which expresses the Bessel function of the second kind in terms of the Legendre function.

Since the most general solution of Bessel's differential equation of order n consists of a linear combination of $J_n(z)$ and $Y_n(z)$, it follows that this can be expressed as a linear combination of

$$z^{\frac{1}{2}} \int_0^1 \cos zt P_{n-\frac{1}{2}}(t) dt$$

and

$$z^{\frac{1}{2}} \int_0^1 \sin zt P_{n-\frac{1}{2}}(t) dt.$$

But this result, which is thus obtained for the case in which n is integral, is the same as the result already obtained for the case in which n is not integral; and therefore we see that in all cases the general solution of Bessel's equation is a linear combination of

$$\left. \begin{aligned} & z^{\frac{1}{2}} \int_0^1 \cos zt P_{n-\frac{1}{2}}(t) dt \\ & z^{\frac{1}{2}} \int_0^1 \sin zt P_{n-\frac{1}{2}}(t) dt \end{aligned} \right\} \quad (\text{V.})$$

and

*On Groups which are Linear and Homogeneous in both Variables and Parameters.** By W. BURNSIDE. Received and Read November 13th, 1902.

In this paper I propose to discuss the nature of the characteristic determinant, first of any simply transitive linear homogeneous group, and secondly of any transitive linear homogeneous group. The result of this discussion, leading to a quite general form for the

* Prof. L. E. Dickson has called my attention to two errors in my paper "On the Continuous Group defined by any given Group of Finite Order" (*Proc. Lond. Math. Soc.*, Vol. xxix., pp. 546-565). These occur in §§ 5, 6, dealing with particular cases of the characteristic determinant of a simply transitive linear homogeneous group. The induction in § 5 is faulty; and, as Prof. Dickson has shown, it is *not* the case that, if the characteristic determinant is the power of a single linear factor, the group is necessarily Abelian. The error in § 6 occurs in the third line from the bottom of p. 553. The correct inference from the previous reasoning, there, should obviously be not $\nu = \mu$, but that ν is a multiple of μ . The results of §§ 5, 6 are used in § 10 in establishing one of the main results of the paper. In order that this should rest on a sound basis it is essential that these errors should be corrected.

determinant in each case, is given in the first two sections of the paper. A short third section is added to apply the results so obtained to giving a materially simpler and shorter proof of the main propositions contained in the second section of my paper "On the Continuous Group defined by any given Group of Finite Order" (*Proc. Lond. Math. Soc.*, Vol. xxix., pp. 546-565).

The question considered in the first section of this paper has been dealt with at considerable length by Herr Molien in his memoir "Ueber Systeme höherer complexer Zahlen" (*Math. Ann.*, Vol. xli., pp. 83-156, 1892); and most of the results arrived at in that section are contained in, or may be deduced from, those given by Herr Molien. I do not think, however, that this is the case with all of them. For instance, Herr Molien nowhere shows that the characteristic determinants of two reciprocal simply transitive groups are identical. It is not, in fact, very easy to find exactly what is and what is not contained in Herr Molien's memoir; for he treats the different parts of what is one subject from the point of view first of complex numbers, then of linear groups, and lastly of matrices. The methods used here are, in any case, quite distinct from those of Herr Molien: they depend solely (i) on the existence of the reciprocal group, and (ii) on the fact that the only self-conjugate subgroup of the general linear group is the cyclical group multiplying each variable by the same factor.

I.

By a linear homogeneous group in the following pages is meant a group whose finite equations are linear and homogeneous both in the parameters and in the variables. When, of the parameters entering in the equations, the number that are linearly independent is equal to the number of variables, the group is called simply transitive. Corresponding to any simply transitive group there is always a second linear homogeneous group (identical with the first, if the first is Abelian, but not otherwise) simply transitive in the same variables, every one of whose operations is permutable with every operation of the first group. This is called the reciprocal group.

Let G and G' be reciprocal simply transitive groups in n variables. An arbitrarily chosen substitution of G' may be reduced to the canonical form

$$\begin{aligned} y'_1 &= \lambda_1 y_1, & y'_2 &= a_2 y_1 + \lambda_1 y_2, & \dots & y'_p &= a_p y_{p-1} + \lambda_1 y_p, \\ z'_1 &= \lambda_2 z_1, & z'_2 &= b_2 z_1 + \lambda_2 z_2, & \dots & z'_q &= b_q z_{q-1} + \lambda_2 z_q, \\ &\dots & \dots & \dots & \dots & \dots & \dots \end{aligned}$$

where $\lambda_1, \lambda_2, \dots$, are the different roots of the characteristic equation of the substitution and $p, q, \dots$, their multiplicities. Every substitution which is permutable with this substitution, and therefore every substitution of G , must transform the y 's, z 's, $\dots$, separately among themselves. Hence G is certainly reducible. The linear transformation of the y 's effected by G will constitute a group which, since G is transitive, is certainly transitive, but is not necessarily simply transitive. This may or may not be reducible. If it is so, similar considerations will apply to the transitive group of transformations on the $p' (< p)$ linear functions of the y 's which are transformed among themselves. Continuing thus, we must eventually arrive at a set of independent linear functions

$$x_{11}, x_{12}, \dots, x_{1r} \quad (r \geq 1)$$

of the original variables, which are transformed among themselves by every operation of G such that it is impossible to form $r' (< r)$ linear functions of them which are transformed among themselves. In other words, the group of transformations on the r functions is irreducible.

Suppose now that some substitution S' of the reciprocal group G changes

$$x_{11}, x_{12}, \dots, x_{1r},$$

into

$$x_{21}, x_{22}, \dots, x_{2r}.$$

Since the first set are linearly independent among themselves, so also must the second set be; but the $2r$ symbols may possibly be connected by one or more linear equations. If the linear equation

$$\Sigma b_i x_{2i} = \Sigma a_i x_{1i}$$

holds, S' changes the linear function $\Sigma b_i x_{2i}$ of the first set into another linear function $\Sigma a_i x_{1i}$ of the first set. Now, S being any

operation of G , $S'SS'^{-1} = S$.

Hence, if S changes $\Sigma a_i x_{1i}$ into $\Sigma c_i x_{1i}$, and $\Sigma b_i x_{2i}$ into $\Sigma d_i x_{2i}$, then S' changes $\Sigma d_i x_{2i}$ into $\Sigma c_i x_{1i}$. Now G contains operations which severally change $\Sigma b_i x_{2i}$ into r linearly independent functions of the first set. Hence, since S' changes each of these into another function of the first set, S' must transform the first set among themselves. Unless S' replaces each of the symbols $x_{11}, x_{12}, \dots, x_{1r}$ by the same multiple of itself, G , so far as it affects this set, would be reducible, which is contrary to the supposition made. Hence either

$$x_{2i} = ex_{1i} \quad (i = 1, 2, \dots, r),$$

where e is a constant multiplier, or the second set is independent of the first. Further, since S' is permutable with every operation of G , every operation of G effects exactly the same transformation on

$$x_{21}, x_{22}, \dots, x_{2r},$$

as it does on

$$x_{11}, x_{12}, \dots, x_{1r}.$$

If some other transformation S'' of G' changes the first set into

$$x_{31}, x_{32}, \dots, x_{3r},$$

precisely similar reasoning shows that either these are linearly independent of the first two sets, or else

$$x_{3i} = ex_{1i} + fx_{2i}, \quad (i = 1, 2, \dots, r),$$

where e and f are constant multipliers. Moreover, the third set are transformed among themselves by G in exactly the same way as the first and second sets. By considering further substitutions of G' this process can be continued till no new independent set can be formed. Suppose that just s sets thus arise

$$x_{11}, x_{12}, \dots, x_{1r},$$

$$x_{21}, x_{22}, \dots, x_{2r},$$

$$\dots \dots \dots$$

$$x_{s1}, x_{s2}, \dots, x_{sr}.$$

The symbols in each line are transformed among themselves by each substitution of G in exactly the same way, and the most general substitution of G' changes x_{1i} into

$$e_1x_{1i} + e_2x_{2i} + \dots + e_sx_{si}, \quad (i = 1, 2, \dots, r).$$

Further, the substitutions S'^{-1} and $S'^{-1}S''$ of G' change

$$x_{21}, x_{22}, \dots, x_{2r},$$

into

$$x_{11}, x_{12}, \dots, x_{1r},$$

and

$$x_{31}, x_{32}, \dots, x_{3r},$$

respectively. Hence the most general substitution of G' changes x_{2i} into

$$f_1x_{1i} + f_2x_{2i} + \dots + f_sx_{si}, \quad (i = 1, 2, \dots, r).$$

It follows thus that the symbols in each column of the preceding scheme are transformed among themselves by every operation of G' , and that any operation of G' gives the same transformation for the set of symbols in each column. The groups of transformations

effected by G and G' on the sr symbols in the scheme must each be transitive, since G and G' are transitive; and, since G and G' are reciprocal, the two groups must each be simply transitive. Hence, if ρ is the number of independent parameters in the group of trans-

formations of $x_{11}, x_{12}, \dots, x_{1r},$

and σ the number in that of

$$x_{11}, x_{21}, \dots, x_{r1},$$

$$\rho = \sigma = rs.$$

But, in any case, $\rho \leq r^2, \sigma \leq s^2.$

Hence $r = s$

and $\rho = \sigma = r^2.$

Whatever the simply transitive group G in n variables may be, it must therefore be possible to choose r^2 ($r \geq 1$) independent linear functions of the variables and an equal number of independent linear functions of the parameters, so that the first r^2 of the finite equations of the group take the form

$$x'_{ij} = \sum_{k=1}^{k=r} e_{jk} x_{ik} \quad (i, j = 1, 2, \dots, r);$$

while those of the reciprocal group are

$$x'_{ij} = \sum_{k=1}^{k=r} e'_{ik} x_k \quad (i, j = 1, 2, \dots, r).$$

So far as it affects this set of variables G is simply isomorphic with the general linear homogeneous group in r variables.

2. If, when G and G' have been transformed so that the first r^2 equations of each have the form arrived at in the preceding section, $x_{11}, x_{12}, \dots, x_{rr}, x'_{11}, x'_{12}, \dots, x'_{rr}$ are all made zero, the first r^2 equations of each are satisfied identically, and the remaining $n - r^2$ equations of each are the equations of linear homogeneous groups in $n - r^2$ variables. These groups must be transitive, or G and G' themselves would be intransitive; and, since G and G' are reciprocal, they must be simply transitive. Hence the process of the preceding section may be applied to them. It is therefore possible to find r_1^2 ($r_1 \geq 1$) linear functions of the original variables, y_{ij} ($i, j = 1, 2, \dots, r_1$), linearly independent of each other and of the preceding r^2 functions

such that when the x_{ij} 's are made zero the y 's undergo the substitution

$$y'_{ij} = \sum_{k=1}^{k=r_1} f_{jk} y_{ik} \quad (i, j = 1, 2, \dots, r_1),$$

corresponding to the general substitution of G , and

$$y'_{ij} = \sum_{k=1}^{k=r_1} f'_{ik} y_{kj} \quad (i, j = 1, 2, \dots, r_1),$$

corresponding to that of G' . The r_1^2 symbols f_{ij} represent linearly independent functions of the original parameters; but whether the f_{ij} 's are independent of the e_{ij} 's requires further discussion. The general equations of the group, so far as these symbols are concerned, will contain for each y'_{ij} a linear function $f_{ij}(x)$ of the x_{ij} 's with linear functions of the original parameters as coefficients.

The process already used may be repeated for the simply transitive group that arises on making all the x_{ij} 's and y_{ij} 's zero, leading to r_2^2 new variables independent of each other and of the x 's and the y 's, whose transformations are given by

$$z'_{ij} = g_{ij}(x) + G_{ij}(y) + \sum_{k=1}^{k=r_2} g_{jk} z_{ik} \quad (i, j = 1, 2, \dots, r_2),$$

where $g_{ij}(x)$ and $G_{ij}(y)$ represent linear functions of the x_{ij} 's and y_{ij} 's with coefficients which are linear functions of the original parameters; and the process may be continued till the whole of the original variables are accounted for.

3. To discuss the question of the independence of the parameters $e_{ij}, f_{ij}, \dots$, consider first the two sets of equations

$$x'_{ij} = \sum_{k=1}^{k=r} e_{jk} x_{ik} \quad (j = 1, 2, \dots, r),$$

$$y'_{ij} = \sum_{k=1}^{k=r_1} f_{jk} y_{ik} \quad (j = 1, 2, \dots, r_1).$$

These constitute a group. Suppose first that r and r_1 are not equal, and let r_1 be greater than r . The totality of the substitutions which replace each x by the same multiple of itself form a self-conjugate sub-group. For this sub-group

$$e_{11} = e_{22} = \dots = e_{rr},$$

$$e_{rs} = 0 \quad (r \neq s).$$

The $r_1^2 f$'s are linearly independent among themselves, and hence, even when there are as many linear relations as possible between them and the e 's, at least $r_1^2 - r^2 + 1$ of them remain arbitrary when the e 's are subject to the above $r^2 - 1$ relations. The group in the y 's would therefore contain a self-conjugate sub-group of order $r_1^2 - r^2 + 1$ at least. Now the general linear group has no self-conjugate sub-group except that which replaces every symbol by the same multiple of itself. Hence, if r and r_1 are unequal, the e 's and the f 's are linearly independent. If $r_1 = r$, and if the e 's and the f 's are not independent, suppose that there are s linear relations connecting them. Then when the e 's satisfy the preceding relations at least $r^2 - s + 1$ of the f 's remain arbitrary. Hence the condition as regards the self-conjugate sub-group can only be satisfied if $s = r^2$, i.e., if the f 's are linearly independent functions of the e 's. But when this is the case to every operation of the group in the x 's will correspond a definite operation of the group in the y 's. Hence the group of the y 's can be so transformed that each f is equal to the corresponding e .

Suppose next that $r = r_1 = r_2$. If each f is equal to the corresponding e , then from the previous case either the g 's must be independent of the e 's or the z 's can be so chosen that each g is equal to the corresponding e . If the e 's and the f 's are distinct, the possibility arises that the g 's may be functions of the e 's and f 's together. If this is so, consider the sub-group given by

$$e_{11} = e_{22} = \dots = e_{rr}, \quad f_{11} = f_{22} = \dots = f_{rr}, \\ e_{rs} = f_{rs} = 0 \quad (r \neq s).$$

Of the g 's then just two would be independent, and the group of the z 's would have a self-conjugate sub-group of order 2, which is not the case. Hence the g 's must be independent, or each g must be equal to the corresponding e , or each g must be equal to the corresponding f . The same method may be used to deal with any number of equalities among the r 's, each fresh set of parameters being shown to be either independent of the preceding ones or identical with one of the preceding sets.

4. The linear functions denoted in the second section by $f_{ij}(x)$, $g_{ij}(x)$, $G_{ij}(y)$, ..., remain now to be discussed. For this purpose suppose again that all the variables except those of two sets, say the y 's and the z 's (these are taken to preserve the notation, and not as

representing necessarily the second and the third sets), are made zero. Suppose also that the f 's and the g 's are independent. The equations of G for the y 's and z 's then become a set of $r_1^2 + r_2^2$ equations constituting a linear group, as also do those of G' . Since the f 's and the g 's give $r_1^2 + r_2^2$ parameters, these groups are transitive, and, since G and G' are reciprocal, they must be simply transitive.

$$\begin{aligned} \text{The equations} \quad y'_{ij} &= \sum_1^{r_1} f_{jk} y_{ik} \quad (i, j = 1, 2, \dots, r), \\ z'_{ij} &= G_{ij}(y) + \sum_1^{r_2} g_{jk} z_{ik} \quad (i, j = 1, 2, \dots, r_2), \end{aligned}$$

in which the f 's and the g 's are linearly independent, constituting a simply transitive group, the coefficients of the y 's in $G_{ij}(y)$ must be linear functions of the f 's and g 's.

Now the values of the parameters

$$\begin{aligned} f_{11} = f_{22} = \dots = f_{r,r} &= \alpha, \quad f_{ij} = 0 \quad (i \neq j), \\ g_{11} = g_{22} = \dots = g_{r_2,r_2} &= \beta, \quad g_{ij} = 0, \end{aligned}$$

constitute a sub-group of order 2, all of whose operations are self-conjugate. For these values the above equations become

$$\begin{aligned} y'_{ij} &= \alpha y_{ij}, \\ z'_{ij} &= \alpha g_{ij}(y) + \beta h_{ij}(y) + \beta z_{ij}, \end{aligned}$$

where $g_{ij}(y)$ and $h_{ij}(y)$ represent certain linear functions of the y 's; and, unless all the functions $g_{ij}(y)$ and $h_{ij}(y)$ are zero, these equations have not the group property. This is verified at once by combining the substitutions with α , β and α' , β' as parameters. Hence the self-conjugate operations are

$$y'_{ij} = \alpha y_{ij}, \quad z'_{ij} = \beta z_{ij}.$$

Every operation of the group, being permutable with these operations, must interchange the y 's and z 's separately among themselves. Hence all the functions denoted by $G_{ij}(y)$ are identically zero. From this it follows that no two sets of the variables which occur in connection with distinct sets of parameters can appear in each other's equations.

To carry the discussion further than this would involve the discussion and discrimination between particular cases, which it is not

proposed here to enter on. If the set of parameters

$$e_{11}, e_{12}, \dots, e_{1r},$$

$$e_{21}, e_{22}, \dots, e_{2r},$$

$$\dots \dots \dots$$

$$e_{r1}, e_{r2}, \dots, e_{rr}$$

occurs in connection with kr sets, of r variables each,

$$x_{ijt}, \quad (i, j = 1, 2, \dots, r; t = 1, 2, \dots, k),$$

the finite equations corresponding to these variables can, from the preceding discussion, be written in the form

$$x'_{ijt} = f_{ij}^{1,t}(x_1) + f_{ij}^{2,t}(x_2) + \dots + f_{ij}^{t-1,t}(x_{t-1}) + \sum_{k=1}^{k-r} e_{jk} x_{ikt}$$

$$(i, j = 1, 2, \dots, r; \dots; t = 1, 2, \dots, k),$$

where $f_{ij}^{s,t}(x_s)$ represents a linear function of the symbols

$$x_{ijs}, \quad (i, j = 1, 2, \dots, r)$$

with coefficients depending on the original parameters.

Corresponding to each distinct set of parameters, *i.e.*, to each irreducible factor of the determinant of the original equations, there will be a similar system of equations; and the nature of the reduction of the group is then determined, so far as it depends on the distinct factors of the determinant. How far the above equations, which correspond to a particular irreducible factor, may be further simplified or modified will depend on the value of k in each case, and will lead to a discussion similar to that I have undertaken, for the case $r = 1$ and the group Abelian, in a paper "On Linear Homogeneous Groups whose Operations are Permutable" (*Proc. Lond. Math. Soc.*, Vol. XXIX., p. 325).

5. The results obtained so far may be conveniently summarized here as follows:—

(i) The determinant D , of the finite equations of a simply transitive linear homogeneous group in n variables with respect to the

variables is always of the form

$$D_s = \prod_{t=1}^{t=s} \begin{vmatrix} e_{11t} & e_{12t} & \dots & e_{1r_t t} \\ e_{21t} & e_{22t} & \dots & e_{2r_t t} \\ \dots & \dots & \dots & \dots \\ e_{r_t 1t} & e_{r_t 2t} & \dots & e_{r_t r_t t} \end{vmatrix}^{k_t r_t} = \prod_{t=1}^{t=s} E_t^{k_t r_t},$$

where

$$n = \sum_{t=1}^{t=s} k_t r_t^2,$$

and the r_t ,

$$= \sum_{t=1}^{t=s} r_t^2,$$

symbols e_{ijk} are linear and independent functions of the n parameters entering in the equations of the group.

(ii) The variables may be chosen so that both G and the reciprocal group G' interchange them among themselves in the same s sets, the number in the t -th set being, $k_t r_t^2$. The finite equations for G as regards the t -th set are of the form given near the end of § 4; and their determinant with respect to the variables is $E_t^{k_t r_t}$, the power of a single irreducible factor.

Both G and G' interchange among themselves the symbols of the t -th set, while the determinant of the equations of G for this set is the power of a single irreducible function $E_t^{k_t r_t}$. Now the determinants of any two reciprocal simply transitive groups differ only, if at all, in the power to which the different irreducible factors enter into them. Further, the determinant of G with respect to the parameters is obtained by writing variables for parameters in the determinant of G' with respect to the variables. Hence—

(iii) D_s and D'_s , the determinants of the finite equations of G and G' with respect to the variables, are identical with each other; and D_s , the determinant of G with respect to the variables, is obtained from D_s by replacing each symbol e_{ijk} by the x_{ijk} defined in § 4.

To these three general results the following special ones are added on account of their importance:—

The determinant D_s is expressed as a function of r independent linear functions of the original n parameters. If no linear relation connects the n differential coefficients of D_s with respect to the

original parameters, ν must be equal to n and then each of the k 's must be unity. Hence—

(iv) If D_c cannot be expressed in terms of less than n linear functions of the original parameters, every irreducible factor of D_c enters into D_c to a power equal to its degree; and G is then the direct product of a number of groups each of which is simply isomorphic with a general linear group.

(v) If D_c cannot be expressed in terms of less than n linear functions of the parameters, and if D_c is a power of one irreducible factor, then n must be a square, and G is simply isomorphic with the parameter group of the general linear group in $\sqrt{n}$ variables.

II.

The preceding results may be applied to deal with the determinant of any transitive linear group, homogeneous both in the variables and in the parameters, and with the reduction of such a group.

Let G be such a group in n variables and m ($n^2 > m > n$) parameters. The parameter group of G is a simply transitive linear homogeneous group in m variables. Denote its determinant with respect to the variables by D_c and that of G by Δ_c . Since $\Delta_c = 0$ determines the systems of values of the parameters for which the finite equations of G cannot be solved, while $D_c = 0$ gives all inadmissible systems of values of the parameters, the irreducible factors of Δ_c must be some or all of the irreducible factors of D_c . The form of Δ_c is thus determined from the previous sections as

$$\Delta_c = \Pi \begin{vmatrix} e_{11} & e_{12} & \dots & e_{1r} \\ e_{21} & e_{22} & \dots & e_{2r} \\ \dots & \dots & \dots & \dots \\ e_{r1} & e_{r2} & \dots & e_{rr} \end{vmatrix}^k,$$

where all the e 's in the different factors of the determinant are linearly independent linear functions of the original parameters. Hence Δ_c must have more than one irreducible factor; for, being of the n -th degree, it would, if irreducible, be a determinant of n rows and columns all of whose elements were independent; and this is impossible, m being less than n^2 . Choose now new parameters, so that the set $e_{11}, \dots, e_{rr}$ occur among them. When we make

$$e_{11} = e_{12} = \dots = e_{1r} = 0,$$

the finite equations of the group cannot be solved for the original variables. Hence these values of the parameters must either make one at least of the original variables disappear from the equations, or must involve at least one linear relation among the transformed variables. If together with the original group we consider the "transposed" group that arises on interchanging the rows and columns of coefficients in the finite equations, when of these two possibilities one holds for the original group, the other must hold for the "transposed." It is therefore sufficient to consider one of them. Suppose then that the assumed conditions for the e 's lead to the relation

$$\Sigma a_i x'_i = 0$$

among the transformed variables. If new variables are taken, of which

$$\xi_1 = \Sigma a_i x_i$$

is the first, then ξ'_1 vanishes when

$$e_{11} = e_{12} = \dots = e_{1r} = 0,$$

whatever the other parameters may be. Moreover, since to the identical operation of the group there correspond the values

$$e_{11} = e_{22} = \dots = e_{rr} = 1, \quad e_{rs} = 0 \quad (r \neq s),$$

of the e 's, it follows that when

$$e_{11} = 1, \quad e_{12} = \dots = e_{1r} = 0,$$

ξ'_1 must be equal to ξ_1 . Hence we may take

$$\xi'_1 = e_{11}\xi_1 + e_{12}\xi_{12} + \dots + e_{1r}\xi_r,$$

$\xi_2, \xi_3, \dots, \xi_r$ being linearly independent functions of the original variables.

If now the transformations corresponding to accented and unaccented values of the e 's are carried out in succession, the resulting substitution is that for which

$$e''_{11} = e_{11}e'_{11} + e_{12}e'_{21} + \dots + e_{1r}e'_{r1},$$

$$e''_{12} = e_{11}e'_{12} + e_{12}e'_{22} + \dots + e_{1r}e'_{r2},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$e''_{1r} = e_{11}e'_{1r} + e_{12}e'_{2r} + \dots + e_{1r}e'_{rr}$$

are the first r parameters; this being given directly by the parameter group. But

$$e''_{11}\xi_1 + e''_{12}\xi_2 + \dots + e''_{1r}\xi_r \equiv e_{11}\xi'_1 + e_{12}\xi'_2 + \dots + e_{1r}\xi'_r,$$

where $\xi'_1, \xi'_2, \dots, \xi'_r$ are the transformed values of $\xi_1, \xi_2, \dots, \xi_r$ corresponding to accented values of the e 's. It follows that

$$\xi'_1 = e'_{11}\xi_1 + e'_{12}\xi_2 + \dots + e'_{1r}\xi_r,$$

$$\xi'_2 = e'_{21}\xi_1 + e'_{22}\xi_2 + \dots + e'_{2r}\xi_r,$$

$$\dots \dots \dots \dots \dots$$

$$\xi'_r = e'_{r1}\xi_1 + e'_{r2}\xi_2 + \dots + e'_{rr}\xi_r.$$

The first r variables are therefore transformed among themselves by every operation of G , and the group so far as it affects these variables is the general linear group in r symbols.

If, in the equations of G , $\xi_1, \xi_2, \dots, \xi_r, \xi'_1, \dots, \xi'_r$ are made zero, the remaining $n-r$ equations give a transitive group in $n-r$ variables, and this may be dealt with as G has been. Moreover, as in the case of the simply transitive group, this process may be continued till all the variables are accounted for. Further, exactly as in § 4 of the first section, it may be shown that, if

$$\xi_1, \xi_2, \dots, \xi_r,$$

$$\eta_1, \eta_2, \dots, \eta_s$$

are two sets of variables thus arising in connection with two distinct sets of parameters (or, in other words, in connection with two distinct irreducible factors of Δ), neither set can occur in the equations giving the transformed values of the other. The results thus obtained may be summarized in the following form:—

(i) The determinant with respect to the variables, Δ_e , of a transitive linear homogeneous group in n variables, other than the general linear group, is never an irreducible function of the parameters. It can always be expressed in the form

$$\Delta_e = \prod_{t=1}^{t=s} \begin{vmatrix} e_{11t} & e_{12t} & \dots & e_{1r,t} \\ \dots & \dots & \dots & \dots \\ e_{21t} & e_{22t} & \dots & e_{2r,t} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ e_{r,1t} & e_{r,2t} & \dots & e_{r,r,t} \end{vmatrix}^{k_t},$$

where the symbols e_{ijk} represent independent linear functions of the original parameters.

(ii) The variables may be so chosen that they are transformed

among themselves in s distinct sets. The number in the t -th set is $k_t r_t$; and, if they are denoted by

$$x_{ijt} \quad (i = 1, 2, \dots, r_t; j = 1, 2, \dots, k_t),$$

the general operation of G , so far as it affects the symbols of this set, is

$$x'_{ijt} = f_{j-1} + \sum_{k=1}^{k=r_t} e_{ikt} x_{kjt} \\ (i = 1, 2, \dots, r_t; j = 1, 2, \dots, k_t),$$

where f_{j-1} represents a linear function of the symbols

$$x_{ipt} \quad (i = 1, 2, \dots, r_t; p = 1, 2, \dots, j-1),$$

with coefficients linear in the parameters.

(iii) An irreducible homogeneous function of degree m in μ variables, between whose first different coefficients with respect to the variables no linear relation with constant coefficients exists cannot be a relative invariant for a transitive group, linear both in the variables and in the parameters, unless $\mu = m^2$; and the function must be of the form

$$\begin{vmatrix} x_{11}, & x_{12}, & \dots, & x_{1m} \\ x_{21}, & x_{22}, & \dots, & x_{2m} \\ \dots & \dots & \dots & \dots \\ x_{m1}, & x_{m2}, & \dots, & x_{mm} \end{vmatrix},$$

where the elements of the determinant are linearly independent linear functions of the variables which are transformed by the group.

III.

I wish to draw attention to the great simplification which the results of the first section of the present paper permit in proving the main propositions of the second section of my paper "On the Continuous Group defined by any given Group of Finite Order" (*Proc. Lond. Math. Soc.*, Vol. xxix., p. 526) already referred to.

If D_y is the determinant with respect to the variables of the group G ,

$$x'_i = \sum_{s=1}^{i=n} y_{si} x_s \quad (i = 1, 2, \dots, n),$$

there considered, and if

$$y_1, y_2, \dots, y_n,$$

$$z_1, z_2, \dots, z_n,$$

are the values of the parameters corresponding to an arbitrary operation T of G and to its inverse T^{-1} , it may be shown, almost exactly as on pp. 556, 557, that

$$z_{i-1} = \frac{1}{nD_y} \frac{\partial D_y}{\partial y_i}.$$

Now, if the y 's have arbitrary values, so also have the z 's; and therefore the n differential coefficients $\frac{\partial D_y}{\partial y_i}$ are linearly independent.

Hence, by (iv) of § 5, every irreducible factor of D_y enters into D_y to a power equal to its degree; and, if there are s such irreducible factors, G is the direct product of s general linear groups in $m_1, m_2, \dots, m_s$ variables, where

$$m_1^2 + m_2^2 + \dots + m_s^2 = n.$$

This being the case, the self-conjugate operations of G constitute a sub-group of order s . But it had previously been proved that this sub-group had r for its order. Hence $s = r$, and the variables may be chosen so that every operation of both G and G' interchange them in the same r sets, the number in each set being a square. These are the main results on which the remainder of the discussion turns.

The Propagation of Light in a Uniaxial Crystal. By A. W.

CONWAY. Received November 5th, 1902. Read November 13th, 1902.

Introduction.

The following paper is an adaptation of the analysis used by Prof. A. E. H. Love ("Integration of the Equations of Propagation of Electric Waves," *Phil. Trans.*, Series A, 1901) to the case of a uniaxial crystalline medium, together with some deductions from the