258. A Proof That $\sum u_n$ Is Divergent When u_n/u_{n-1} Can Be Expanded in a Series of Descending Integral Powers of n Beginning with $1 - 1/n$

Author(s): F. J. W. Whipple

Source: *The Mathematical Gazette*, Vol. 4, No. 72 (Jun., 1908), p. 281

Published by: [Mathematical Association](#)

Stable URL: <http://www.jstor.org/stable/3603381>

Accessed: 30-10-2015 10:20 UTC

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*.

<http://www.jstor.org>

MATHEMATICAL NOTES.

258. [D. 2. a. a.] *A proof that Σu_n is divergent when u_n/u_{n-1} can be expanded in a series of descending integral powers of n beginning with $1 - \frac{1}{n}$.*

It is well-known that when the ratio u_n/u_{n-1} can be expanded in a series of descending integral powers of n beginning with $1 - \frac{a}{n}$ the convergency can be determined by comparison with $\Sigma 1/n^a$ by choosing κ between a and 1 . In the case in which $a=1$ the comparison with $\Sigma 1/n$ breaks down and it is apparently necessary to use the next type in the scale of convergency, viz. $\Sigma 1/n \log n$. It is, however, possible to get over the difficulty by comparing the terms of Σu_n with later terms of $\Sigma 1/n$.

$$\begin{aligned} \text{Write} \quad v_n &= \frac{1}{n - \kappa}, \\ v_n/v_{n-1} &= 1 - \frac{1}{n} - \frac{\kappa}{n^2} - \frac{\kappa^2}{n^3} \dots \end{aligned}$$

Whatever be the value of β an integral value of κ can be found such that

$$u_n/u_{n-1} \equiv 1 - \frac{1}{n} - \frac{\beta}{n^2} - \frac{\gamma}{n^3} \dots > 1 - \frac{1}{n} - \frac{\kappa}{n^2} - \frac{\kappa^2}{n^3} \dots$$

for all values of n above a certain finite limit.

When κ takes this value, then Σu_n diverges more rapidly than $\Sigma \frac{1}{n - \kappa}$ which is known to be a divergent series.

This theorem is not quite as general as the one enunciated by Mr. Roseveare (*Gazette*, p. 246), who omits the word 'integral' in the enunciation, but it covers all the cases of frequent occurrence.

The most important example of the theorem is the limiting case of the Hypergeometric Series for which

$$\begin{aligned} u_n &= \frac{a(a+1) \dots (a+n-1) \beta(\beta+1) \dots (\beta+n-1)}{\gamma(\gamma+1) \dots (\gamma+n-1) \delta(\delta+1) \dots (\delta+n-1)} \\ u_n/u_{n-1} &= \frac{(a+n-1)(\beta+n-1)}{(\gamma+n-1)(\delta+n-1)} \\ &= \left(1 + \frac{a-1}{n}\right) \left(1 + \frac{\beta-1}{n}\right) \left(1 + \frac{\gamma-1}{n}\right)^{-1} \left(1 + \frac{\delta-1}{n}\right)^{-1} \\ &= 1 + \frac{a + \beta - \gamma - \delta}{n} + \dots, \end{aligned}$$

an expansion in negative integral powers of n . Our theorem shews us accordingly that the Hypergeometric Series is divergent when $a + \beta - \gamma - \delta = -1$. (Cf. Chrystal, Part II., p. 116, where the higher criterion is used.) As a numerical example we may take the case $a = -\frac{1}{4}$, $\beta = \frac{3}{4}$, $\gamma = \frac{1}{2}$, $\delta = 1$,

$$\begin{aligned} u_n &= - \frac{[3 \cdot 7 \cdot 11 \dots (4n-5)]^2 (4n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \dots 4n} \\ u_n/u_{n-1} &= 1 - \frac{1}{n} - \frac{3}{16} \cdot \frac{1}{n^2} \end{aligned}$$

In this case the series diverges less rapidly than $\Sigma \frac{1}{n}$ but more rapidly than $\Sigma \frac{1}{n-1}$ and it is therefore divergent. F. J. W. WHIPPLE.