92. Solution of Problem 372, and Notes Suggested Thereby

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Source: *The Mathematical Gazette*, Vol. 2, No. 25 (Jan., 1901), pp. 9-10

Published by: Mathematical Association

Stable URL: <http://www.jstor.org/stable/3602924>

Accessed: 29-12-2015 13:06 UTC

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MATHEMATICAL NOTES.

92. [K. 13. a.] *Solution of Problem 372, and notes suggested thereby.*

No solution of this having yet appeared, I add the following analytical notes to my previous statement.

The general (finite) displacement of a rigid body by a screw about the axis of z is given by

$$x' = \alpha x, \quad y' = \frac{1}{\alpha} y, \quad z' = z + \beta,$$

where

$$x = X + iY, \quad y = X - iY;$$

X, Y, z are a rectangular Cartesian set of coordinates and α, β are constants, of which β is real and the absolute value of α is unity; in fact $\alpha = e^{i\phi}$, if ϕ be the angle of rotation. Then a point midway between $(x'y'z')$ and (xyz) is

$$x_0 = \frac{1}{2}(1 + \alpha)x, \quad y_0 = \frac{1}{2\alpha}(1 + \alpha)y, \quad z_0 = z + \frac{1}{2}\beta,$$

so that if (xyz) is in a given plane

$$Ax + By + Cz + D = 0,$$

then $(x_0y_0z_0)$ is in another plane.

The plane bisecting at right angles the line joining (xyz) , $(x'y'z')$ is

$$(x' - x)(\eta - y_0) + (y' - y)(\xi - x_0) + 2(z' - z)(\xi - z_0) = 0,$$

(ξ, η, ζ) being current coordinates in the plane. This reduces to

$$(1 - \alpha)\left(\frac{1}{\alpha}\xi y - x\eta\right) + \beta(2\xi - 2z - \beta) = 0.$$

Hence with (xyz) in the given plane, the latter plane contains the fixed point (O)

$$\frac{-(1 - \alpha)\eta}{A} = \frac{(1 - \alpha)\xi}{\alpha B} = \frac{-2\beta}{C} = \frac{\beta(2\xi - \beta)}{D},$$

which completes the solution of the problem.

If O has been displaced from the point O' , we have

$$\xi' = \frac{1}{\alpha}\xi = -\frac{2\beta}{1 - \alpha} \frac{B}{C}, \quad \eta' = \alpha\eta = \frac{2\beta\alpha}{1 - \alpha} \frac{A}{C}, \quad \zeta' = \zeta - \beta = -\frac{\beta}{2} - \frac{D}{C},$$

and with these values it can be proved (just as above) that the plane

$$Ax + By + Cz + D = 0$$

bisects OO' at right angles; this is the property of O stated in my last note (*Gazette*, Oct., 1900).

If we desire to evaluate the *velocities* due to an instantaneous screw about the axis of z , we have to make β, ϕ infinitesimal, and then we find

$$\frac{dx}{dt} = i\theta x, \quad \frac{dy}{dt} = -i\theta y, \quad \frac{dz}{dt} = \gamma,$$

where $\theta = \frac{d\phi}{dt}$, $\gamma = \frac{d\beta}{dt}$, so that both are real; and the pitch of the screw is γ/θ .

In these coordinates we can prove very easily a theorem due to Cayley, that every point $(x_0y_0z_0)$ can move instantaneously along the line joining $(x'y'z')$, (xyz) for all positions of (xyz) . For, in terms of $(x_0y_0z_0)$,

$$x' - x = 2\frac{\alpha - 1}{\alpha + 1}x_0, \quad y' - y = -2\frac{\alpha - 1}{\alpha + 1}y_0, \quad z' - z = \beta,$$

and thus the point $(x_0y_0z_0)$ can be screwed along this line by an infinitesimal screw about the axis of z of pitch $\frac{\beta i}{2} \frac{a+1}{a-1} = \frac{\beta}{2} \cot \frac{\phi}{2}$ (using the expressions for the velocities just found). So that, in particular, if the original motion be infinitesimal, this value of the pitch is (β/ϕ) as it ought to be.

In concluding, it may be well to point out wherein the three-dimensional results differ from the two-dimensional. In the two-dimensional case, we have only to ignore z and β in the above; so that the line bisecting at right angles the join of (xy) , $(x'y')$, or

$$(x' - x)(\eta - y_0) + (y' - y)(\xi - x_0) = 0,$$

reduces to

$$\frac{1}{a} \xi y - x \eta = 0,$$

which passes through the origin whatever the point (x, y) may be; thus the origin is a marked point of the body which does not move in the displacement. This is of course a familiar property, and is only mentioned now for comparison with the other results.

If we assume that the usual conditions for orthogonality of lines, etc., can be extended to space of n dimensions, merely by adding on terms of the same type; I have proved that when n is *odd* there is no real point at a finite distance which does not move in a general displacement of a rigid body; but if n is *even*, there is always at least one such point (cf. R. F. Muirhead, *Proc. Edin. Math. Soc.*, Vol. xvi., p. 70). Of course by specializing the motion we can always keep a certain number of points fixed; but if n be odd, and a point is kept fixed, then a line is thereby also fixed.

Before leaving this subject, I should like to remark that the use of coordinates such as (x, y) above (*circular coordinates*) is often advantageous in geometrical problems that involve metrical relations; thus, properties of the epicycloids, etc., have been very simply obtained from these coordinates by Prof. F. Morley (*Am. Journal of Maths.*, Vols. xiii., xvi.): see also Vol. i., No. 2, of the *Transactions of the American Mathematical Society*.

T. J. F.A. BROMWICH.

93. [K.] Prof. Hill's Eucl. v. and vi., Definition of Ratios, and Incommensurables.

I am glad to have the opportunity of showing that my proposal of *measure* as the definition of ratio is free from the difficulties suggested by Prof. Hill* (Review of Prof. Hill's Eucl. v. and vi. Vol. I. No. 24, p. 411).

ART. 1. If A and B are commensurable so that $A = sC$, $B = rC$, then the measure, $A|B$ say, is the rational fraction s/r , since $A = s/r \cdot B$.

When A and B are incommensurable we find, as in the decimal system of weighing, a terminating decimal $\mu_n = \omega \cdot a_1 a_2 \dots a_n$, such that, the decimal μ'_n being the same as μ_n with $a_n + 1$ for a_n , and μ the true measure $A|B$,

$$A = \mu B > \mu_n B < \mu'_n B.$$

The process is supposed to be carried far enough to show any required difference between μ and any different decimal ν .

ART. 2. *Theorem I.* If the measures $A|B$, $C|D$ are μ , ν , and if, r and s being integers, whenever $rA > sB$, $rC > sD$, and when $<$, $<$, then $A|B = C|D$.

*The less important points raised by Prof. Hill are passed over for lack of space, as well as a detailed account of measurement in Art. 1 justifying (a) in Art. 2.