XVI. On the equilibrium of arches

Mr. John Southern

To cite this article: Mr. John Southern (1801) XVI. On the equilibrium of arches , Philosophical Magazine Series 1, 11:42, 97-107, DOI: [10.1080/14786440108675963](https://doi.org/10.1080/14786440108675963)

To link to this article: <http://dx.doi.org/10.1080/14786440108675963>

Published online: 18 May 2009.

Submit your article to this journal [↗](#)

Article views: 2

View related articles [↗](#)

XVI. *On the Equilibrium of Arches.* By Mr. JOHN SOUTHERN, Engineer, Soho, Birmingham *.

SIR,

Soho, Birmingham, Nov. 2, 1801.

AT the present moment, when the grand project of throwing a single arch over the Thames (proposed by Messrs. Telford and Douglas) is in contemplation, whatever tends to render the theory of the equilibrium of arches more easily to be understood, will, it is reasonable to suppose, be acceptable to many of your readers; and if you think the following paper of this description, you will, by inserting it in the Philosophical Magazine, oblige,

Sir,

Your obedient servant,

Mr. Tilloch.

JOHN SOUTHERN.

1. LET the right lines, fig. 1. ZBCDEF be situate in a vertical plane, and moveable about the angles B, C, D, E as joints, and also about the points Z and F; let ZB be horizontal, and let such weights be laid upon the joints BCDE as will keep the whole in equilibrio, the points Z and F not yielding: required what such weights are, and what the forces exerted in each line.

From any point B in the line ZB draw Bp perpendicular to that line, continue the said line to any distance, a, whence draw the line aM parallel to Bp; by the equilibrium of forces, the weight (represented by the line) aM laid upon the joint B will be counterbalanced by a force aB in the direction ZB, and a force MB in the direction MB, and of course the point B remain at rest. But action and reaction being equal and contrary, the force aB will not only be exerted against B, but also be returned and act against the point Z; also the force MB will be returned, and act against the joint C. Continue BC to b, so that bC = MB, and from b draw the line bN parallel to Bp, meeting the line CD in N; then will the force bC, in the direction BC, be just balanced by a weight bN laid upon the joint C, and a force NC in the direction NC. In like manner continue the lines CD and DE to c and d respectively, making the lines cD = NC, and dE = OD; and from the points c and d drawing lines parallel to Bp meeting the line DE and EF in O

* Communicated by the Author.

VOL. XI.

G

and

No. 41.

November 1801.

and P; so will the weight cO laid upon the joint D, and the force OD, balance the force cD in the direction CD; and the weight dP laid upon the joint E, and the force PE balance the force dE acting in the direction DE. The force PE will, by its reaction, be exerted upon the point F.

Make $BA = aB$, and draw Am parallel to BC; Bm will therefore $= aM$, and $Am = BM$; and consequently the lines AB, Bm , mA , will represent the forces and their directions of action which keep the joint B in equilibrium. Draw An parallel to DC, then because $BC = MB = mA$, and bN is parallel to mn (included in Bp), the triangle Amn is equal and similar to the triangle CbN , and consequently the lines Am , mn , nA , represent the forces and their directions of action which keep the joint C in equilibrium. So, AO and Ap , being respectively drawn parallel to DE and EF, the triangle $Ano = DcO$, and $Aop = EdP$; and consequently their sides represent the forces which keep the joints D and E respectively in equilibrium.

2. It is obvious that the weight pressing on any joint is as the *difference of the tangents* of the angles which are formed by each of the legs of that joint with the horizontal line: for the weight on the joint D is as $cO = no$; but Bn is the tangent of the angle nAB (to the radius AB) = the angle which the leg DC of the joint forms with the horizontal line AB; and Bo is the tangent of the angle oAB (to the same radius) = the angle which the other leg DE of the joint forms with the horizontal line AB; consequently, no is the difference of these tangents.

3. It is equally obvious that the force exerted through any leg of any of the joints, and which may be called the *compressing force*, is as the *secant* of the angle which that leg forms with the horizontal line: for the force through the leg FE is as pA , the force through ED is as oA , through DC as nA , through CB as mA , and through BA as $B\bar{A}$; respectively the secants of the angles, with the radius AB, which each leg makes with the horizontal.

4. The horizontal force, or compressing force at the crown, (which is constant,) is $AB = \text{radius}$; because the leg at the crown is horizontal, and therefore the angle is nothing there, the secant of which is equal to radius.

5. If the weights which are disposed upon any *number of contiguous joints* be united and laid upon one joint, and if the legs which form this joint have the same declination from the horizontal line, and the same forces exerted through them that the *extreme* legs of the two extreme joints of the said contiguous number have, the equilibrium will still be exact.

Continue

Continue the line BC to G and e , and the line FE to G; make $Ge = MB =$ the force acting towards G from B; draw eg parallel to Bp . By construction, Ge is parallel and equal to Am , and gG is parallel to pA ; therefore the triangles Geg and Amp are equal and similar, and represent the forces which keep the joint G at rest; but the force or weight $eg = mp = mn + no + op$ is equal to the sum of the weights upon the angles CDE; and, by construction, the legs FG and BG of the joint G have the same declination from the horizontal that the extreme legs (FE, BC) of the two extreme joints (EC) of the contiguous joints CDE have.

6. It is hence evident that the *sum* of the weights upon *any number* of successive joints (whether they be numerous or few), beginning at the vertex, is *as the tangent* of the angle formed by the *ultimate* leg of the last joint with the horizontal line.

7. Consequently, if the legs be extremely numerous and short, so as to form a curve, or a line differing insensibly from a curve, the whole weight pressing on that curve must be *as the tangent of the angle* which a tangent to the curve at the lowest point makes with the horizontal line, because the said tangent will coincide with the ultimate leg of the curve.

8. If the curve be the arch of a circle, the *angle*, which a tangent to the curve forms with the horizontal line, will be equal to that at the centre, contained between the vertical radius and that drawn to the point whence the tangent is drawn, and may be measured by the arch itself.

Let adf , fig. 2. be the arch of a circle from the centre z , and za vertical; draw aF horizontal, a tangent to the arch at a , which is consequently at right angles to the radius az . From any point in the arch, as e , draw em a tangent to the curve at e ; the angle emF is therefore that which the tangent to that point makes with the horizontal line; and because em , one of its legs, is at right angles to ez , and Fm , (the other leg,) by construction, at right angles to az , the angle eza is equal to the angle emF , and may be measured on the arch from a to e . So the angle dza may be proved to be equal to that which a tangent to the arch at d would, if drawn, make with the horizontal. And so of any other point.

9. Let the arch adf , in the last figure, represent the intrados of a bridge; if it be required what weights should be laid upon any given portions of it to effect the equilibrium, draw through the given points $bcdef$ radii from the centre z to the horizontal line aF ; so are the portions aB, BC, CD, DE , and EF , proportional to the weights required to be laid

on the respective portions ab , bc , cd , de , and ef , of the arch. For Ea is (7.) proportional to the whole weight on the arch ea , Da is proportional to the whole weight on the arch da , and therefore ED is proportional to the weight on that portion of the arch ed corresponding with it, and is as the difference of the tangents of the angles formed by tangents to the arch at the points e and d with the horizontal. (2.)

10. The horizontal force, or compressing force at the crown, is as radius (4.) $= za$; the compressing force at any point, c or d , is as zC and zD respectively (3.); and, of course, the pressure on the abutment in direction of a tangent to the arch there, as zF .

11. It is easy, from what has hitherto been demonstrated, to calculate from tables of tangents the actual weights which should be laid upon given portions of arches, provided the weight upon any one portion be given. Suppose, for instance, an arch to contain from the summit or crown to the abutment 30° , and that on the first degree 10 tons be laid, (including the weight of the arch, road, &c.) then say, As the tangent of one degree is to 10 tons, so is the tangent of two degrees to the weight upon two degrees, from which subtract the weight upon the first, and there will remain the weight due to the second degree; and so on.

Thus, as tang.	1°	-	co. ar.	1.75808
is to	10 tons	-	-	1.00000
so is tang.	2°	-	-	8.54308

Weight upon 2°	20,006 tons	-	1.30116
Subtract	10 the weight upon the first degree.		
	10,006 tons upon the second degree.		

Again, as tang.	1°	co. ar.	1.75808
is to	10 tons	-	1.00000
so is tang.	15°	-	9.42805

Weight upon 15°	153.51 tons	2.18613
------------------------	-------------	---------

As tang.	1°	co. ar.	1.75808
is to	10 tons	-	1.00000
so is tang.	16°	-	9.45750

Weight upon 16°	-	164.28 tons	-	2.21558
Subtract weight upon 15°		153.51 tons		

Weight upon the 16th degree 10.77 tons

Again,

Again, as tang.	1°	co. ar.	1.75808
is to	10 tons	-	1.00000
so is tang.	30°	-	9.76144

The whole weight upon 30° = 330.77 tons - 2.51952
 Double this will be the whole weight of the bridge.

12. It was shown (3.) that the compressing force of any leg in a system of angles was as the secant of the angle which that leg formed with the horizontal line. The leg at any point of a curve will coincide with the tangent at the same point, and therefore the compressing force at any point of the arch of a circle is as the secant of the arch between that point and the vertex (8.) Thus the compressing force at 30° from the crown, is as the secant of 30°; and this is the force exerted upon the abutment in direction of the tangent to the arch there. The compressing force at 2° from the crown is as the secant of 2°, and at the crown (10.) as the secant of 0 degrees.

Thus, as the tang. of	1°	co. ar.	1.75808
is to	10 tons	-	1.00000
so is secant of	0°	-	10.00000

The compressing force at the crown = the *horizontal* force, or thrust, or push, against the abutment } 572.90 tons - 2.75808

Again, as the tang. of	1°	co. ar.	1.75808
is to	10 tons	-	1.00000
so is secant of	30° *	-	10.06247

The compressing force exerted against the abutment in direction of the tangent to the arch at that point } 661.53 tons 2.82055

13. This seems to be all that is requisite to be known of the theory of the equilibrium of arches of a circle, so far as it relates to practice: the builder will naturally desire to construct his bridge with the fewest materials consistent with stability, and will probably first determine the dimensions and weight of those at the crown, which he will fix at a minimum, and from that, by the rules just given, the weight

* To some of those who may not have a logarithmic table of secants it may be useful to say, that the logarithmic secant of an angle may be had by subtracting the co-sine of the angle from 20.00000.

over other parts of the arch: in most cases wherein the span of the arch is large, the spandrels will require to be hollow, and the weights, deduced as above, will show him how much. To carry the theory a little further,

14. Let AmB , fig. 3, be any curve equilibrated by the wall $ABCD$ built upon it, and sustained by a horizontal force applied at A the crown, and a tangential one at B ; let S be the centre of curvature at A and SAD vertical. It was shown (9. 10.) that the weight of incumbent matter on any portion of the arch of a circle, is to the horizontal force as the tangent of that portion of the arch is to radius. Let Am be a very small portion of the curve, which may therefore be supposed not to differ sensibly from the arch of a circle of the radius SA , nor indeed from the tangent of that arch. Therefore the weight upon Am will be to the horizontal force as Am is to radius AS : but as the weight is proportionate to the altitude AD of the wall, so is the horizontal force also as that altitude; and as the former may be expounded by the small area An (mn being parallel to AD) $= Am \times AD$, so may the latter be expounded by $AS \times AD$: in other words, the horizontal force, push, or thrust, is equal to a wall whose length is AS and altitude AD . Make BG a tangent to the curve at B , and draw BP horizontal; also draw sad parallel to SD , and make $sa = SA$, and $ad = AD$; draw ab and dc horizontal, and sb at right angles to BG . The angle bsa is therefore equal to the angle GBP , which is the angle made by the curve at B with the horizontal line; and as ab is the tangent of this angle with the radius as ($= AS$), the area $abcd$ is equal to $ABCD$ (7.) $= ab \times ad$.

Draw FI parallel and very near to BC , and FH a tangent to the curve at F ; also at right angles to FH draw sf , and from f parallel to bc draw fi . It follows, from what has been said, that the small trapezium BI is equal to the small parallelogram bi .

15. Hence the *general equation* for the altitude of the wall above of all curves of equilibration, which are so poised by the matter acting upon them *vertically only*, as if standing on vertical pillars close together upon the arch, viz. $ai = wj$, wherein

a = height of the wall at the crown.

i = the fluxion of the tangent of the angle GBP , formed by the tangent to any point of the curve (as B) with the horizontal line, the radius being AS .

w = the altitude of the wall above the said point (as BC), which determines the figure of the extrados.

j =

$\dot{y}$ = the fluxion of the ordinate to the same point of the curve = the horizontal distance of the lines BC and FI.

16. Let the curve be the arch of a circle, and let it be required to find w . Let r = radius, and $\dot{y}$, $\dot{z}$, and i , the contemporary fluxions of the sine, the arch, and the tangent, (see fig. 4.); also, let s = the secant, and u be drawn parallel to $\dot{z}$.

$$\text{then, } \dot{y} : \dot{z} :: r (= AS) : s (= Sb)$$

$$\dot{z} : u :: r : s$$

$$u : i :: r : s$$

$$\text{therefore, } \dot{y} : i :: r^3 : s^3 \text{ and } i = \frac{s^3 \dot{y}}{r^3}$$

Substitute this value for i in the general equation above, and it will be,

$$\frac{as^3 \dot{y}}{r^3} = w \dot{y}; \text{ therefore}$$

$$w = \frac{as^3}{r^3}.$$

$\frac{a}{r^3}$ being a constant quantity, w is proportional to s^3 , as found by others.

17. Let the curve be the parabola, fig. 5.

A the vertex or crown.

B the point on which w is required.

BK a tangent to the curve at B.

x = AP (and by the nature of the parabola $2x$ = PK).

y = PB; other letters as before.

$$y : 2x :: r : t$$

$$t = \frac{2xr}{y}$$

$$i = \frac{2ry\dot{x} - 2rx\dot{y}}{y^2};$$

and substituting this value for i in the general equation $at = w\dot{y}$, we have

$$\frac{2ary\dot{x} - 2arx\dot{y}}{y^2} = w\dot{y}, \text{ hence } wy^2\dot{y} = 2ary\dot{x} - 2arx\dot{y}.$$

By the nature of the figure $y^2 = px$, where p the parameter = $2r$ (r being the radius of curvature at its vertex)

$$\text{therefore } y^2 = 2rx \text{ and } x = \frac{y^2}{2r};$$

$$\text{also } \dot{x} = \frac{y\dot{y}}{r}.$$

G 4

Substituting

Substituting these values in the equation just above,

$w = a$; so that the extrados and intrados are precisely similar and equal figures, having their vertical distance constantly the same $= a$, the altitude of the wall at the crown.

18. Let the extrados be a horizontal line DC, fig. 6, required the intrados, a being given, and x, y, w , as before.

By the figure $w = a + x$, and the general equation will be

$$wy\dot{y} = \overline{a+x}\dot{y} = at;$$

$$\text{therefore } \dot{y} = \frac{ai}{a+x}.$$

$$\text{Also, } \dot{y} : \dot{x} \text{ (see figure) } :: r : t.$$

$$\text{Therefore } \dot{y} = \frac{r\dot{x}}{t} = \frac{ai}{a+x}$$

$$ati = ar\dot{x} + rx\dot{x},$$

$$at^2 = 2arx + rx^2,$$

$$t^2 = \frac{r}{a} \times \overline{2ax + x^2},$$

$$t = \frac{r^{\frac{1}{2}}}{a^{\frac{1}{2}}} \sqrt{2ax + x^2},$$

$$\dot{y} = \frac{r\dot{x}}{t} \text{ (as above) } = \frac{r\dot{x}}{\frac{r^{\frac{1}{2}}}{a^{\frac{1}{2}}} \sqrt{2ax + x^2}} =$$

$$= \sqrt{ar} \frac{\dot{x}}{\sqrt{2ax + x^2}},$$

$$y = \sqrt{ar} \times \text{hyp. log. of } a+x + \sqrt{2ax + x^2} + C.$$

But when $x = 0, y = 0$, and $C = -\sqrt{ar} \times \text{hyp. log. } a$.

The correct fluent is therefore

$$y = \sqrt{ar} \times \text{hyp. log. } \frac{a+x + \sqrt{2ax + x^2}}{a},$$

which is the same that Dr. Hutton makes it, page 44, new edition of his Principles of Bridges, wherein Q there $= ar$, here $=$ the horizontal force.

If a, x , and y , be given in any case, as, for instance, the altitude of the wall at the crown above the intrados and the altitude

altitude and half span of the arch ; then r may be found as follows, and thence the ordinates to any other point.

$$\text{Let } c = \text{hyp. log.} \left(\frac{a + x + \sqrt{2ax + xx}}{a} \right).$$

$$\text{Then } y = \sqrt{ar} \times c,$$

$$y^2 = arc^2,$$

$$r = \frac{y^2}{ac^2}.$$

Example. Suppose the span of the arch to be 100, the altitude of it 40, and the height of the wall above the crown 6 (same case as Dr. Hutton's, page 47, Principles of Bridges),

$$a = 6, x = 40, y = 50, \text{ therefore}$$

$$\frac{a + x + \sqrt{2ax + xx}}{a} = 15.268, \text{ the hyperbolic logarithm}$$

of which is $2.7257^* = c$, $c^2 = 7.4297$, $ac^2 = 44.578$, and

$$\frac{y^2}{ac^2} = \frac{2500}{44.578} = 56.081 = r = \text{the radius of curvature of the arch at the crown.}$$

19. The WEIGHT, of the incumbent matter on a curve loaded to the equilibrium, contained between the crown and any point B, fig. 3. and which may be expounded by the area AC of the wall above it, has been shown (14.) to be always $= ab \times ad = at$. Let A = said area AC.

20. If the curve be the arch of a circle, and T = the tabular tangent of the number of degrees contained in the arch, then will $t = rT$, and consequently $A = arT$.

21. If the curve be the parabola, see fig. 5 ;

$$\text{then } y : 2x :: r : t = \frac{2xr}{y} \left. \begin{array}{l} \text{and } y^2 = px = 2rx \\ \text{and } y^2 = px = 2rx \end{array} \right\} \text{ by the nature of the figure.}$$

$$\text{Therefore } t \left(= \frac{2rx}{y} = \frac{y^2}{y} \right) = y,$$

$$\text{and } A = ya.$$

* The hyperbolic logarithm of a number may be found by multiplying the common or Briggs's logarithm of the same number by 2.302585, &c. the product is the required hyperbolic logarithm.

22. If

22. If the top or extrados be a horizontal straight line, see fig. 6;

$$t = \frac{r^{\frac{1}{2}}}{a^{\frac{1}{2}}} \sqrt{2ax + xx}, \text{ (18.) therefore}$$

$$ta = A = r^{\frac{1}{2}} a^{\frac{1}{2}} \sqrt{2ax + xx} = \sqrt{rax \times 2a + x}.$$

Example. Take the case of Dr. Hutton's last mentioned, wherein r was found equal to 56.081; therefore

$$rax = 13459, \text{ and } 2a + x = 52.$$

$$\sqrt{13459 \times 52} = 836.60 = A.$$

23. It is well known to mechanicians that if a heavy body be sustained by two forces, their directions must meet either at the centre of gravity of that body, or in a *vertical* line which passes through it. Let A, fig. 7, be the body, g its centre of gravity; ea a string by which it is suspended, which again is sustained by the strings ab, ac . ae is of necessity vertical, and, if continued, must pass through the centre of gravity g of the body. If this string were either longer or shorter, the point a would still be at rest, and the strings ab and ac under the same tension. If the body were removed vertically to the situation dotted, it would be sustained in the same position by the strings bf and cg , which would suffer the same tension as when they were united in the point a . If instead of strings, props bi, kl , applied in the same directions on opposite sides of the body, were substituted; or if, instead of these props, others parallel to them were applied, which, if continued, would meet in the vertical line, (as mn or op , which meet in g , do,) the body would still be supported at rest and in the same position, and the props would suffer the same force of compression in these latter cases, as the strings on the opposite sides suffered tension in the former.

24. Hence the CENTRE OF GRAVITY of the weights upon the joints E, D, C, (fig. 1.) must be in a vertical line passing through G, because they are sustained in equilibrium, or at rest, by two forces in the direction BC and FE, which lines, continued, meet in G. Also, the centre of gravity of all the weights, say on B, C, D, and E, must be in a vertical line that passes through b , because they are sustained by two forces in the direction FE and ZB, which, continued, meet in b .

25. Hence also the centre of gravity of the materials upon a curve in equilibrium will be in a vertical line that passes through the point of intersection of the tangents to the extreme points of the curve.

Let

Let abc , fig. 8. be the curve loaded to the equilibrium; let ad be a tangent to the curve at a , cd a tangent to the curve at c , and be a tangent to the curve at b (the crown). The centre of gravity of the whole materials is in a vertical line which passes through d ; and the centre of gravity of the materials over the arch a,b , is in a vertical line that passes through the point e ; because the points d and e are respectively the points of intersection of the tangents, drawn from the extreme points of the portions in question of the curve.

26. To find the horizontal distance af of the centre of gravity of the materials contained between the *crown* and the abutment, from the latter. Drop a vertical line ef , from e , upon the ordinate y . Let $af = d$, and $\dot{x}$ and $\dot{y}$ be parallel to, and cotemporary fluxions of, x and y respectively. Then it is evident that $\dot{x} : \dot{y} :: x (\cdot ef) : d (=af)$;

$$\text{therefore generally } d = \frac{xy}{\dot{x}}.$$

27. If the curve be the arch of a circle, e,b (see last figure) is evidently the tangent of half the arch, which subtracted from half the span, leaves d , = sine of the arch — tangent of half the arch.

28. If the curve be the parabola,

$$y : 2x :: \dot{y} : \dot{x} = \frac{2y\dot{y}}{y}, \text{ by the nature of that fig.}$$

$$\text{therefore } d = \frac{\frac{xy}{\dot{x}}}{\frac{2xy}{y}} = \frac{1}{2}y.$$

29. If the curve be the equilibril curve with a horizontal extrados.

$$\text{It was shown (18) that } \dot{y} = \sqrt{ar} \left(\frac{\dot{x}}{\sqrt{2ax + xx}} \right) =$$

$$\sqrt{\left(\frac{ar}{2ax + xx} \right)} \dot{x};$$

$$\text{therefore } d (= \frac{xy}{\dot{x}}) = \sqrt{\frac{arxx}{2ax + xx}} = \sqrt{\frac{arx}{2a + x}}.$$

Example. Take the case of Dr. Hutton's mentioned before, wherein r was found = 56.081, $rax = 13459$, and $2a + x = 52$,

$$d = \sqrt{\frac{13459}{52}} = 16.088.$$