INDUCTANCE OF STRAIGHT CONDUCTOR.

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I.

THE inductance of a length l of a straight conductor is usually given by the equation

$$L = 2l \lg \frac{D}{r} \times 10^{-9}. \quad (1)$$

Where D = distance of return conductor, r = radius of the conductor and the total length of the conductor is assumed as infinitely great compared with l and D . This is approximately the case with the conductors of a long distance transmission line.

For infinite distance D of the return conductor, that is, a conductor without return conductor, equation (1) gives $L = \infty$, that is, a finite length of an infinitely long conductor without return conductor has an infinite inductance L and so inversely zero capacity C .

In equation (1), the magnetic field is assumed as instantaneous, that is, the velocity of propagation of the magnetic field is neglected. With alternating currents traversing the conductor, this is permissible, when the distance to the return conductor is a negligible fraction of the wave-length. That is, if D is negligible compared with v/N , where v = velocity of light, N = frequency of alternating current. It obviously is not permissible in a conductor having no return conductor.

If a conductor conveying an alternating current has no return conductor, its circuit is closed by electrostatic capacity, either the distributed capacity of the conductor, or capacity connected to the ends of the conductor. To produce in such a case considerable currents, either the conductor must be very long, or the frequency and E.M.F. very high.

No conductor extending parallel to the ground, as a telegraph or transmission wire can be considered as having no return, since

even if the conductor is isolated from the ground, secondary currents induced in the ground (or in the higher regions of the atmosphere) act inductively as return currents.

Hence the case of the conductor without return conductor can physically be realized only by a conductor perpendicular to the ground, as the sending and receiving antennæ of a wireless telegraph station, and even then completely only if there are no other vertical conductors, as trees, mountains, etc., which may act as inductive returns, in the space, that is on the ocean.

Since a vertical conductor is limited in length, very high frequencies are required, and so the wave is of moderate length, that is, the velocity of propagation of the magnetic (and electrostatic) field must be considered in investigating the self-induction and the mutual induction of such a conductor.

That is, the magnetic field at a distance u from the conductor, and at time t , corresponds to the current in the conductor at the time $t - t'$, where t' is the time required for the electric field to travel the distance u , that is, $t' = u/v$, where v = velocity of light. Or: the magnetic field at distance u and time t corresponds to the current in the conductor at the time $t - u/v$.

Representing the time t by angle: $\varphi = 2\pi Nt$, where N = frequency of alternating current in the conductor, and denoting

$$a = \frac{2\pi N}{v} = \frac{2\pi}{\lambda},$$

where :

$$\lambda = \text{wave-length of electric current.} \quad (2)$$

The field at distance u and time angle φ corresponds to time angle $\varphi = au$, that is, lags in time behind the current in the conductor by the phase angle au .

II. SECTION I OF INFINITELY LONG STRAIGHT CONDUCTOR WITHOUT RETURN CONDUCTOR.

Let :

$$i = I \sin \varphi = \text{current, abs. units.} \quad (3)$$

The magnetic induction at distance u then is :

$$B = \frac{2I}{u} \sin (\varphi - au), \quad (4)$$

hence, the total magnetic flux surrounding the conductor, from distance r to infinity :

$$\begin{aligned}\Phi &= l \int_r^\infty \frac{2I}{u} \sin(\varphi - au) du \\ &= 2Il \left\{ \sin \varphi \int_r^\infty \frac{\cos au}{u} du - \cos \varphi \int_r^\infty \frac{\sin au}{u} du \right\} \quad (5)\end{aligned}$$

or, substituting the symbols :

$$\left. \begin{aligned} \int_r^\infty \frac{\cos au}{u} du &= \text{col } ar \\ \int_r^\infty \frac{\sin au}{u} du &= \text{sil } ar \end{aligned} \right\} \quad (6)$$

it is :

$$\Phi = 2Il \{ \sin \varphi \text{col } ar - \cos \varphi \text{sil } ar \}. \quad (7)$$

Fig. 1.

The E.M.F. consumed by this magnetic flux, or E.M.F. consumed by self-induction, then is :

$$e = \frac{d\Phi}{d\varphi},$$

hence :

$$e = 2lI \{ \cos \varphi \text{col } ar + \sin \varphi \text{sil } ar \} \quad (8)$$

and since the current is :

$$i = I \sin \varphi,$$

the E.M.F. consumed by self-induction contains an *energy component* :

$$e_1 = 2lI \text{sil } ar \sin \varphi \quad (9)$$

and a *wattless component* :

$$e_2 = 2lI \text{col } ar \cos \varphi \quad (10)$$

which latter leads the current by a quarter period.

The energy component e_1 gives rise to an *effective resistance* :

$$r_1 = e_1/i = 2l \text{ sil } ar. \quad (11)$$

and a *reactance* :

$$x = \left| \frac{e_2}{i} \right| = 2l \text{ col } ar. \quad (12)$$

When considering the finite velocity of propagation of the magnetic field, self-induction is thus not wattless, but contains an energy component, and can be represented by an impedance :

$$\begin{aligned} Z &= r_1 - jx, \\ &= 2l(\text{sil } ar - j \text{ col } ar), \\ &= 2l(\text{sil } ar - j \text{ col } ar) 10^{-9} \text{ ohms.} \end{aligned} \quad (13)$$

The *inductance* L would then be given by :

$$\begin{aligned} L &= \frac{jZ}{2\pi N}, \\ &= \frac{l}{\pi N} (\text{col } ar + j \text{ sil } ar), \\ &= \frac{l}{\pi N} (\text{col } ar + j \text{ sil } ar) 10^{-9} \text{ henrys.} \end{aligned} \quad (14)$$

The effective resistance r_1 represents the power

$$P_1 = i^2 r_1, \quad (15)$$

which is sent out by the conductor as electromagnetic radiation.

The functions :

$$\begin{aligned} \text{col } ar &= \int_r^\infty \frac{\cos au}{u} du, \\ \text{sil } ar &= \int_r^\infty \frac{\sin au}{u} du, \end{aligned}$$

or :

$$\left. \begin{aligned} \text{col } y &= \int_r^\infty \frac{\cos y}{y} dy, \\ \text{sil } y &= \int_r^\infty \frac{\sin y}{y} dy, \end{aligned} \right\} \quad (16)$$

can in general not be expressed in finite form, and so would have to be recorded in tables.

In some of the most important cases however, close approximations can be derived of these functions $\text{col } y$ and $\text{sil } y$, in finite form.

III. SELF-INDUCTANCE OF CONDUCTOR.

When considering the reaction of the magnetic field on the conductor, which produces it, if:

l = length of a section of the infinitely long conductor,

r = radius of conductor,

even at extremely high frequency, r is small compared with $1/\alpha$, hence ra a very small quantity, and then $\text{sil } ar$ and $\text{col } ar$ can be approximated.

The integral

$$y' = \int_0^\infty \frac{e^{-bu} \sin au}{u} du, \quad (17)$$

differentiated gives

$$\frac{dy'}{da} = \int_0^\infty e^{-bu} \cos au du. \quad (18)$$

By partial integration, it is:

$$\int e^{-bu} \cos au du = \frac{e^{-bu}(a \sin au - b \cos au)}{a^2 + b^2}.$$

Hence,

$$\frac{dy'}{da} = \frac{b}{a^2 + b^2}$$

and

$$y' = b \int \frac{da}{a^2 + b^2} + C = \tan^{-1} \left(\frac{a}{b} \right) + C. \quad (19)$$

But y' and $\tan^{-1}(a/b)$ vanish with $a = 0$ and we get $C = 0$.

The integral (17) becomes

$$y' = \int_0^\infty \frac{e^{-bu} \sin au}{u} du = \tan^{-1} \left(\frac{a}{b} \right). \quad (20)$$

In the limit case of $b = 0$, we get

$$y'' = \int_0^\infty \frac{\sin au}{u} du = \frac{\pi}{2}. \quad (21)$$

If in the integral:

$$y''' = \int_0^r \frac{\sin au}{u} du. \quad (22)$$

r is a very small quantity, expanding the sine function into a series and integrating the series, gives :

$$y''' = \int_0^r \frac{1}{u} \left(au - \frac{a^3 u^3}{3} + \frac{a^5 u^5}{5} \dots \right) du$$

hence :

$$y''' = ar - \frac{a^3 r^3}{3} + \frac{a^5 r^5}{5} \dots \quad (23)$$

Combining (21) and (23), we get

$$\int_r^\infty \frac{\sin au}{u} du = y'' - y''' = \frac{\pi}{2} - ar + \frac{a^3 r^3}{3} - \frac{a^5 r^5}{5} + \dots \quad (24)$$

In integral calculus, we have the relation between these two definite integrals

$$\int_\delta^\gamma \frac{\cos a\theta}{\sin \frac{1}{2}\theta} d\theta = 2 \int_{a\delta}^{a\gamma} \frac{\cos \varphi}{\varphi} d\varphi,$$

provided that $\gamma > \delta$, $a\gamma = \infty$ and γ and δ are very small.

From trigonometry, we get

$$\begin{aligned} 2\{\sin \theta + \sin 2\theta + \dots + \sin S\theta\} &= \frac{2 \sin \frac{1}{2}(1+S)\theta \cdot \sin \frac{1}{2}S\theta}{\sin \frac{1}{2}\theta} \\ &= \frac{\cos \frac{1}{2}\theta - \cos (\frac{1}{2} + S)\theta}{\sin \frac{1}{2}\theta}. \end{aligned}$$

Integrating this equation, we get

$$\begin{aligned} \int_\delta^\pi \frac{\cos (\frac{1}{2} + S)\theta}{\sin \frac{1}{2}\theta} d\theta &= \int_\delta^\pi \cot \frac{1}{2}\theta d\theta \\ &+ 2 \left[\cos \theta + \frac{1}{2} \cos 2\theta + \dots + \frac{1}{S} \cos S\theta \right] \\ &= 2 \lg \frac{\sin \frac{\pi}{2}}{\sin \frac{\delta}{2}} + 2 \left(-1 + \frac{1}{2} - \frac{1}{3} + \dots + \frac{(-1)^S}{S} \right) \\ &- 2 \left(\cos \delta + \frac{1}{2} \cos 2\delta + \dots + \frac{1}{S} \cos S\delta \right) \end{aligned}$$

or

$$\int_\delta^\pi \frac{\cos (\frac{1}{2} + S)\theta}{\sin \frac{1}{2}\theta} d\theta = -2 \left\{ 1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{(-1)^S}{S} - \lg 2 \right\}$$

$$\begin{aligned}
& -2 \lg \frac{2 \sin \frac{\delta}{2}}{\delta} - 2 \lg S \delta + 2 \left\{ \lg S - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{S} \right\} \\
& + 2 \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{S} - \left(\cos \delta + \frac{1}{2} \cos 2\delta + \cdots + \frac{1}{S} \cos S\delta \right) \right\}.
\end{aligned}$$

Now let us increase S infinitely great, and $S\delta$ remaining finite, so that δ decreases infinitely small, this combined with the preceding equation, gives

$$\begin{aligned}
\text{Limit}_{S=\infty} \int_{\delta}^{\pi} \frac{\cos(\frac{1}{2} + S)\theta}{\sin \frac{1}{2}\theta} d\theta &= 2 \int_{S\delta}^{S\pi} \frac{\cos \varphi}{\varphi} d\varphi = -2 \lg S \delta \\
&+ 2 \left\{ \lg S - 1 - \frac{1}{2} - \frac{1}{3} \cdots - \frac{1}{S} \right\} \\
&+ 2 \left\{ \left[\frac{\delta^2}{2} + \frac{(2\delta)^2}{2 \cdot 2} + \cdots + \frac{(S\delta)^2}{2 \cdot S} \right] - \left[\frac{\delta^4}{4} + \frac{(2\delta)^4}{2 \cdot 4} + \cdots \right] + \cdots \right\},
\end{aligned}$$

or

$$\begin{aligned}
\text{Limit}_{S=\infty} \int_{S\delta}^{S\pi} \frac{\cos \varphi}{\varphi} d\varphi &= -\lg S \delta - \left\{ \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{S} \right) - \lg S \right\} \\
&+ \frac{(S\delta)^2}{2 \cdot 2} - \frac{(S\delta)^4}{4 \cdot 4} + \cdots
\end{aligned}$$

Let us denote the second term of the above expression by q , or

$$q = \text{Limit}_{S=\infty} \left\{ \sum_1^S \frac{1}{S} - \lg S \right\}.$$

This series has a finite value and has been computed to be

$$0.5772156 \dots$$

Putting $S\delta = d$ and $S\pi = \infty$, we get

$$\int_d^{\infty} \frac{\cos \varphi}{\varphi} d\varphi = -q - \lg d + \frac{d^2}{2 \cdot 2} - \frac{d^4}{4 \cdot 4} + \dots$$

Hence, we obtain

$$\int_r^{\infty} \frac{\cos au}{u} du = -q - \lg ar + \frac{(ar)^2}{2 \cdot 2} - \frac{(ar)^4}{4 \cdot 4} + \dots \quad (25)$$

Combining the results obtained in (24) and (25), we get

$$\Phi = 2Il \left\{ \sin \varphi \cdot \left(-q - lg ar + \frac{(ar)^2}{2 \cdot 2} - \frac{(ar)^4}{4 \cdot 4} + \dots \right) - \cos \varphi \cdot \left(\frac{\pi}{2} - ar + \frac{(ar)^3}{3 \cdot 3} - \frac{(ar)^5}{5 \cdot 5} + \dots \right) \right\}. \quad (26)$$

With the dimension of common wire, we can neglect all terms containing ar except $lg ar$, and we have

$$\Phi' = -2Il \left\{ (q + lg ar) \cdot \sin \varphi + \frac{\pi}{2} \cos \varphi \right\}, \quad (27)$$

or

$$\Phi' = -2Il \sqrt{(q + lg ar)^2 + \frac{\pi^2}{4}} \sin(\varphi - \omega), \quad (28)$$

where

$$\tan \omega = \frac{\pi}{2(q + lg ar)} \quad (29)$$

and

$$q = 0.5772156 \dots \quad (30)$$

Therefrom follows: (13)

$$\begin{aligned} Z &= 2l \left\{ \frac{\pi}{2} + j(q + lg ar) \right\} 10^{-9} \\ &= 2l \left\{ \frac{\pi}{2} - j \left(lg \frac{1}{ar} - q \right) \right\} 10^{-9} \text{ ohms} \end{aligned} \quad (31)$$

and, (14)

$$\begin{aligned} L &= \frac{l}{\pi N} \left\{ -(q + lg ar) + j \frac{\pi}{2} \right\} 10^{-9} \\ &= \frac{l}{\pi N} \left\{ \left(lg \frac{1}{ar} - q \right) + j \frac{\pi}{2} \right\} 10^{-9} \text{ henrys} \end{aligned} \quad (32)$$

as the effective impedance, and inductance, respectively, of a section l of an infinitely long conductor at frequency N , where:

$$a = \frac{2\pi N}{v}$$

and

v = velocity of light.

$$q = .5772156 \dots$$

It follows herefrom :

The power radiated by the conductor, section l , as electromagnetic wave :

$$\begin{aligned} P &= i^2 r_1 \\ &= 2l \frac{\pi}{2} i^2 \\ &= \pi l i^2 \end{aligned} \quad (33)$$

is independent of the frequency N , and the conductor dimensions, as the radius r , as long as ar is negligible, that is, $2\pi Nr$ small compared with the velocity of light v , or $2\pi r$ small compared with the wave-length λ .

Instance :

$$l = 200 \text{ ft} = 6,000 \text{ cm.}$$

$$r = .4'' = 1 \text{ cm.}$$

$$N = 400,000 \text{ cycles,}$$

hence

$$\lambda = 2500 \text{ ft.}$$

$$ar = 85 \times 10^{-6}$$

and

$$Z = (18.9 - 105.5j) 10^{-6} \text{ ohms.}$$

At

$$i = 1000 \text{ amps.}$$

$$P = 18.9 \text{ watts.}$$

IV. SELF-INDUCTANCE OF STRAIGHT CONDUCTOR OF FINITE LENGTH l .

Let

$$i = I \sin \varphi = \text{current.}$$

$$l = \text{length of the wire.}$$

$$a/2\pi N = \text{reciprocal of the velocity of the propagation of the electromagnetic wave.}$$

$$ds = \text{equivalent current element on the axis of the wire.}$$

$$u = \text{vector drawn from } ds \text{ to the point } P \text{ outside the wire.}$$

$$\omega = \text{the angle between the vector } u \text{ and the current } i.$$

The magnetic induction due to the current element at any point in free space is represented by

$$B = \frac{I \sin(\varphi - au) \cdot \sin \omega \cdot ds}{u^2}.$$

The total flux of magnetic induction around the wire is given by the following integral

$$\Phi = \int_0^l \int_0^\pi \int_{r/\sin \omega}^\infty \frac{I \sin(\varphi - au) \cdot u \cdot \sin \omega}{u^2} du d\omega ds \quad (34)$$

or

$$\Phi = I \int_0^l \int_0^\pi \int_{r/\sin \omega}^\infty \frac{\sin(\varphi - au) \cdot \sin \omega}{u} du d\omega ds. \quad (35)$$

The first integral of the above equation is just the same form as treated in the preceding case, and we have

$$\Phi = I \int_0^l \int_0^\pi \left\{ \sin \varphi \cdot [-q - lg ar + lg(\sin \omega)] - \cos \varphi \cdot \frac{\pi}{2} \right\} \sin \omega d\omega ds. \quad (36)$$

Consider the integral

$$\int_0^\pi \sin \omega \cdot lg(\sin \omega) d\omega,$$

we have

$$X \equiv \int_0^\pi \sin \omega \cdot lg(\sin \omega) d\omega = 2 \int_0^{\pi/2} \sin \omega \cdot lg(\sin \omega) d\omega$$

and

$$Y \equiv \int \sin \omega \cdot lg(\sin \omega) d\omega = \cos \omega - \cos \omega \cdot lg(\sin \omega) + \frac{1}{2} lg \frac{1 - \cos \omega}{1 + \cos \omega}.$$

Hence, we obtain

$$\begin{aligned} X &= 2L \lim_{\epsilon=0} [Y]_{\epsilon/2}^{\pi/2} = 2L \lim_{\epsilon=0} \left\{ lg 2 - 1 + \left(lg \frac{\epsilon}{2} - lg \frac{\epsilon}{2} \right) \right\} \\ &= 2(lg 2 - 1), \end{aligned}$$

or

$$\int_0^\pi \sin \omega \cdot lg(\sin \omega) d\omega = 2(lg 2 - 1).$$

Consequently, we get

$$\Phi = 2I \int_0^l \left\{ -(q + lg ar + 1 - lg 2) \sin \varphi - \frac{\pi}{2} \cos \varphi \right\} ds,$$

or

$$\begin{aligned}\Phi &= 2Il \left\{ \left(lg \frac{1}{ar} - q - 1 + lg2 \right) \sin \varphi - \frac{\pi}{2} \cos \varphi \right\} \\ &= 2Il \left\{ \left(lg \frac{1}{ar} - s \right) \sin \varphi - \frac{\pi}{2} \cos \varphi \right\}\end{aligned}\quad (37)$$

where

$$s = q + 1 - lg2 = .8841 \dots$$

or

$$\Phi = 2Il \sqrt{\left(lg \frac{1}{ar} - s \right)^2 + \left(\frac{\pi}{2} \right)^2} \sin(\varphi - \omega) \quad (38)$$

where

$$\tan \omega = \frac{\pi}{2 \left(lg \frac{1}{ar} - s \right)} \quad (39)$$

hence

$$Z = 2l \left\{ \frac{\pi}{2} - j \left(lg \frac{1}{ar} - s \right) \right\} 10^{-9} \text{ ohms}, \quad (40)$$

and

$$L = \frac{l}{\pi N} \left\{ \left(lg \frac{1}{ar} - s \right) + j \frac{\pi}{2} \right\} 10^{-9} \text{ henrys}. \quad (41)$$

Instance :

Length of conductor, $l = 200 \text{ ft.} = 6000 \text{ cm.}$

Diameter of conductor, $2r = .8'' = 2 \text{ cm.}$

Conductor grounded at bottom, connected at top to capacity about three times that of the conductor, so as to reduce its frequency of oscillation to $N = 400,000$ cycles. This gives $Z = (18.9 - 102j) 10^{-6}$ ohms. At $i = 1,000$ amps., $P = i^2 r_1 = 18.9$ watts.

V. MUTUAL INDUCTANCE BETWEEN TWO FINITE STRAIGHT CONDUCTORS AT CONSIDERABLE DISTANCE FROM EACH OTHER.

This problem is of interest in relation to wireless telegraphy, in that it investigates one component, the electromagnetic, of the electric field which transmits the message between sending and receiving station.

The distance between the two conductors is supposed to be very great compared with the lengths of the wires, and the angles subtended by one of the wires to any point in another wire may be considered equal to each other.

Let :

l and l_1 = lengths of the straight conductors in space.

D = length of any vector connecting the wires.

α and α_1 = angles between D and the wires.

As we have assumed D to be very great compared with l and l_1 , the angles α and α_1 may be considered remain constant independent of D .

Fig. 2.

Also, let us denote by δ the angle subtended by any point in l carrying electric current i to the projection of l_1 upon the plane determined by l and D ; and θ the angle between the planes determined by the vector D and each of the wires l and l_1 .

The projection of l_1 upon the plane of l and D can be decomposed into two components, one parallel to and another perpendicular to the vector D ; thus, we get

$$l_1 \cos \alpha_1 = \text{parallel component.}$$

$$l_1 \sin \alpha_1 \cos \theta = \text{normal component.}$$

$$\delta = \frac{l_1 \sin \alpha_1 \cos \theta}{D}.$$

Hence, we get the following integral for the expression of the total flux of the magnetic induction which has just passed over the conductor l_1 at the instant t .

$$\Phi = \int_0^l \int_D^\infty \frac{I \sin (2\pi Nt - au) \cdot \sin \alpha \cdot u \cdot \delta}{u^2} du ds$$

or

$$\Phi = \frac{l_1 \sin \alpha_1 \sin \alpha \cos \theta}{D} \int_0^l \int_D^\infty \frac{\sin (\varphi - au)}{u} du ds.$$

Let us suppose that the distance D is very great compared with the lengths of conductors l and l_1 and the wave-length λ .

Putting $D = \lambda(m + s)/4$, where m is a large positive integer and s the positive fractional part less than one, we have

$$\begin{aligned}
 X &\equiv \int_D^\infty \frac{\sin(\varphi - au)}{u} du \\
 &= \int_D^\infty \frac{\sin \varphi \cos au}{u} du - \int_D^\infty \frac{\cos \varphi \sin au}{u} du \\
 &= \sin \varphi \left[\int_D^{\lambda(m+1)/4} + \int_{\lambda(m+1)/4}^{\lambda(m+2)/4} + \dots \right] \frac{\cos au}{u} du \\
 &\quad - \cos \varphi \left[\int_D^{\lambda(m+1)/4} + \int_{\lambda(m+1)/4}^{\lambda(m+2)/4} + \dots \right] \frac{\sin au}{u} du
 \end{aligned}$$

(i) $m = 2n$

$$\begin{aligned}
 X_1 &\doteq (-1)^n \left(1 - \sin \frac{s\pi}{2} \right) \frac{\sin \varphi}{\frac{\lambda}{4}na} + (-1)^n \left(1 - \cos \frac{s\pi}{2} \right) \frac{\cos \varphi}{\frac{\lambda}{4}(2n+1)a} \\
 &\quad - (-1)^n \frac{\sin \varphi}{\frac{\lambda}{4}a} \left\{ \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} \dots \right\} \\
 &\quad - (-1)^n \frac{\cos \varphi}{\frac{\lambda}{4}a} \left\{ \frac{1}{n+\frac{1}{2}} - \frac{1}{n+1+\frac{1}{2}} + \frac{1}{n+2+\frac{1}{2}} \dots \right\}.
 \end{aligned}$$

(ii) $m = 2p + 1$

$$\begin{aligned}
 X_{11} &\doteq (-1)^p \frac{\sin \varphi}{\frac{\lambda}{4}(2p+2)a} \left(1 - \sin \frac{s\pi}{2} \right) \\
 &\quad + (-1)^p \frac{\cos \varphi}{\frac{\lambda}{4}(2p+1)a} \left(1 - \cos \frac{s\pi}{2} \right) \\
 &\quad - (-1)^p \frac{\sin \varphi}{\frac{\lambda}{4}a} \left\{ \frac{1}{p+1} - \frac{1}{p+2} + \dots \right\} \\
 &\quad + (-1)^p \frac{\cos \varphi}{\frac{\lambda}{4}a} \left\{ \frac{1}{p+1+\frac{1}{2}} - \frac{1}{p+2+\frac{1}{2}} + \dots \right\}.
 \end{aligned}$$

These results have already a relative error of $(\lambda/4D)^2$ order and the summation of the fractional series can be obtained only in approximate ways.

It can be shown that the summation of the series,

$$S \equiv \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \dots \equiv \sum_{p=0}^{\infty} \frac{(-1)^p}{n+p},$$

can be represented by $\frac{1}{4} \log (n^2 - 1)/n(n-2)$ with a relative error of $(1/n)^3$ order.

For the sake of simplicity, we shall follow another method of the approximate calculation.

Let us consider the following series

$$S_0 = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} \dots$$

$$S' = \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} \dots,$$

then we have the following relations

$$S > S'$$

$$S > \frac{1}{(n+1)^2} + \frac{1}{(n+3)^2} + \frac{1}{(n+5)^2} + \dots$$

$$S' < \frac{1}{(n+1)^2} + \frac{1}{(n+3)^2} + \frac{1}{(n+5)^2} + \dots$$

$$S < \frac{1}{n^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+4)^2} + \dots$$

$$\therefore S - S' < \frac{1}{n^2},$$

moreover we have

$$S + S' = \frac{1}{n}.$$

Therefore, neglecting the small quantity $(1/n)^2$, we get

$$S \doteq \frac{1}{2n}.$$

Substituting this approximate result in the preceding equations, we get the generalized formulæ

$$\int_{D=\lambda(m+S)/4}^{\infty} \frac{\sin(\varphi - au)}{u} du \doteq -(-1)^m \frac{\cos(\varphi - aD)}{aD + \pi/2}.$$

Hence, we get

$$\Phi \doteq (-1)^{m+1} \frac{M_1 \sin a \sin \alpha_1 \cos \theta}{D(aD + \pi/2)} \cos(\varphi - aD)$$

where

$$D = \frac{\lambda}{4}(m + s) \text{ and } 0 < s < 1$$

and the formula shall give the result with a relative error of $(1/m)^2$ order.

Taking the differential coefficient of Φ with respect to t , we obtain the expression for the induced E.M.F.

$$e = -\frac{d\Phi}{dt} \doteq (-1)^{m+1} \frac{2\pi N M_1 \sin a \sin \alpha_1 \cos \theta}{D(Da + \pi/2)} \sin(\varphi - aD),$$

or

$$e \doteq (-1)^{m+1} \frac{v M_1 \sin a \sin \alpha_1 \cos \theta}{D\left(D + \frac{\lambda}{4}\right)} \sin \frac{2\pi}{\lambda}(vt - D),$$

where v is the velocity of the propagation of the electromagnetic wave in the air and λ the wave-length.

Writing the equation in another form, we get

$$e \doteq (-1)^{m+1} \left(1 - \frac{\lambda}{4D}\right) \frac{v M_1 \sin a \sin \alpha_1 \cos \theta}{D^2} \sin \frac{2\pi}{\lambda}(vt - D).$$

Instances:

$$l = l_1 = 200 \text{ ft.} \doteq 6100 \text{ cm.}$$

$$r = .4 \text{ inch} \doteq 1 \text{ cm.}$$

$$N = 4 \times 10^5.$$

$$I/\sqrt{2} = 1000 \text{ amp.}$$

$$D_1 = 10 \text{ miles} = 1.61 \times 10^6 \text{ cm.}$$

$$D_2 = 200 \text{ miles} = 3.22 \times 10^7 \text{ cm.}$$

$$D_3 = \text{Quadrant of the earth} \doteq 6214 \text{ miles} \doteq 10^9 \text{ cm.}$$

The induced E.M.F. become

$$(i) \quad \frac{E_1}{\sqrt{2}} = 0.427 \text{ volt.}$$

$$(ii) \quad \frac{E_2}{\sqrt{2}} = 1.08 \text{ millivolts.}$$

$$(iii) \quad \frac{E_3}{\sqrt{2}} = 1.12 \text{ microvolts.}$$

The preceding investigation applies only to the electromagnetic component of the electric field of a straight conductor. Similar equations, however, would be derived in an analogous manner for the electrostatic component of the electric field, that is, for the capacity.

From the relation between the inductance L and the capacity C of a finite length l of an infinitely long conductor.

$$LC = \frac{l^2}{v^2}$$

where v = velocity of light; it follows, that when neglecting the velocity of propagation of the electric field in space, the capacity of a finite length of an infinitely long conductor without return conductor equals zero :

$$C = 0$$

since :

$$L = \infty .$$

Considering the finite velocity of propagation of the electric field in space, L becomes finite, but a function of the frequency of the alternating current, and so the capacity C also becomes finite, and a function of the frequency, which may be evaluated by a similar investigation.