On the Proper Method of Loading the Premiums required for the Assurance of Sums at Death, &c. By WILLIAM MATTHEW MAKEHAM, Fellow of the Institute of Actuaries.

OUR predecessors adopted a ready and (all things considered) a sufficiently effectual method of regulating the charges to be made for granting assurances on lives. They satisfied themselves on the one hand that the mortality table adopted allowed for a much higher mortality than would probably be experienced, and on the other they assumed in their calculations a rate of interest below that which they knew would be realized on their investments. With these precautions they could rely pretty safely upon realizing a considerable profit, and this consideration not unnaturally proved sufficient for the scientific requirements of the time.

The chief defect of this system, as is the case with all the so-called “practical” methods, not founded upon true principles, consisted in the fact that while it answered sufficiently well in ordinary cases, there were others in which it proved totally

inadequate to the end in view. In dealing with assurances effected upon one life against another, for instance, it was quite impossible to say in which direction the assumed exaggerated mortality might operate; and thus an opportunity was afforded, as soon as more correct tables became accessible, for speculations disadvantageous to the Companies; opportunities of which there would be no lack of shrewd calculating men ready to take advantage.

The defect in question proved fatal to the system as soon as the direction of opinion fell to men possessed of the requisite scientific attainments. The writings and practice of Milne, Gompertz, De Morgan, Babbage and others, quickly abolished a practice, not only in the highest degree unphilosophical, but fraught with danger, not the less real because it was not recognised by the practical men by whom the system had been upheld.

The difference between the two schools, if I may so term them, is clearly explained in Gompertz's paper, read before the Royal Society on the 29th June, 1820. He says, "It appears a point of great importance to decide what ought to be the demands of those Companies, so that the public may reap the greatest benefit from them. And the only means of answering this question is the possession of the mathematical and philosophical principles by which those Institutions ought to be guided. In the present improved state of the science of life contingencies it is not sufficient for a proper regulation to follow old customs and calculations drawn from a less perfect experience than we have now the means of obtaining. . . . The tables should be as accurate as they can possibly be made, and the interest should be calculated at that rate which shall appear to be the average interest to be made for money; but such additional demands should be made by the Company or Institution as to leave an adequate portion for its security, profit and expenses, for it does not seem possible, in the various beneficial applications which can be made from a proper knowledge of this branch of the mathematics, to judge invariably how to adapt tables which are not correct in themselves, connected with a rate of interest which is not the average rate made in reality, so that the advantage may tend in one direction. In granting assurances on lives it is a practice to use a certain table of mortality and to calculate at a certain rate of interest without making any additional charge; in the presumption that the tables are in themselves incorrect but that their deviation from the truth is in favour of the Society; and that the interest of money is less

"than they can really make; but such a plan does not appear
 "to me sufficiently scientific to be followed by Companies
 "concerned with life contingencies generally."

Thus originated the method now universally acknowledged (if not universally acted upon) of basing the rates of premium upon what is termed a "true" table of mortality and a "true" rate of interest, and making a direct and independent provision for profit and expenses. The recognition of the *principle* was the great end attained; the method of determining the amount of such premium in each case, technically called the "loading" of the premium, being a matter of secondary although by no means inconsiderable importance.

To the principle thus introduced no exception can be taken. Like many other instances which occur in the progress of science, its truth is so obvious that we find it difficult to understand how it could ever have been disputed. The mode adopted of carrying it out, however—which I believe consisted invariably in the first instance of a percentage addition to the premium—was not, I venture to think, by any means so happily devised. Indeed it soon became apparent that the adoption of the new system involved us in difficulties and inconsistencies from which the old system was free. The most prominent was the excessively high scale of premiums required for assurances effected at advanced ages,—more especially if made by a single payment,—an evil which Mr. Jellicoe proposed to remedy by the ingenious suggestion of combining a proportional with a constant addition to the net premium. This palliative sufficiently accomplished the particular end in view, but left other—less obvious but not less real—anomalies untouched.

It is undoubtedly true that an Assurance Office is an establishment opened for the sale of reversions, and the great fundamental law of exchange that price varies so as to maintain the equality between supply and demand, applies just as much as in any other business. At first sight, therefore, it might seem perfectly reasonable that the margin for profit and contingencies should, as in most other cases, be proportional to the price of the thing brought to market.

But although it may sometimes be expedient to view an Assurance Office in this light, yet it must not be overlooked that the nature of its business differs in an essential point from the general case. Generally speaking, the price of anything in the market represents its *cost*,—*i.e.*, the outlay incurred in producing and

bringing it to market,—plus the *profit*, the expectation of which is the inducement for bringing it there. It follows that the amount of the profit properly consists of a percentage on the price. Now, in our case we have nothing analogous to cost, but the current expenses of the establishment and the capital at risk, neither of which, as I shall show, is proportional to the price of the benefits secured.

Let us take a simple instance. Suppose the chance of a given individual dying before the expiration of one year to be $\cdot 1$, or one-tenth—the rate of interest on the investments of the Company, 4 per cent,—and first let the question be to determine the premium to be charged for an assurance against the contingency mentioned. The true value of the risk for the sum of £100 is $\cdot 1 \times (1\cdot 04)^{-1} \times 100 = 9\cdot 615$. Now, seeing that the Office may be a loser to the extent of 86·539 by the transaction, it will, I think, be admitted that the margin or loading in this case should not be much less than 2·404, or 25 per cent of the net premium, or 20 per cent of the premium charged. For the sake of argument, I will assume that this is the proper ratio of profit for this case, according to the existing state of supply and demand.

I will now suppose that the benefit to be secured consists of the same sum payable in the event of the individual *surviving* the year. Here the true premium is $\cdot 9 \times (1\cdot 04)^{-1} \times 100 = 86\cdot 539$, and adding a loading of 25 per cent, as before, we arrive at the singular conclusion that £108·174 is the proper premium per cent to be charged in this case.

The explanation of this absurdity is easily seen to be that the capital at risk in the two cases instead of being *directly* is *inversely* proportional to the price. In the first case, the amount risked is 86·539, and the premium, 9·615; while in the second the figures are reversed, the amount risked being 9·615, and the premium, 86·539.

This effectually disposes of the question as regards the capital at risk. With respect to expenses it is evident that there is only one item in the charges of an Office—viz., the commission allowed to agents, which is proportional to the amount of the premiums received. I think it is very desirable that the present practice of allowing the commission on the entire premium should be discontinued, and that a higher rate, computed on the loading only, should be substituted for it,—but in the meantime of course the charge must be provided for by a proportional addition to the premium. This particular item therefore is excluded from the conclusions hereafter arrived at.

It will now, I think, be admitted that the percentage method of loading the premium is not at all applicable to the case, and that I have made good my assertion that the choice of this method was not a happy one. It now however devolves upon me to propose another which, while sufficiently convenient for practical use, shall at least be less obnoxious to censure than that which it is proposed to supersede.

It is an obvious and well-known fact that actuarial calculations are based upon two perfectly distinct and almost equally important elements,—viz. the rate of mortality at successive ages, and the rate of interest for money. These two elements enter into all the calculations connected with the business of life assurance,—sometimes in nearly equal proportions, at other times in proportions differing greatly from each other.

To resort to an extreme case,—which is generally the best for elucidating a principle,—let us suppose one in which the element of mortality is entirely eliminated,—leaving the element of interest only to be dealt with. For instance let us suppose that the transactions of the Office consist exclusively of deposits made from time to time by its policyholders as premiums to secure a sum of money at the expiration of a given time, independently of any contingency. Upon what principle should the relation between the deposits and the sum assured be determined?

This question I think admits of but one answer. An Office transacting business of the description referred to would first ascertain what rate of interest it might expect to realize upon its investments, and then calculate its premiums at a rate proportionately lower.

So far, then, our course seems clear enough. But let us now suppose that it is the element of interest which is annihilated,—the element of mortality remaining only to be dealt with.

I will first take the case of a simple Immediate Life Annuity. Now it is a peculiarity attaching to this benefit that the two elements of mortality and interest affect the value in precisely the same way. Thus, if we suppose a constant addition to be made to the force of mortality at all ages, the effect upon the calculated value of the annuity will be the same as if the addition had been made to the force of interest instead. Again, a table of the mean duration of life may be used as a table of the values of annuities certain (in perpetuity) at a variable force of interest equal to the force of mortality at each age.

This circumstance points out the way to deal with this case.

If the effect of mortality is identical with that of interest, then in loading a table in which the former is substituted for the latter, we have only to operate upon the mortality precisely as we should have done upon the interest,—that is, we must reduce the force of mortality in a constant proportion at all ages.

Reasoning by analogy, we might perhaps proceed at once to apply the principle thus deduced to the case of a life assurance, or sum payable at death,—of course *increasing* the force of mortality in a constant ratio instead of *diminishing* it. I prefer, however, to deduce this result independently from an entirely different point of view. It was, indeed, by the following course of reasoning that the principle itself was first suggested.

To take a simple case,—suppose an assurance effected for the term of one year only. Here the transaction between the Company and the policyholder is precisely of the nature of a wager, and the question resolves itself into an investigation of the proper mode of determining the odds to be taken or given by a gamester who pursues his avocation as a matter of business, and accordingly regulates his procedure upon strictly commercial principles.

Let a represent the chances in favour of the event upon which the claim depends, and b the chances against it;—the probability of the event consequently being $\frac{a}{a+b}$. Now, if we multiply this probability by a constant factor (which would correspond to our percentage loading on the premium), we involve ourselves in this dilemma. Let $1+\lambda$ be the constant multiplier, then for all values of λ greater than $\frac{b}{a}$ the probability of the event exceeds unity. This absurdity may, however, be avoided by a simple contrivance,—viz., by multiplying not the probability of the event but the chances in favour of it. The probability would then become $\frac{a(1+\lambda)}{a(1+\lambda)+b} = \frac{a}{a + \frac{b}{1+\lambda}}$, a quantity which, however great λ may

be taken, will always be less than unity.

Applying this to the case of an assurance payable at death, we have for and against the event of death in one year, at age x , d_x , and l_{x+1} respectively. If therefore, in lieu of these, we assume the chances in question to be $(1+\lambda)d_x$, and l_{x+1} , we shall have for the probability of dying in the year,

$$\frac{d_x(1+\lambda)}{l_{x+1} + d_x(1+\lambda)} = \frac{d_x + \lambda d_x}{l_x + \lambda d_x} \text{ instead of } \frac{d_x}{l_x}.$$

The probability of living a year becomes

$$1 - \frac{d_x + \lambda d_x}{l_x + \lambda d_x} = \frac{l_{x+1}}{l_x + \lambda d_x},$$

and the probability of living n years,

$$\frac{l_{x+1}}{l_x + \lambda d_x} \times \frac{l_{x+2}}{l_{x+1} + \lambda d_{x+1}} \times \dots \times \frac{l_{x+n}}{l_{x+n-1} + \lambda d_{x+n}}$$

from which expression a table of annuity values might be computed, and the premium for a whole life assurance determined therefrom in the usual way.

Instead of proceeding by yearly intervals, however, it will be better to resort to the "continuous" method—which one may safely predict will form the basis of the future system of life contingencies. The expression $\frac{d_x + \lambda d_x}{l_x + \lambda d_x}$ then becomes $-\frac{dl_x}{l_x}(1 + \lambda)$, for λd_x becomes insignificant in comparison with l_x , as d_x is diminished without limit, but not so in comparison with d_x . Hence the effect of the loading is to convert the force of mortality from $-\frac{dl_x}{l_x dx}$ to $-\frac{dl_x}{l_x dx}(1 + \lambda)$, that is, the force of mortality is increased proportionately at all ages.

In consideration, then, of the foregoing results, the method which I would propose for loading premiums is to operate upon the forces of interest and mortality, by dividing the former by $1 + \lambda$ in all cases, and by multiplying or dividing the latter by the same quantity, according to the nature of the contingency upon which the claim depends,—the value of λ being determined according to the judgment of the computer and the circumstances of the case.

If we double the force of mortality at all ages, and then calculate a table of annuities therefrom, the result will be the same as if we computed a table of annuities on two joint lives of equal ages. And if we do the same with the force of mortality trebled at all ages, we get a table of annuities on three joint lives of equal ages. Consequently, if the loading in respect of mortality were supposed to be 100 per cent, and it were desired to find the premium to be charged for an assurance on a given life, we should merely have to calculate the premium for an assurance on two joint lives of the same age, and so on for other rates of loading.

Hence it appears that if we have tables of annuities calculated for equal ages, say, up to four lives, at various rates of interest, we should be able to determine the premium to be charged according

to any given rate of loading, in respect both to interest and mortality; for by taking out the results corresponding to $\lambda=0, 1, 2, 3$, and 4, the value corresponding to any other value of λ could be found by interpolation. However, as it would be desirable to have a shorter method of attaining the required results, I proceed to show how this object may be accomplished.

The conclusions hitherto deduced are applicable to all tables of mortality, quite irrespective of their mode of construction. I shall now however assume that the law of mortality is expressed by the formula $F_x = a + bq^x$; *i.e.* that the total force of mortality is composed of two partial forces, the one a constant, and the other increasing according to the Gompertzian theory of a geometrical progression.

Waiving on this occasion the question whether the formula referred to is a true exponent of the normal law of mortality, it will be allowed to be at least sufficient for practical purposes, if the tables at present in use be taken as the standard. No one I imagine will contend that anything is gained by retaining the irregularities exhibited by the Carlisle Table, or that a reconstructed table derived by means of the formula in question will not be at least as trustworthy as the original table. Such being the case there can be no reason why we should not avail ourselves of the extreme simplicity which the formula exhibits to deduce useful and important general principles to guide us in our professional pursuits.

As a case in point,—let us suppose that the quantity q is unity. The force of mortality, and therefore the annual premiums for assurances, are the same at all ages, and consequently no reserve fund would be required for the liabilities of an Office beyond what might be considered necessary to provide for an unusual mortality, as in the case of assurances against fire. But the series represented by $a + bq^x$ may be made as nearly constant as we please by increasing the constant a . Hence it follows that a constant addition to the force of mortality at all ages, while it will necessarily increase the annual premium, will at the same time diminish the reserve required,—not *relatively* to the increased premium only,—but *absolutely*. This consideration shows very forcibly the fallacy of judging of the relative positions of Offices by the number of years premiums in hand, even when (as can rarely be the case) the circumstances are in all other respects parallel.

A consideration of the expression $a + bq^x$ enables us also to

expose the popular fallacy that the reserve is required because the event of death must, sooner or later, take place. Death would be just as certain if the force of mortality were constant, precisely as if a coin be continually tossed it must, by the law of chances, *eventually* fall head, notwithstanding that the chance of its falling head is the same at each throw. Nevertheless, in the case of a constant mortality, as we have seen, no accumulation would be necessary; and hence we see that it is not the certainty of the event, but the continual increase in the force of mortality from age to age which alone necessitates the reserve.

Further, the expression $a + bq^x$ shows us how it is that the rates of premium increase so much faster at higher ages than in youth. At the earlier ages the constant a is much the greater of the two terms, and hence the force of mortality at these ages is nearly constant. At the higher ages the term bq^x exceeds the constant, whence the force of mortality increases rapidly.

To return from this digression to the subject in hand, let us see what useful deductions can be drawn respecting the effect of multiplying the force of mortality by the constant loading $1 + \lambda$.

In the expression $a(1 + \lambda) + b(1 + \lambda) \cdot q^x$, which the force of mortality becomes, the addition $a\lambda$ made to the constant term is equivalent to the introduction of the element of interest,— $a\lambda$ corresponding precisely to the force of discount. Hence we see that this part of the mortality loading merges into the loading on account of interest.

Again, the multiplication of the term bq^x has the effect of a constant addition to the age. Thus, putting $b(1 + \lambda) \cdot q^x = b \cdot q^{x+d}$ we have $1 + \lambda = q^d$, whence d (the addition to the age) $= \frac{\log(1 + \lambda)}{\log q}$ a constant quantity.

We thus see that the method of loading proposed resolves itself, in the case of tables constructed by means of the formula $F_x = a + bq^x$, into a constant addition (positive or negative) to the rate of interest, together with a constant addition to the age.

Having thus arrived at a very simple method for loading assurance premiums—such method resulting directly (in the case of the particular law of mortality referred to) from a rational and consistent principle—I would suggest that the method in question may with propriety be adopted as the proper one in all cases, whether the table of mortality be constructed precisely according to the given law, or not.

The following are specimens of the application of the method. They are the single and annual premiums for an ordinary life assurance, calculated from the Carlisle Table, upon the supposition that the true rate of mortality is increased in the ratio of 1:1.4, and the true rate of interest (supposed to be about $4\frac{1}{2}$ per cent.) diminished in the ratio of 1.4:1.

Age.	Single Premium.	Annual Premium.
20	37.107	1.833
30	42.899	2.335
40	49.741	3.075
50	59.017	4.475
60	69.208	6.984

In reference to other descriptions of assurances, it may be sufficient to refer here to one or two of the more obvious results of the application of the theory. In the case of term assurances we should have an addition to the premium nearly equal to the addition made to the force of mortality. Again, in the case of an assurance on life against life, the principle of the method would require that the age of the one life should be increased, and that the age of the other should be diminished, the effect of which would doubtless be to increase the loading annually applied to these cases.

I propose on another occasion to go somewhat fully into these matters, when a complete set of specimen tables, embracing every description of benefit, will be given. In the meantime I will conclude this portion of my subject with the observation that in applying the theory to the business of a Reversionary Society, or an Office established for the *purchase* of interests depending upon the contingencies of life, we should of course have to reverse the process. The rate of interest, instead of being diminished, would have to be increased; and the force of mortality increased in the case of a life annuity, and diminished in that of a sum payable at death. So far as regards the rate of interest, this agrees with the course suggested by Mr. Sprague in his recent paper on this subject. The plan now proposed of operating upon the force of mortality also, by means of a constant addition to or deduction from the age, will, I think, be found desirable, more especially where advanced ages are concerned, in which cases the addition to the rate of interest has generally but little effect upon the result.