean fibre to lie in a straight line, to calculate, first, the bending moments due to the dead weight w per unit of length and the counterpoise g ; next, the moments produced by the moving load; then the changes produced by the alterations in level caused by bringing the bridge on to its proper bearings; and, lastly, by a combination of the various forces simultaneously acting, to obtain the actual strains at each point of the girder considered as having the same section throughout.

The permanent load on the bridge is w per unit of length for the portions l and l' (Fig. 4), and, neglecting the counterpoise on

FIG. 4.

the portion a , $w+g$ for β . The moment at the point C is

$\mu_2 = \frac{w+g}{2} \beta^2$; then by the theory of moments (the moment at A being zero) $\mu_2 l' + 2 \mu_1 (l + l') = w \frac{l^3 + l'^3}{4}$, or $\mu_1 = w \frac{l^2 - ll' + l'^2}{8} - \frac{l' \beta^2 w + g}{4(l + l')}$; and for any points along l and l' , at distances x and x' respectively from B, the moments will be

$$\mu = \left(\frac{\mu_1}{l} - \frac{wx}{2} \right) (l - x), \text{ and } \mu' = \frac{\mu_2 x'}{l'} + \left(\frac{\mu_1}{l'} - \frac{wx'}{2} \right) (l' - x').$$

The reactions at A, B, and C are

$$\begin{aligned} & w \frac{3l^2 + ll' - l'^2}{8l} + \frac{\beta^2 l' w + g}{4l(l + l')}; \\ & w \frac{l + l'}{8} \left(3 + \frac{l}{l'} + \frac{l'}{l} \right) - \frac{w + g}{4ll'} (2l + l') \beta^2; \\ \text{and } & (w + g) \beta \left(1 + \frac{\beta}{2l'} + \frac{\beta}{4(l + l')} \right) + w \frac{3l'^2 + ll' - l^2}{8l'}. \end{aligned}$$

The weights, g , acting on each unit of length of a produce no moment at C (Fig. 5), but only at B; let μ_1 be the moment, and

FIG. 5.

$g du'$ the weight on an infinitesimal length at a distance u' from B so that $\mu_1 = \int g du' u' \frac{l' - u'}{2l'} \frac{2l' - u'}{l + l'}$; and taking the sum between the limits $l' - a$ and l' the moment is $g a^2 \frac{2l'^2 - a^2}{8l'(l + l')}$. The locus of the moments along l is a straight line, and along l' is straight from B to $l' - a$ from B, and parabolic from that point to C (Fig. 5).

The reactions at A, B, and C, due to the weight g along a , are

$$\begin{aligned} & -g a^2 \frac{2l'^2 - a^2}{8ll'(l + l')}; \quad g a^2 \frac{4ll' + 2l'^2 - a^2}{8ll'^2}; \text{ and} \\ & g a \left(1 - \frac{a}{8l'^2} \frac{6l'^2 + 4ll' - a^2}{l + l'} \right). \end{aligned}$$

The total moment, M , due to the dead load when the bridge is closed, obtained by combining the above results, is

$$\frac{1}{8(l+l')} \left\{ w(l^3 + l'^3) - 2(w+g)l'\beta^2 + g a^2 \frac{2l'^2 - a^2}{l'} \right\};$$

and the moment at any point at a distance x from B is

$$\left(\frac{M}{l} - \frac{wx}{2} \right) (l-x);$$

and where $x = \frac{2M}{wl}$ there is a point of inflection, beyond which point the moment becomes negative. Along the portion l' the moment due to the permanent load when the bridge is closed is equal to

$$\left(\frac{M}{l'} - \frac{wx'}{2} \right) (l' - x') + \frac{(w+g)\beta^2 - g a^2}{2l'} x'$$

between B and a distance $l' - a$ from B, and beyond this it becomes

$$\left\{ \frac{M}{l'} - \frac{wx'}{2} + g \frac{(l' - a)^2 - l'x'}{2l'} \right\} (l' - x') + \frac{w+g}{2l'} \beta^2 x'.$$

The strains due to the weight of the moving load on the bridge when closed must next be investigated, and combined with those due to the permanent load. The bending moments produced by the moving load are sometimes positive and sometimes negative, according to the position of the load. In order to ascertain what parts of the bridge must be loaded to obtain, at any given point, a positive or negative moment, the effect of a weight, P , acting at a single point on one or other of the two spans l and l' , must be considered, neglecting the small portion of the load that might affect the overhanging end β .

When two weights, P and P' , are placed on the two spans, l and l' , of a beam resting on three supports at the same level, the weights being applied at distances u and u' respectively from the intermediate support, a moment is produced which presses down the beam on this support, and whose value is

$$\frac{1}{2(l+l')} \left\{ P u (l-u) \frac{2l-u}{l} + P' u' (l'-u') \frac{2l'-u'}{l'} \right\} = M_1;$$

and at any point x along the length u of the span l the moment is represented by the equation

$$\mu = M_1 \frac{l-x}{l} - P \frac{l-u}{l} x;$$

and from the point of application of P to C (Fig. 6) by the equation

$$\mu = \frac{l-x}{l} (M_1 - Pu).$$

These are the equations to the two straight lines AB and BC (Fig. 6). It is evident that M_1 , the positive moment at the

FIG. 6.

intermediate support, increases with P and P' wherever they are placed along l and l' ; and therefore the maximum strain at this point is produced by fully loading both spans, which makes $M_1 = w' \frac{l^2 - ll' + l'^2}{8}$, where w' is the moving load per unit of length.

It also follows that to obtain the greatest positive value of μ at any other point where $x < u$ the span l' must be always fully loaded, since M_1 is thus increased without the introduction of negative terms; but, as regards the load on the span l , for a given point at a distance x from the intermediate support the greatest effect in increasing the positive value of μ is produced by loading l from its extremity to a point where $u \geq l - \sqrt{l^2 - 2lx \frac{l+l'}{l-x}}$; and when $x \geq \frac{l^2}{3l+2l'}$ the span l must be entirely unloaded to produce the maximum effect.

The intermediate point of support, where $x = 0$, is the only point where the moment is a maximum with both spans fully loaded.

To obtain the maximum negative value of μ at a point x the span l' must be unloaded, and the span l must be loaded between the points $u = 0$ and $u = l - \sqrt{l^2 - 2lx \frac{l+l'}{l-x}}$, where x is less than $\frac{l^2}{3l+2l'}$, but for points beyond this distance l must be fully loaded.

The same observations are applicable to the exactly parallel case of the span l' , and the same formulæ by transposing l and l' .

From the foregoing considerations it appears that the greatest positive moment produced by a load w' per unit of length, at a

point in the span l at a distance x from the origin, where the value of x is between zero and $\frac{l^2}{3l+2l'}$ is obtained by taking the sum of the moments produced by the weights $w' du$ within the limits $u = l$ and $u_1 = l - \sqrt{l^2 - 2lx \frac{l+l'}{l-x}}$, and by the weights $w' du'$ distributed all along l' , which give

$$\mu = \frac{w'}{l} \int_{u_1}^l \left(u \frac{2l-u}{2l} \frac{l-x}{l+l'} - x \right) (l-u) du + \frac{w'}{2l'l} \frac{l-x}{l+l'} \int_0^{l'} u' (l'-u') (2l'-u') du',$$

and integrating,

$$\mu = \frac{w'}{2} \left(\frac{l^2 - ll' + l'^2}{4} \frac{l-x}{l} + \frac{l+l'}{l-x} x^2 - lx \right).$$

When x is greater than $\frac{l^2}{3l+2l'}$ the span l' alone must be loaded to obtain the greatest positive moment, which is $\mu = \frac{w' l'^3}{8l} \frac{l-x}{l+l'}$ the equation to a straight line which starts as a tangent from the preceding curve at the point $x = \frac{l^2}{3l+2l'}$.

As the positions of the loads which produce the greatest negative moments are exactly supplementary to those which give the greatest positive moments, the negative maxima may be obtained by deducting the two values just obtained for μ from the value of μ when the spans are fully loaded, which is $w' \frac{l-x}{8} \left(l - l' + \frac{l'^2}{l} - 4x \right)$, and the resulting negative maxima moments are $\mu = -\frac{w' x^2 l' + x}{2} \frac{l+x}{l-x}$, when the value of x is between zero and $\frac{l^2}{3l+2l'}$, and $\mu = w' \frac{l-x}{8} \left(\frac{l^2}{l+l'} - 4x \right)$ between this latter point and $x = l$. The greatest negative moment is at the point where $x = \frac{l}{8} \frac{5l+4l'}{l+l'}$, and its value is $-\frac{w'}{2} \left(\frac{l}{8} \frac{3l+4l'}{l+l'} \right)^2$.

To obtain corresponding formulæ for the span l' , it is only necessary to transpose l and l' .

The effects of the load on the overhanging portion β have not been taken into consideration, but they would be small, and where producing any appreciable result would tend to diminish the strains on the bridge.

The results obtained for the permanent and moving loads might be combined, but the Author proposes merely to do this in a special example, as for the more general case the strains can be more simply represented by the graphic method; and moreover the curves showing the strains on the bridge when closed and when open, can be put on the same diagram.

The arbitrary strains due to wedging up the bridge remain to be considered. M. Bresse has stated¹ the general theorem of three moments, allowing for the alterations in level, that the supports may experience, but it is easy to obtain a special formula suitable to this particular case.

Let A B C (Fig. 7) be a beam spanning two openings and

FIG. 7.

originally straight, but depressed at the ends A and C in a vertical direction through distances h and h' , which depression produces a bending moment m at B, and a reaction F at A, and consequently at a point x in the span l , $\epsilon \frac{d^2 y}{d x^2} = -F(l-x) = m \frac{l-x}{l}$, since the bending moment must equal m when $x = 0$. In the equation, ϵ is the product of the modulus of elasticity and the moment of inertia of the cross section, and the second differential co-efficient $\frac{d^2 y}{d x^2}$ of the curve of flexion expresses approximately the curvature of the mean fibre.

Integrating the equation, it becomes $\epsilon y = m x^2 \frac{3l-x}{6l} + cx$; and since $y = h$ when $x = l$ $c = \epsilon \frac{h}{l} - \frac{m l}{3}$, which gives the inclination of the mean fibre at B; and for the span l' the expression is $c' = \epsilon \frac{h'}{l'} - \frac{m l'}{3}$, and since the beam is continuous the inclination at B must in both cases be a tangent to the curve of deflection, and therefore $\epsilon \left(\frac{h}{l} + \frac{h'}{l'} \right) = m \frac{l+l'}{3}$, and supposing all the displacement to occur at C it becomes $\epsilon \frac{h'}{l'} = m \frac{l+l'}{3}$.

¹ Vide "Mécanique appliquée." M. Bresse. Vol. iii., p. 6.

When C is raised a height H' a negative moment m is produced at B which reduces the positive moment at B, caused by the load, by an amount $\frac{3 \epsilon}{l'} \frac{H'}{l+l'}$. This involves a special lifting force at C, whose value is $\frac{3 \epsilon}{l'^2} \frac{H'}{l+l'}$, in addition to the reaction at this support when kept level with the two other supports.

The reaction at C due to the dead weight, which is the only load acting before it is raised, is

$$Q' = \frac{w l'}{2} + (w + g) \beta + g \alpha \frac{2 l' - a}{2 l'} + \frac{\mu_2 - \mu_1}{l'},$$

where μ_1 and μ_2 are the moments due to the permanent load.

In raising the extremity C it is first necessary to exert a force Q' , by means of the wedges or screws, which causes the deflection near the pivot to disappear, and then an additional pressure, $\frac{3 \epsilon}{l'^2} \frac{H'}{l+l'}$, to raise it the height H' . The total distance through which C is raised is $H' + Q' l'^2 \frac{l+l'}{3 \epsilon}$, or calling the total force exerted Q'' the total distance traversed is $l'^2 \frac{l+l'}{3 \epsilon} Q''$.

The bending moments caused by the wedging up are represented by the straight lines A D and C D (Fig. 8), the ordinate B D representing the negative moment m at B due to raising C through a height H' . The lifting of C produces a considerable diminution

FIG. 8.

in the strains near the central support when the bridge is loaded, and allowing that it is advantageous that the strains at that part should be less when the bridge is loaded than when being swung, a minimum value can be inferred for m , and consequently for H' , and also Q'' which is equal to $Q' + \frac{m}{l'}$. The highest limit of the

value of m is $(w + w') \frac{l^2 - ll' + l'^2}{8} - \frac{l' \beta^2 w + g}{4} \frac{w + g}{l+l'} + \frac{g \alpha^2 2l'^2 - a^2}{8 l' \frac{l+l'}{l+l'}}$,

the value of the positive moment of the bridge when closed and its supports at the same level, as if this limit was passed the extremities of the bridge would tend to bend upwards.

If the extremity A was raised instead of C the pressure exerted would be less, but the distance traversed would be greater, so that there would be no saving in mechanical work.

Taking a special case where the length $a + \beta$, over which the counterpoise is distributed, is very short, and where the distance, δ , between the pivot and the intermediate support is equal to zero,¹ the counterpoise $g(a + \beta)$ concentrated on the support of the tail-end has the balancing effect of $w \frac{l^2 - l'^2}{2 l'}$, and the reaction to be

overcome in raising the tail-end, including the counterpoise as well as the force required to raise the end C a height H' , will be reduced to the simple expression $Q'' = w \frac{3 l^2 + l l' - l'^2}{8 l'} - \frac{m}{l'}$. The limit

of the wedging up, so that the bridge when unloaded may not be lifted off the pivot or the central support, is given by the equation

$m = -w \frac{l^2 + 3 l l' + l'^2}{8}$; the resistance to be overcome, or Q'' , is

$w l \frac{l + l'}{2 l'}$, and the total lift to be given to the extremity C is

$\frac{w l l'}{6 \epsilon} (l + l')^2$. The pressure on the extremity of the tail-end must

at least be sufficient to prevent its rising when the other portion of the bridge alone is loaded, so the holding-down pressure when the bridge is unloaded $Q'' \geq \frac{w'}{8 l'} \frac{l^3}{l + l'}$.

Also the wedging-up force at C should at least suffice to prevent the strain at the central support from being greater when the bridge is closed and loaded than when it is open, and hence

$Q'' \geq w' \frac{l^2 - l l' + l'^2}{8 l'}$.

In the particular case under consideration the locus of the absolute maxima moments is not influenced by the counterpoise when the bridge is closed, which simplifies the algebraic solution. The total strains due to w, w' , and the wedging are obtained

¹ This subject has been thoroughly investigated by M. Brockhaus, and the calculations published in the "Annales des Travaux publics de Belgique," and in the "Annales du Génie civil," 1874.

from the following values of the moments. Between $x = 0$ and

$x = \frac{l^2}{3l + 2l'}$, along l the maximum moment μ is

$$\frac{w + w'}{2} (l - x) \left(\frac{l^2 - ll' + l'^2}{4l} - x \right) + \frac{w' x^2}{2} \frac{l' + x}{l - x} + m \frac{l - x}{l};$$

and beyond this point,

$$\mu = \frac{l - x}{8l} \left\{ \frac{(w + w') l^3 + w l^3}{l + l'} - 4wlx + 8m \right\};$$

and the minima moments, or negative maxima, are respectively between the same limits,

$$\mu = \frac{l - x}{2l} \left(w \frac{l^2 - ll' + l'^2}{4} - wlx + 2m \right) - \frac{w' x^2}{2} \frac{l' + x}{l - x};$$

$$\mu = \frac{l - x}{2l} \left\{ \frac{w + w'}{4} l \left(\frac{l^2}{l + l'} - 4x \right) + \frac{w}{4} \frac{l^3}{l + l'} + 2m \right\}.$$

As long as the supports maintain their original level the absolute maxima moments shift at the abscissa $\frac{l^2 - ll' + l'^2}{4l}$; up to

this point the positive maxima are the only moments to be considered, and beyond it only the negative maxima.

A diagram representing the moments for a particular instance of a swing bridge (Plate 5, Figs. 9, 10, 11) has been drawn, where $l = 65.6$ feet, $l' = 46$ feet, $\delta = 16.4$ feet, $w = 0.36$ ton per lineal foot, $w' = 0.42$ ton. With $\beta = 3.28$ feet, and $\alpha = 13.12$ feet, the exact balancing weight would require $g = 2.52$ tons; but supposing g be increased to 3 tons, there would be an excess of about 8 tons, which would insure the proper action of the rollers supporting the tail-end. Considering, first, the diagrams of moments in the case of the open bridge, the equation $\mu = .18 (65.6 - x)^2$ gives, when $x = -16.4$, the maximum moment at the pivot equal to 1,210.

The curve is a parabolic arc with its axis vertical, and its vertex at the extremity A. With regard to the moments of the tail-end the weight g may be taken as 2.52 tons, as the rest of the counterpoise is borne by the guiding rollers, and the curve of the moments is given by the equation $\mu' = .18 (49.21 - x')^2 + 41.328 (41.08 - x')$ from the pivot, where $x' = 16.4$ and the moment 1,210, to the point where $x' = 32.8$, making $\mu' = 390$, and then to the extremity, where $x' = 49.21$, there is another curve $\mu' = 1.44 (49.21 - x')^2$, which is a tangent

to the preceding one. When the bridge is closed the whole weight of the counterpoise must be taken, or $g = 3$ tons, as in this case it tends to increase the strains, being situated within the end support. By substituting the given values of l, l', w, w' , and g in the general equations previously given, the numerical values of the other moments are similarly found; and the curves representing their values, at any point of the bridge when open, closed, and loaded, are shown on the diagram (Plate 5, Fig 11). After having drawn the portions of the curves, whether positive (above the axis of x), or negative (below the axis of x), representing the moments produced by the fixed and moving loads the effects are combined, and to the fixed ordinates due to the permanent load are added the ordinates subsequently obtained (positive or negative), which tend to increase the strains, and the results are shown by the dotted curves in the diagram. It is evident from these curves that there are only very short spaces within which the elastic curvature can change its sign, although w' considered alone always produces a positive and a negative moment. The closed bridge has been assumed to have all its supports at the same level, and this position necessitates the existence of a wedging-up force capable of neutralising the deflection of the open bridge, producing a reaction at C equal to 49 tons. The amount of lift is $\frac{4409232}{\epsilon}$,

a quantity whose value depends on the moment of inflexibility, ϵ , which is afforded by the section of the girder employed. After the supports have been brought to a level the inferior limit of the additional wedging-up force is given by the consideration that the tail-end must not be liable to rise when the other portion of the bridge alone is loaded, so that Q'' must be greater than 2.9 tons; this limit, however, is considerably lower than it would be ever wise to adopt.

Again, for the superior limit the wedging-up force must not exceed $19\frac{1}{4}$ tons, in order that the bridge, when unloaded, may not be lifted off its central support, which joined to the pressure of 49 tons previously exerted gives $68\frac{1}{4}$ tons as the total force of wedging. Another superior limit is when, with the bridge loaded, the negative moment due to wedging neutralises the positive moment at the central support. This gives a force of $8\frac{1}{4}$ tons, which joined to the previous pressure of 49 tons makes the limiting force $57\frac{1}{4}$ tons. Within these limits the wedging-up force may be fixed as desirable. For instance, if the most advantageous wedging force for the bridge when closed is required (leaving out of consideration the strains on the bridge when open), by taking

the broken line A' B' C, instead of the straight line A B C, as the axis from which to measure the ordinates, the maximum positive moment on the support B will be made equal to the maximum negative moment F G. The moment of wedging, B' B, satisfying this condition is about 77, and consequently the reaction of 49 tons has only to be increased by $\frac{77}{46} = 1\frac{2}{3}$ ton, so that the whole force is $50\frac{2}{3}$ tons. The space D E, in which, with the old axis, the curves were liable to change signs, is reduced to D' E'.

It would appear at first sight useless to endeavour to reduce somewhat the strains on the bridge when closed, since in the example given they are considerably less than when the bridge is open, in which latter case the moment is 1,210 over the pivot. In consideration, however, of the disproportion between the two kinds of strains, exaggerated in the example by reason of the distance 16·4 feet between the pivot and the support B of the bridge when closed, it appears desirable to support the strains by two separate means. In the first case, where the bridge is closed, the ordinary resistance of the wrought-iron plate girder to flexion would bear the strain; and in the second case, where the bridge turns on the pivot and rollers, a tension-bar, or suspension-chain, would be used as represented in the bridge (Plate 5, Fig. 9) which serves as the example, and which is a copy of the bridge at Birkenhead.¹ Assuming that t is the tension of the rod when the bridge is open, and that its perpendicular distance from the central point of the girder is 19·68 feet, as shown on the drawing (Plate 5, Fig. 9), it follows that the rod produces, with reference to the pivot, a negative moment $-19\cdot68\ t$, by the help of which the bending moment exerted on the girder when the bridge is open can be reduced to the limited amount which exists when the bridge is closed. For this purpose, taking on the diagram a length N O = L B' = 300 approximately, measured from N the end of the ordinate corresponding to the pivot, the remaining portion O P = 910 being made equal to the auxiliary moment $19\cdot68\ t$ gives the tension required. By drawing, moreover, the line A O on the diagram, the parabola A Q N will still represent by its vertical ordinates the bending moments of the swinging portion of the open bridge, but the ordinates must be measured from the new axis A O instead of from the horizontal axis A B, so that between A and Q the curve will be negative.

¹ Another instance is furnished by the bridge over the river Wensum on the Norwich railway: *vide* Minutes of Proceedings Inst. C.E., vol. v., p. 434.

A force of longitudinal compression, equal to the horizontal projection of t , will be added to the force of flexion, but its relative value is very insignificant. The section of plate girder adopted (Plate 5, Fig. 10) has a solid web plate, and if allowed to be strained to 3.82 tons per square inch will afford a moment of resistance equal to 300, which is sufficient for the case of the bridge when closed, and also, consequently, for the open bridge when supported in the manner described above, neglecting the longitudinal compression. The value of this force is $\frac{910}{19.68} =$

46 tons approximately, which spread over a sectional area of 50.22 inches would increase the strain by 0.91 ton per square inch. This additional strain might easily be provided for either by adding near the pivot a short plate to the lower flange only of the girder which is in compression, or by merely raising a little the point O in the diagram, which indicates an increase in the tension on the supporting rod.

The tension-rod acts as a substitute for the support, of which the extremity of the swing portion of the bridge is deprived when open, by means of the vertical component of the tension t , which is equal to about 10 tons. As this component only slightly exceeds the reaction developed at the support at A by the unloaded bridge when closed, which is $8\frac{3}{4}$ tons when the supports are at the same level, it follows that when the wedging up has only slightly raised the end above the level the tension-rod will be quite relieved from strain, and the flexion of the whole girder when the bridge is closed will follow the laws shown on the diagram. This is based on the assumption that the section of the girder is kept constant throughout its whole length, which, moreover, is rendered advisable by the fact that the portions capable of being made less strong would be very short, as the diagram indicates. The case would be different if, instead of adopting an auxiliary method of suspension, it was desired to construct a girder capable of supporting the whole strain of the bridge when open. Suppose, for example, that a symmetrical bridge, spanning two equal openings, and turning on its central support, had to be constructed, and that, moreover, the bridge was to carry a railway, and consequently to be subjected to a relatively heavy rolling load, there would in this case be no longer such a marked difference between the limits of the strains of the bridge when unloaded and open, and when closed and loaded. The best plan in such a case would be to construct a simple girder bridge without tension-rods, but strengthened at the pivot by additional plates riveted to the

flanges, which would give a variable section of girder to which the formulæ contained in this Paper would not strictly apply; but in practice such exactness is rarely required, and usually the theories applicable to girders of constant section are extended to those whose section is variable.

It should be observed that there is no absolute contradiction of terms in the expressions "equal resistance" and "equal flexibility": ϵ being constant, or the moment of inertia constant, does not imply a constant section, the height only has to vary conformably. Suppose, for instance a double T section where the height a varies, the thickness e of the web being constant, the flanges having a constant breadth b , and a variable thickness z . The moment of inertia, $I = \frac{e a^3}{12} + \frac{b z a^2}{2}$ approximately, can remain constant as the variations of z can neutralise those of a , and therefore the resistance $\frac{2 R I}{a}$ can vary according to an arbitrary law. Let it decrease uniformly to a third of its original value, M , from one end to the other of the swinging portion of the bridge, then $\frac{2 R I}{a} = M \frac{3 l - 2 x}{3 l}$, which makes a increase as the ordinate of the hyperbola $a = \frac{6 R I l}{M (3 l - 2 x)}$, whilst in reality, owing to the progressive diminution of z , the area of the section, and therefore the weight, diminishes. It would, however, be most inadvisable, for the sake of reconciling theory with economy of materials, to erect such a structure, deceptive to the eye, and increasing in apparent size where in reality it decreases in strength. In spite of the theoretical advantages shown by the formulæ, the true principles of art require a diminution in height where the strength diminishes, unless a constant height is preferred.

The summary process of applying to girders of variable section formulæ established for a constant section is justified by the difficulties which would often attend more exact calculations, and the Author does not propose to undertake a revision of the general formulæ for the purpose of adapting them to the case of girders of equal resistance. The solution of the problem will be more easily obtained by investigating the special case of some large work. As a specimen of the principle, and the method of calculation, two simple instances will be investigated.

The first instance selected is when the bridge is strengthened near its central support, and it is proposed to investigate merely the special effect of the disturbance of the level, by wedging up,

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a straight line and becomes zero at the extremity, and where the bridge would only be subjected to uniform loads over the whole span.

In the span l (Fig. 10), q being the load per unit of length, and

FIG. 10.

M the moment of the central support, the reaction, Q , at the extremity must satisfy the equation

$$-Ql + \frac{ql^2}{2} = M;$$

and therefore

$$Q = \frac{ql}{2} - \frac{M}{l},$$

and the equation of elasticity is

$$\epsilon \frac{d^2 y}{dx^2} = \frac{q}{2} (l-x)^2 - Q(l-x) = (l-x) \left(\frac{M}{l} - \frac{qx}{2} \right);$$

and making $\epsilon = \frac{l-x}{k}$, where k is a constant, it becomes

$$\frac{d^2 y}{dx^2} = k \left(\frac{M}{l} - \frac{qx}{2} \right),$$

and integrating, $y = \frac{kx^2}{2} \left(\frac{M}{l} - \frac{qx}{6} \right) + cx$;

and in order that this expression may become equal to h when $x = l$ it is necessary that

$$c = \frac{h}{l} - \frac{k}{2} \left(M - \frac{ql^2}{6} \right).$$

For the span l' , with a co-efficient $k' = k \frac{l'}{l}$, and consequently $\epsilon = \frac{l' - x'}{k'}$, which restores the value $\frac{l}{k}$ at the pivot, an analogous equation is obtained, $c' = \frac{h'}{l'} - \frac{k'l'}{2l} \left(M - \frac{q'l'^2}{6} \right)$; and as the tan-

gential continuity of the two curves at the point of support gives $c = -c'$, the following relation is obtained, $M(l+l') = 2 \frac{lh' + l'h}{kl'} + \frac{ql^3 + q'l'^3}{6}$. This equation gives the initial moment M , on which the values of all the other moments depend.

As a numerical illustration, assume the case of a bridge having two equal openings, l and $l' = 65.6$ feet, carrying a railway, the dead load supported by one girder being 0.27 ton per lineal foot, and the rolling load 0.75 ton per lineal foot. The maximum moment when the bridge is open is $0.27 \times \frac{(65.6)^2}{2} = 580$. When the bridge is closed with its supports at the same level, and considering only the total loads for each span, the full load $q = q' = 0.27 \text{ ton} + 0.75 \text{ ton} = 1.02 \text{ ton}$, so that $M = 1.02 \times \frac{(65.6)^2}{6} = 731$, and with only one span loaded $q = 1.02 \text{ ton}$, $q' = 0.27 \text{ ton}$, and $M = \frac{q + q'}{12} l^2 = 462$.

As the bridge is symmetrical, the diagram need only be drawn for one span (Plate 5, Fig. 12). The equations of the parabolas representing the moments are—

Both spans loaded,

$$\mu = 731 - 44.6x + .51x^2.$$

Span represented in diagram alone loaded,

$$\mu = 462 - 40.5x + .51x^2.$$

The other span alone loaded,

$$\mu = 462 - 15.9x + .135x^2.$$

These curves give the following abscissæ of inflexion—

21.88 feet, 13.84 feet, and 52.26 feet respectively;

and the following maxima negative moments,

244 at the abscissa $x = 43.73$ feet;

342 " " $x = 39.73$ "

6 " " $x = 58.92$ "

It is evident that in this case the curve for the open bridge does not rise so high at the central support as the curve for the closed bridge subjected to the full load, and it is only in the zone of the inflexions that it is higher than the others.

Having found the moments when the supports are at the same level, it remains to consider what advantage could be gained by further wedging up. As the girder is made of equal resistance

throughout, it is the mean moment, namely, the surface B D K P Q, together with the surface Q R S A, that must be made a minimum by changing the position of the axis A B to A B'. Supposing the displacement of the axis to be infinitely small, and the curve A P K fixed, one portion of the area would decrease by B B' O Q, and the other would increase by O Q A, and it would be advantageous to move the position of the axis either upwards or downwards till the increase and decrease become equal, or $\frac{B B' \times A B}{2} = Q O \times A Q$, which condition is fulfilled when

$A Q = \frac{1}{\sqrt{2}} A B$. Taking then the point Q' satisfying this con-

dition ($A Q' = 46.4$ feet), and drawing the ordinate P' R' to the points P' and R' on the curves of the positive and negative moments, and bisecting it at O', the straight line A O' B' is the axis which would afford the minima mean moments. This geometrical construction, however, applies merely to the curves of moments for the bridge when shut; but for the open bridge the change of axis would only raise the arc P' K to F V, and the advantage of wedging could only be established by comparing experimentally the eliminated area Q' B B' O' with the added areas A Q' O' + F V K P'. But allowing, for the sake of brevity, that B B' represents the moment of the wedging force definitely chosen, by turning up on A B' the limiting curves of the negative moments, the circumscribing envelope D V F G A of the absolute values of the limited moments is obtained.

The question naturally arises, how far theory is in accordance with facts. If the girder has a constant height, its moment of resistance will increase lineally as the moment of inertia, and to make it agree with the assumed hypothesis it could be fixed according to the straight line A D on the diagram; but in that case there would be an excess of strength along the half of the span nearest the centre, and a deficiency of strength along the other half. The tangent D G H, if substituted for D A as the line representing the moments of resistance, would remedy the deficiency, and might, moreover, be redrawn with considerable exactness by constructing for it a fresh diagram of moments intermediate between that drawn and that which would apply to the case of ϵ being constant, as the line D H is a mean between D A,

conformable to the hypothesis $\epsilon = \frac{l-x}{k}$, and D I (parallel to A B') when ϵ is constant.

It must be observed that, with this method of determining the section of the girder, the most advantageous wedging force will no longer be such as would produce a negative moment equal to BB' , as the trapezium $AB'DH$ must be made a minimum interval of the circumscribing envelope of the bending moments.

If the height of the girder varies, as is frequent in the case of large swing bridges, the lineal variation assumed for the moment of inertia I will cause a curved variation of the moment of resistance $\frac{2RI}{a}$, where R is the co-efficient of resistance for a unit of section, and a the height of the section. If, for example, with $I = \frac{l-x}{kE}$, where E is the modulus of elasticity, a varies according to the parabola $a = a' \frac{l^2 - x^2}{l^2}$, the moment of resistance will vary according to the hyperbola,

$$\frac{2RI}{a} = \frac{2R}{a'kE} \times \frac{l^2}{l+x},$$

where a' is the central height chosen.

For instance, let $R = 8,534$ lbs. per square inch in the central section, $E = 25,600,000$, $a' = 11.15$ feet, then $\frac{2Rl}{a'kE} = 731$, the maximum bending moment less the wedging-up force, whose moment, in order to make use of the same diagram, shall be assumed to be BB' ; therefore $\frac{2Rl}{a'kE} = DB' = 613$; and therefore $k = \frac{1}{156,273}$, and the moment of resistance $\frac{2RI}{a} = \frac{40,213}{65.6+x}$. The curve DTU representing these moments of resistance is drawn on the diagram, with AB' as the axis of x , which, besides being theoretically correct, might be adopted with advantage in practice.

The force of wedging and the amount of lift to produce a central moment equal to BB' , on the hypothesis $\epsilon = \frac{l-x}{k}$, are readily obtained. The normal reaction is

$$Q = \frac{q'l}{2} - \frac{M}{l} = \frac{q'l}{3} = \frac{0.27 \times 65.6}{3} = 5.9 \text{ tons,}$$

to which must be added $\frac{BB'}{l} = \frac{118}{65.6} = 1.8$ ton, so that the whole force is 7.7 tons; and both extremities are raised through a

height h either together or one after the other, which latter plan is the best, as then the first is raised with less expenditure of force. The height $h = \frac{k l m}{2} = \frac{65 \cdot 6 \times 118}{2 \times 156,273} = 0 \cdot 025$ foot; and the total amount of lift, including that for neutralising the deflection at the centre of the open bridge, is $\frac{7 \cdot 7}{1 \cdot 8} \times 0 \cdot 025$ foot = $0 \cdot 106$ foot.

The accidental increase of pressure due to curvature caused by dilatation can be calculated in the same manner as the wedging force has been calculated. A case of this kind occurred at the bridge over the Missiessy dock at Toulon. The upper flange of the girder was heated by the sun's rays, whilst the lower flange was sheltered by the planking and by the web plate, causing a curving of the girder, and the pressure on the wedging apparatus was so great that the process of opening was made almost impossible, and the plates were considerably strained. The mischief was repaired by covering the top flange with wood, allowing free circulation for the air.

Diagrams form only part of the work in designing a structure, and detailed drawings must be made before the suitability of the proposed arrangements can be ascertained with certainty; but the general considerations the Author has brought forward show the possibility of investigating in most cases with sufficient accuracy what are the conditions of resistance of a swing bridge, even of variable section, and what are the most suitable arrangements to adopt.

The communication is accompanied by reduced drawings, from which Plate 5 has been engraved.