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ON THE APPLICATION OF SPECTRAL BANDS IN DETERMINING ANOMALOUS DISPERSION AND ACHROMATIC SYSTEMS OF VARIOUS TYPES.

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IF two systems of interfering rays, after undergoing different phase-changes, in whatever way, are examined spectrally, dark bands are found for those frequencies where destructive interference occurs. If two sets be examined in juxtaposition, they may be made to coincide, either throughout the entire range of the spectrum, or only over certain portions. The former indicates perfect, the latter partial, differential achromatism. In general, the first is unrealizable; but the latter may be attained to a greater or less extent, particularly in the visible spectrum, by means of certain elements which constitute, in special combinations, achromatic systems.

If N represents the number of times destructive interference, by whatever mode, occurs for such a system, in varying the wave-length from $\lambda = \lambda_\infty$ to $\lambda = \lambda_D$, say, then $[\partial N / \partial \lambda] \delta \lambda$ represents the number of bands in the spectrum between wave-lengths λ and $\lambda + \delta \lambda$. If, by any change, we can make this number the same for a second element, we should have

$$\frac{\partial N'}{\partial \lambda} = \frac{\partial N''}{\partial \lambda}. \quad (1)$$

The pair is then said to be differentially achromatic for these two wave-lengths. If the bands coincide between these limits, this

couple is perfectly achromatic within this region. In the former, residual spectral effects occur which produce chromatism to a greater or less degree, which latter also occurs outside these limits.

If we integrate (1), we have

$$N' = N'' + k. \quad (2)$$

This shows that if we should vary λ in each element from λ_∞ to λ_D , say, where achromatism occurs, and count the number of times destructive interference occurs; or if we should begin in the spectrum at λ_∞ and count the bands up to λ_D , we should, in general, find a different number or order for each element of the pair.

N may be readily obtained if we can vary it in any way as I have already shown.¹ Thus if N_∞ , N_D , and N_x represent the orders for λ_∞ , λ_D , and λ_x respectively, and N'_∞ , N'_D , and N'_x the orders obtained after any variation, we shall have:

$$\begin{array}{ll} N_D - N_\infty = N_D & N'_D - N'_\infty = N'_D \\ N_x - N_\infty = N_x & N'_x - N'_\infty = N'_x \end{array}$$

or

$$N_x - N_D = p$$

and

$$N'_x - N'_D = np,$$

if $N'_\infty = N_\infty$ and p is the number of bands between the two wavelengths λ_D and λ_x , and np the number when N is changed to N' .

We also have

$$N'_D - N_D = m$$

and

$$\frac{N'_x - N'_D}{N_x - N_D} = n = \frac{N'_D}{N_D} = \frac{N'_x}{N_x},$$

since

$$\frac{N'_D - N'_\infty}{N_D - N_\infty} = \frac{N'_D}{N_D},$$

if

$$N_\infty = N'_\infty = 0$$

where m is the number of bands passing λ_D in the positive direction, toward the red, say, in changing N_D to N'_D .

¹ Phil. Mag., Oct., 1899, p. 345; also Rendtorff, Phil. Mag., May, 1901, p. 539, and Williams, PHYS. REV., April, 1904, p. 280.

Hence,

$$N_D = \frac{m}{n-1}. \quad (3)$$

If for λ_∞ , $N_\infty \neq 0$, or $= \varphi$, say, then (3) holds only for the corresponding portion or phase φ of the bands. Thus if $\varphi = \lambda/2$, we must use the white bands instead of the black bands in the settings and the countings to obtain N . The true order will then be $N + \varphi$. We have thus the methods at hand of determining the degree of achromatism and also the orders which must be used to reduce the chromatism to a minimum. A similar method will give the order of the same space without the medium, or we may find the thickness by determining the number of air-bands N_{E-D} between two wave-lengths λ_D and λ_E , say. We shall then have at once the data from which we can find the thickness. Thus, if

$$\mu_0 = 1, \quad \frac{d}{\lambda_E} - \frac{d}{\lambda_D} = N_{E-D} \quad \text{or} \quad d = N_{E-D} \left(\frac{\lambda_D \lambda_E}{\lambda_D - \lambda_E} \right).$$

We have thus the absolute index needed in the technical applications of these principles, which will be embodied in the usual formulæ after transformation from the above equations.

ANOMALOUS DISPERSION.

In this case we have to deal with relative phase-retardations of the two interfering systems due to a difference in path, neglecting for the moment phase-changes at the bounding surface.

If d is the thickness and μ the index of the medium,

$$N = \frac{d\mu}{\lambda}; \quad \text{and} \quad N_0 = \frac{d\mu_0}{\lambda} \quad (4)$$

for the same space with a medium of constant index μ_0 .

Thus

$$\frac{\partial N}{\partial \lambda} = \frac{d}{\lambda^2} \left(\mu + \lambda \frac{\partial \mu}{\partial \lambda} \right) \quad \text{and} \quad \frac{\partial N_0}{\partial \lambda} = \frac{d\mu_0}{\lambda^2}. \quad (5)$$

If we should be able to adjust the two sets of bands to coincidence at any point of the spectrum, we should have from (1)

$$\frac{\partial N}{\partial \lambda} = \frac{\partial N_0}{\partial \lambda}.$$

Hence,

$$\frac{d\mu}{\lambda} + \frac{\partial(d\mu)}{-\partial\lambda} = \frac{d_0\mu_0}{\lambda} \text{ or } \frac{d\mu}{\lambda} - \frac{d_0\mu_0}{\lambda} = -\frac{\partial(d\mu)}{-\partial\lambda} = k = N - N_0. \quad (6)$$

Since, if there is a coincidence between the bands with constant index and those of the medium to be examined, k must be a whole number, we shall have

$$\frac{\partial(d\mu)}{-\partial\lambda} = 0, \quad (7)$$

if $N = N_0$. Thus, if by trial we can obtain a set of coincidences and then find, by the method of counting given above, that the orders are the same, μ must be constant in this region and hence be a maximum or a minimum, or have a point of inflection; that is, the tangent to the curve of dispersion is parallel to the coördinate axis of λ . For example, taking $\mu_0 = 1$, as in the case of a vacuum

Fig. 1.

(or of air, approximately), let us compare the air-bands with those of fuchsin, say. In the curve of dispersion of the latter, Fig. 1, as determined by Cartmel,¹ equation (7) holds at b and d on curve I. and the order N_1 of the air bands is the same as N_2 that of fuchsin at b ; and N_1' and N_2' are the same at d . Curve II. may illustrate a reduction in the anomaly, in a solution, say, and Curve III. a still further reduction, the points of maximum and minimum coinciding

¹ Phil. Mag., Aug., 1903, p. 213.

at a point of inflexion c . At each of these points a characteristic distribution of the bands occurs from which we can readily interpret the nature of the anomaly. Referring to (5) again, since at these points

$$\frac{d\mu}{\lambda^2} = \frac{d_0\mu_0}{\lambda^2},$$

we have the residual term $d[\lambda \partial\mu] - \partial\lambda$ from which to determine the character of the curve at these points.

Thus $\partial\mu - \partial\lambda$ is + going from a to b and - going from b to c , or

$$\frac{\partial N}{-\partial\lambda} > \frac{\partial N_0}{-\partial\lambda}$$

and

$$\frac{\partial N}{-\partial\lambda} < \frac{\partial N_0}{-\partial\lambda},$$

respectively; *i. e.*, on the red side of a coincidence, the bands crowd together; while, on the violet side, they spread out relatively to the air-bands in case of a maximum. Going from c to d $\partial\mu - \partial\lambda$ is - and from d to e it is +, or

$$\frac{\partial N}{-\partial\lambda} < \frac{\partial N_0}{-\partial\lambda}$$

and

$$\frac{\partial N}{-\partial\lambda} > \frac{\partial N_0}{-\partial\lambda},$$

respectively, *i. e.*, if, on the red side of a coincidence, the bands of the medium spread out and on the violet side they crowd together more than the air-bands of the same order, the region has a minimum index. On Curve III., at a point of inflexion c , $\partial\mu - \partial\lambda$ is + on the red and + on the violet side, or

$$\frac{\partial N}{-\partial\lambda} > \frac{\partial N_0}{-\partial\lambda}$$

in each case; *i. e.*, if the bands shorten with respect to the air bands of the same order on each side of a coincidence, the index has a point of inflexion.

In the region beyond a , since the bands must begin to coincide again before $\lambda = \lambda_\infty$, they must begin to lengthen relatively to the

air-bands so as to coincide again at $\lambda = \lambda_\infty$. These various characteristics are illustrated in Fig. 2.

This apparent crowding together and spreading out of the bands is the more marked, the more pronounced the anomaly, being more marked in band- and line-spectra.

Fig. 2.

If the index of the medium of comparison is not constant, as it is in the case of a vacuum (or air) but varies slowly and uniformly, we can still apply similar tests for any anomalies which may exist. Thus from

$$\frac{\partial N}{\partial \lambda} = \frac{\partial N_0}{\partial \lambda}$$

we should have, in such a case,

$$\frac{d\mu}{\lambda} + \frac{\partial(d\mu)}{\partial \lambda} = \frac{d_0\mu_0}{\lambda} + \frac{\partial(d_0\mu_0)}{\partial \lambda}$$

or

$$\frac{\partial(d\mu)}{\partial \lambda} = \frac{(d_0\mu_0)}{\partial \lambda}, \text{ if } N = N_0,$$

and

$$\frac{\partial \mu}{\partial \lambda} = \frac{d_0}{d} \frac{\partial \mu_0}{\partial \lambda}. \quad (8)$$

Since d_0/d will not be large (rarely, if ever, equal to 2 for liquids and solids) and as the inclination of the dispersion curve to the λ -axis is always small for frequencies which are not near the natural frequencies of the medium, $\partial \mu / \partial \lambda$ will be small; and the crowding and spreading of the bands should occur approximately at a maximum or a minimum point, just as in a case when air is used as a means of comparison. For a point of inflection, the test might indicate a close maximum-minimum region. Thus, water (as a solvent), glass, mica, etc., could be used in detecting anomalous dispersion, or a definite number of air-bands may be added.

If, instead of there being a single medium in the path of each of the systems of interfering rays, there are several, in addition to the anomalous medium to be examined, similar considerations will show that the criteria for the anomalies are the same.

Let the resultant orders for the two sets of bands which are to be compared be represented by N' and N'' where $N' = N + N'_1 + N'_2 + \dots + N'_m$ and $N'' = N_0 + N''_1 + N''_2 + \dots + N''_m$.

Then if

$$\frac{\partial N'}{\partial \lambda} = \frac{\partial N''}{\partial \lambda},$$

we have

$$\frac{\partial N}{\partial \lambda} - \frac{\partial N_0}{\partial \lambda} + \frac{\partial N'_1}{\partial \lambda} - \frac{\partial N''_1}{\partial \lambda} + \dots + \frac{\partial N'_m}{\partial \lambda} - \frac{\partial N''_m}{\partial \lambda} = 0.$$

When the orders of the two sets of interfering bands are made the same before inserting the media to be compared, as can readily be realized in various interfering systems by suitable experimental conditions, we shall have

$$N'_1 = N''_1, N'_2 = N''_2 \dots N'_m = N''_m$$

and hence

$$\frac{\partial N}{\partial \lambda} = \frac{\partial N_0}{\partial \lambda} \quad (9)$$

Thus the crowding and spreading of the resultant bands will indicate the presence of anomalies as before. The additional bands then serve as lines of reference for making measurements, when the orders of the media to be compared are too low to give sufficient bands to make consecutive comparisons throughout the spectrum. The equality $N = N_0$, however, cannot be determined from (3) so readily as before by the process of counting. Either the additional bands must be excluded in order to use (3) as usual, or the bands from these media must be counted when they are introduced into the field from zero order up to the orders to be used. For low orders, less than unity, the latter method is usually preferable. With the previous method we may determine whether we have a true zero order by inserting a compensating element of the order m , say for the D -line. The dispersion of this compensating element must be the same as that of the medium producing the bands. If then we do not find the final band on the D -line, the interfering system may be adjusted until we get coincidence of the order with the D -line. We may then proceed in the usual way, by counting, to obtain the

order of the unknown element, noting, however, any change from the zero order of the interfering system and adjusting, accordingly, if necessary. Should there be any uncertainty as to the proper compensation for other parts of the spectrum, we should note whether the other bands are in their proper places in the rest of the spectrum also. The various characteristics discussed above may be readily represented graphically. From equations (4) we have

$$N\lambda = d\mu, \quad (4a)$$

and

$$N_0\lambda = d_0\mu_0h, \quad (4b)$$

where μ is a constant. (4b) represents a series of equilateral hyperbolæ, with different parameters depending on $d_0\mu_0$ being taken as unity. In Fig. 3 the curve $A B C D E$ is plotted from Cartmel's

Fig. 3.

data. The full lines represent the rectangular hyperbolæ, while dotted lines represent the deviations from the corresponding hyperbolæ, when μ_0 varies but slowly, as it does considerably beyond the anomalous region. The curve $A B C D E$ represents the variations in N due to the fluctuations of μ in its anomalous region. Thus from

A to B , N increases more rapidly than N_0 , from B to D more slowly, then more rapidly from D to E . Beyond A , N increases less rapidly than N_0 , becoming finally asymptotic to the axis of λ at $\lambda = \lambda_\infty$, while beyond E , N increases less rapidly, becoming asymptotic to the axis of N at $\lambda = \lambda_0$ where μ becomes unity. In the case of an inflexion, the curve $B'CD'$ becomes tangent on crossing the hyperbola C . Thus the order N increases more rapidly before and after crossing the hyperbola, but diminishes more rapidly beyond B' and D' .

Fig. 2 is a reproduction of Cartmel's fuchsin-bands, showing the fluctuations in width referred to. Toward the violet end, we see the fuchsin-bands are relatively narrower, while they are wider than the air-bands in the central portion, indicating a minimum in this region. At the red end the fuchsin-bands are narrower than the air-bands, thus indicating a maximum region.

The method of interferences in the study of dispersion has much to recommend it over the usual prismatic methods. The strong absorption in the anomalous region prevents the use of any but the smallest angled prism, which involves serious instrumental errors. The spectral interference methods referred to eliminate these errors, since the factors sought can be determined by a process of counting and are hence independent of the angle of the film, which may vary in any arbitrary way so long as the change in thickness is not so abrupt as to prevent the "count" of the bands. Media through which it would be impossible to obtain sufficient light for measurements, if in prismatic form, can readily be examined in thin films in this way.

The use of spectral bands presents us with a method which unfolds before the eye, in the position of the bands in the spectrum, an exact record of whatever the medium has impressed upon the various trains of waves which have passed through it. Using the criteria mentioned above, we may at once locate the presence of any dispersive characteristic in the spectrum which the medium may possess. Thus, if in addition a quantitative determination is to be made, we may localize our measurements to this particular region.

The special method of interference to be adopted will depend partly on the nature of the media studied. When the films are thick enough and also sufficiently transparent to give the charac-

teristic interference bands by reflection, when examined directly by the spectroscope, observations are simplified. Comparison can be conveniently obtained by means of a double rectangular collimator one half the field of which is illuminated with light from one of its slits directly, and the other half with light from the other slit after total reflection from a rectangular prism placed before the objective similar to the arrangement used by the writer¹ and later by Rendtorff,² in the study of differential double refraction. In this case, total reflecting prisms may be placed at the slits to throw the light upon the films and thence into the collimator, as used by Cartmel³ in finding the thickness of his films. Instead of the above arrangement, two collimators may be arranged with the compound prism as used in the spectrophotometer of the writer.⁴ These arrangements allow of independent movements of each of the two media under examination, and facilitate the adjustments considerably over those made when the two films are placed in juxtaposition before the same slit. The former arrangement also brings the two sets of bands closer together. If the absorption is great, the film to be examined may be placed between semi-transparent mirror-surfaces to equalize the intensities of the two interfering beams. If the order is too low to give sufficient bands for comparison, an auxiliary film may be introduced in series with the film to be examined and the method of comparison of mixed bands described above be used. Under the latter conditions, we may also use an interferometer, preferably of such a type that the light may pass through the film once only. The method adopted by the writer, of converging the light within an interferometer similar in type to that of Jamin and of using a mixed system of bands, should be chosen. Cartmel⁵ has used this arrangement most effectively in his studies on the dispersion of fuchsin. Here again we have the spectral bands after passing the interferometer. This spectral analysis of interferential phenomena has much to recommend it over the customary mode of analyzing the light first and examining afterward, with homogeneous rays, these interference effects.

¹ L. c.

² L. c.

³ L. c.

⁴ Phil. Mag., Nov., 1899, p. 420; Astrophys. Journ., Jan., 1900.

⁵ L. c.

When very slight anomalies are present and the absorption is small, we should resort to high orders and the high resolving power of the grating. Thus, in the examination of weak solutions of varying concentrations of anomalous substances, the growth of the anomaly is at once made evident. Here, several thousand bands may be introduced into the spectrum, and, as the effect is cumulative, any slight relative shift in the bands becomes at once evident. Comparison bands, from the solvent say, may be introduced by means of a second slit arranged in duplication of the system used by Williams¹ in determining refractive indices, the Fraunhofer lines being used for reference along with the bands themselves. In fact, for the study of anomalies in gases, this arrangement should be used on account of the high resolving power required.

ROTARY POLARIZATION.

As is well known, if a rotary medium be placed between two nicol prisms and the light transmitted through this system be examined spectrally, dark bands will appear for those wave-lengths the principal planes of whose vibrations are at right angles to that of the analyzing nicol. This extinction will recur for every increment or decrement in the rotation of 180° . Thus the order of the bands will be proportional to the rotation, or

$$\frac{\partial N}{\partial \lambda} \propto \frac{\partial \rho}{\partial \lambda}.$$

Hence if

$$\frac{\partial N}{\partial \lambda} = \frac{\partial N_0}{\partial \lambda},$$

$$\frac{\partial \rho}{\partial \lambda} = \frac{\partial \rho_0}{\partial \lambda}$$

and

$$\rho = \rho_0 + \text{const.}$$

(10)

is the condition for achromatism of such a couple. The total rotation may be obtained by the process of counting, as described above, to obtain the order, the thickness of the rotary medium being varied. Thus, taking black bands as our basis of calculation, the

¹ L. c.

nicols must be crossed. Since here our order corresponds to a rotation equal to 180° , we have from (3)

$$N_{180^\circ} = \rho_D = \frac{m}{n-1} 180^\circ. \quad (11)$$

The application of this spectral method to achromatic comparisons is practically limited to a very few substances on account of the great rotations necessary to produce sufficient bands.

Thus, quartz would give only about 9 or 10 bands in the visible spectrum for a thickness of 50 mm. As quartz has a much higher rotary power than most active substances, very considerable lengths of the media would be necessary in any comparisons. The same is true of magnetically active media. For this same reason the above tests for anomalous rotary dispersion are impracticable except with mixed systems, using perhaps quartz to give sufficient bands for comparison. Exact quantitative comparison should of course be made by means of half-shade plane polarizing systems. The importance of achromatism to saccharimetry warrants, perhaps, such spectral comparisons, to determine the region in the spectrum giving the best rotary achromatism with quartz; or, in fact to determine a more perfectly achromatizing comparator than quartz itself.

DOUBLE DIFFERENTIAL REFRACTION.

The relative retardation produced by a double-refracting plate placed between two nicols, may also be examined by means of a spectroscop. Thus, if the nicols are crossed and the principal axis of the plate makes an angle of 45° with that of the polarizing system, the light transmitted by such a plate will be cut out when the difference in phase between the two components, expressed in wavelengths, is a whole number. As this difference is a function of the frequency, we should obtain the well known bands in the spectrum corresponding to those frequencies. I have already used the above method¹ of determining the orders of individual plates and have also determined the particular orders to be used in crossed achromatic couples. Rendtorff² has also made similar determinations

¹ L. c.

² L. c.

using a grating. This method has revealed persistent errors in old determinations of the orders of plates arising from the assumption that, if

$$\frac{\partial N'}{\partial \lambda} = \frac{\partial N''}{\partial \lambda},$$

we shall have $N' = N''$, which is seriously in error in the comparison of one crystal with another, as is customary with crystallographers. Thus, if

$$\frac{\partial N'}{\partial \lambda} = \frac{\partial N''}{\partial \lambda},$$

we should have

$$N' = N'' + \text{const.}$$

or

$$d'(\mu_e' - \mu_0') = d''(\mu_e'' - \mu_0'') + \text{const.} \quad (12)$$

where

$$N' = \frac{d'(\mu_e' - \mu_0')}{\lambda} \text{ and } N'' = \frac{d''(\mu_e'' - \mu_0'')}{\lambda}$$

and μ_e and μ_0 are the two principal indices.

Mouton¹ claimed to have obtained anomalous differential dispersion in selenite. An examination of Rendtorff's bands by means of the maximum-minimum criteria, shows no such fluctuations. That such anomalies are present in other regions may be expected, since the natural frequencies of the two rectangular components are usually different, as exists in the case of tourmaline, Iceland Spar, etc. That such bands may be tested by our criteria is evident from the factors constituting N , which is evidently a mixed system and, hence, conformable to our rules, being a difference instead of a summation system, the one having a slow variation in the dispersion since it is not near its natural frequency. Hence our comparison bands may be bands from thin films, or they may be differential bands from another crystal having no anomaly in this region. The applications of this method in the case of crystals, as well as the experimental arrangements for making the comparisons, may be found in the papers already referred to.

¹ Wien, Ak. Ber., XXXVI. (2), p. 793, 1877.

ACHROMATIC COMPENSATION AND INTERFERENCE.

In many of the determinations of the velocities of systems of waves in different media, white light is used in an interferometer and the measurements are made to depend upon the position of the so-called "central" or white band. When a zero method is adopted, an optical compensation is used to bring back this "central" band to the fiducial line. If this is not done the apparent displacement of this band is usually measured directly. Either of these methods may introduce serious errors in the results. Considerations, similar to those mentioned above as to the achromatism of the two systems of interfering rays, will enable us to determine when such errors do enter, and their proper corrections. In the "zero" method, the compensation is produced either by an adjustment of one of the surface elements of the interferometer or by the introduction of an additional amount of a retarding medium. An example of the former practice is shown in the Michelson type and of the latter in the Jamin type of interferometer. It is evident that both the zero and the displacement methods involve essentially the same chromatic differentiation, since in both we have the introduction of like compensating components. We may therefore discuss them under the same head. Such systems are mixed systems the consideration of which may be reduced to that of two elements, namely, the one to be examined and the other the compensating factor. The compensating medium is thus either a vacuum (or air) or a solid (like glass, mica, etc.) or a liquid (such as the solvent in a solution, etc.). Referring to equations (9), we have now, as the condition of a white band, our equation of reference

$$\frac{\partial N}{\partial \lambda} = \frac{\partial N_0}{\partial \lambda}$$

or

$$N = N_0 + k.$$

Thus our so-called white band will be achromatic to a greater or less degree, depending on the extent of the region over which the first equality holds. If then sN_D is the order of the compensating elements $s(N_0 + k)_D$ must be the order of effect introduced by the unknown element. The constant k should be determined beforehand

under the same conditions of luminosity, absorption, etc., as those of the compensation. Thus, if the silicate crown glass strips, usually mounted in connection with Jamin's interferometer, were to be used to compensate displacements due to borate flint specimens, errors would enter in, since, in this case, $k \neq 0$. Similar errors would occur in comparing the solution of certain substances with the solvent itself. When the compensation consists of a displacement of one of the surface elements, the compensating medium is usually air (where of course the paths are so small that dispersion does not enter), and we should not have an achromatic band, unless a region of inflection of the refractive index of the medium existed over the most luminous part of the spectrum used, so as to give a nearly constant index. If this condition did not obtain, the so-called central band would be imperfectly achromatized. Stokes¹ has already called attention to the error in the latter instance in determining refractive indices by the displacement of the bands. This phenomena, referred to by Stokes, is, however, accompanied by a certain amount of chromatism of the band, and for this reason the error is readily detected. In the examples mentioned above this error cannot be thus detected. Our only resource is the use of spectral bands, if a preliminary examination of the two elements is not made beforehand. Mention may again be made of the convenience of this system on account of our ability to determine the order by the method of counting. Thus, if we are determining the index (and dispersion) of a gas, it is quite unnecessary to count the total number of bands, on the introduction of the gas into the path of one of the interfering rays. We only need to vary the quantity by a slight amount, in order to get the number of passages m of the bands at any point in the spectrum and note the ratio of n . The same would hold in the case of solids, solutions, etc. The tediousness of many of the old measurements could thus be greatly reduced.

Attention might also be called to the test for anomalies, in such measurements on gases, which would occur near the region of absorption.

The fact that we may have achromatism without the orders being the same, *i. e.*, $k \neq 0$, if

¹ Math. and Phys. Papers, Vol. II., p. 361.

$$\frac{\partial N}{\partial \lambda} = \frac{\partial N_0}{\partial \lambda},$$

shows that in the experiments on so-called Talbot's bands we should not obtain them, even when the retardation is introduced in the appropriate way.

ACHROMATIC REFRACTING SYSTEMS.

We may extend to achromatic systems a treatment¹ similar to that which I have already given, in discussing the variation in direction which is produced by any arbitrary system of reflecting and refracting surfaces upon a ray which has been subjected to a change in its velocity or in the phase of its vibration. Consider the refraction of paraxial rays, N' and N'' , at a series of surfaces, $s_1, s_2 \dots s_m$, very near together. The order of the undeviated ray is $N' = N'_0 + N'_1 + N'_2 + \dots + N'_m$, and that of a ray at a distance ∂s from the axis is

$$\begin{aligned} N'' = N''_0 + N''_1 + N''_2 + \dots + N''_m = N'_0 + N'_1 + N'_2 \\ + \dots + N'_m + \frac{\partial N'_0}{\partial s} \partial s + \frac{\partial N'_1}{\partial s} \partial s + \frac{\partial N'_2}{\partial s} \partial s + \dots + \frac{\partial N'_m}{\partial s} \partial s. \end{aligned} \quad (13)$$

In all cases of refraction at a surface, destructive interference must occur at all points in space, corresponding to this interval, beyond the surfaces, except at a point on the undeviated ray which is designated as the focus of the system of rays. All disturbances must therefore arrive in the same phase at this point. Hence

$$N' = N''$$

or

$$\frac{\partial N}{\partial s} = \frac{\partial}{\partial s} (N_0 + N_1 + N_2 + \dots + N_m) = 0. \quad (14)$$

This corresponds to the condition of zero spherical aberration.

The order of the axial ray for some other wave-length, $\lambda + \partial \lambda$, is

$$\begin{aligned} N' = N'_0 + \frac{\partial N'_0}{\partial \lambda} \partial \lambda + N'_1 + \frac{\partial N'_1}{\partial \lambda} \partial \lambda \\ + N'_2 + \frac{\partial N'_2}{\partial \lambda} \partial \lambda + \dots + N'_m + \frac{\partial N'_m}{\partial \lambda} \partial \lambda \end{aligned}$$

¹ Phil. Mag., Apr., 1901, p. 464.

and for a ray of the same wave-length at distance δs is

$$\begin{aligned}
 N'' &= N_0'' + \frac{\partial N_0''}{\partial \lambda} \delta \lambda + N_1'' + \frac{\partial N_1''}{\partial \lambda} \delta \lambda + N_2'' + \frac{\partial N_2''}{\partial \lambda} \delta \lambda + \dots N_m'' \\
 &+ \frac{\partial N_m''}{\partial \lambda} \delta \lambda = N_0' + \frac{\partial N_0'}{\partial s} \delta s + \frac{\partial N_0'}{\partial \lambda} \delta \lambda + \frac{\partial^2 N_0'}{\partial \lambda \partial s} \delta \lambda \delta s \\
 &+ N_1' + \frac{\partial N_1'}{\partial s} \delta s + \frac{\partial N_1'}{\partial \lambda} \delta \lambda + \frac{\partial^2 N_1'}{\partial \lambda \partial s} \delta \lambda \delta s + \\
 &\dots \dots \dots \\
 &+ N_m' + \frac{\partial N_m'}{\partial s} \delta s + \frac{\partial N_m'}{\partial \lambda} \delta \lambda + \frac{\partial^2 N_m'}{\partial \lambda \partial s} \delta \lambda \delta s.
 \end{aligned}$$

As $N' = N''$ for every wave-length, we must have as before

$$\frac{\partial N}{\partial s} = 0$$

and also

$$\frac{\partial^2 N}{\partial \lambda \partial s} = 0 \quad \text{or} \quad \frac{\partial N}{\partial \lambda} = \text{const.} \quad (15)$$

We shall now transform these equations so as to embody the factors of index, curvature, etc., used in the ordinary treatment of refracting surfaces. Let there be a single refracting surface s_1 , Fig. 4, whose radius of curvature is r_1 .

Since

$$N = \frac{\mu l}{\lambda},$$

we have

$$\frac{\partial N_0}{\partial s} = \frac{\mu_0}{\lambda} \frac{\partial l_0}{\partial s}. \quad (16)$$

Fig. 4.

By geometry we have also

$$\frac{\partial l}{\partial s} = \frac{\partial s}{2r}. \quad (17)$$

Since we are dealing with paraxial systems in which we neglect second-order small quantities, if we put $\delta s = 2$, we have

$$\frac{\partial l}{\partial s} = \frac{1}{r}. \quad (18)$$

Hence if u_0 is the distance from the radiant to the surface s_1

$$\frac{\partial N_0}{\partial s} = \frac{\mu_0}{\lambda} \left(\frac{1}{u_0} - \frac{1}{r_1} \right). \quad (19)$$

Similarly if u_1 is the distance from s_1 to the image of the radiant,

$$\frac{\partial N_1}{\partial s} = \frac{\mu_1}{\lambda} \left(\frac{1}{r_1} - \frac{1}{u_1} \right). \quad (20)$$

But from (14)

$$\frac{\partial N}{\partial s} = \frac{\partial}{\partial s} (N_0 + N_1) = 0$$

or substituting (19) and (20), we have

$$\frac{\mu_0}{\lambda} \left(\frac{1}{u_0} - \frac{1}{r_1} \right) + \frac{\mu_1}{\lambda} \left(\frac{1}{r_1} - \frac{1}{u_1} \right) = 0 \quad \text{or} \quad \frac{\mu_1}{u_1} - \frac{\mu_0}{u_0} = \frac{\mu_1 - \mu_0}{r_1}. \quad (21)$$

For a vacuum, $\mu_0 = 1$ and we have

$$\frac{\mu_1}{u_1} - \frac{1}{u_0} = \frac{\mu_1 - 1}{r_1}. \quad (22)$$

If there be a single medium, bounded by two surfaces s_1 and s_2 , within a second medium, we have similarly

$$\frac{\partial N_0}{\partial s} = \frac{\mu_0}{\lambda} \left(\frac{1}{u_0} - \frac{1}{r_1} \right) \quad (19)$$

$$\frac{\partial N_1}{\partial s} = \frac{\mu_1}{\lambda} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \quad (20)$$

$$\frac{\partial N_2}{\partial s} = \frac{\mu_2}{\lambda} \left(\frac{1}{r_2} - \frac{1}{u_2} \right). \quad (23)$$

From (14),

$$\begin{aligned} \frac{\partial N}{\partial s} = \frac{\partial}{\partial s} (N_0 + N_1 + N_2) = 0 &= \frac{\mu_0}{\lambda} \left(\frac{1}{u_0} - \frac{1}{r_1} \right) \\ &+ \frac{\mu_1}{\lambda} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) + \frac{\mu_2}{\lambda} \left(\frac{1}{r_2} - \frac{1}{u_2} \right) = 0. \end{aligned} \quad (24)$$

Since $\mu_0 = \mu_2$, we have finally

$$\frac{1}{f_1} = \frac{1}{u_2} - \frac{1}{u_0} = \left(\frac{\mu_1}{\mu_0} - 1 \right) \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

or, if $\mu_0 = 1$

$$\frac{1}{f_1} = \frac{1}{u_2} - \frac{1}{u_0} = (\mu_1 - 1) \left(\frac{1}{r_1} - \frac{1}{r_2} \right). \quad (25)$$

Consider now two lenses in contact; equation (15) gives us the condition for achromatism for the two wave-lengths λ and $\lambda + \partial\lambda$. We must therefore substitute in (15) the values of $\partial N/\partial s$ given in (19), (20) and (23), and differentiate with respect to λ . Thus for the first lens,

$$\frac{\partial}{\partial \lambda} \left(\frac{\partial N_0}{\partial s} \right) = \frac{1}{\lambda} \left(\frac{1}{u_0} - \frac{1}{r_1} \right) \frac{\partial \mu_0}{\partial \lambda} - \frac{\mu_0}{\lambda^2} \left(\frac{1}{u_0} - \frac{1}{r_1} \right) \quad (26)$$

$$\frac{\partial}{\partial \lambda} \left(\frac{\partial N_1}{\partial s} \right) = \frac{1}{\lambda} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \frac{\partial \mu_1}{\partial \lambda} - \frac{\mu_1}{\lambda^2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \quad (27)$$

$$\frac{\partial}{\partial \lambda} \left(\frac{\partial N_2}{\partial s} \right) = \frac{1}{\lambda} \left(\frac{1}{r_2} - \frac{1}{u_2} \right) \frac{\partial \mu_2}{\partial \lambda} - \frac{\mu_2}{\lambda^2} \left(\frac{1}{r_2} - \frac{1}{u_2} \right) \quad (28)$$

For the second lens,

$$\frac{\partial}{\partial \lambda} \left(\frac{\partial N_2}{\partial s} \right) = \frac{1}{\lambda} \left(\frac{1}{u_2} - \frac{1}{r_3} \right) \frac{\partial \mu_2}{\partial \lambda} - \frac{\mu_2}{\lambda^2} \left(\frac{1}{u_2} - \frac{1}{r_3} \right) \quad (29)$$

$$\frac{\partial}{\partial \lambda} \left(\frac{\partial N_3}{\partial s} \right) = \frac{1}{\lambda} \left(\frac{1}{r_3} - \frac{1}{r_4} \right) \frac{\partial \mu_3}{\partial \lambda} - \frac{\mu_3}{\lambda^2} \left(\frac{1}{r_3} - \frac{1}{r_4} \right) \quad (30)$$

$$\frac{\partial}{\partial \lambda} \left(\frac{\partial N_4}{\partial s} \right) = \frac{1}{\lambda} \left(\frac{1}{r_4} - \frac{1}{u_4} \right) \frac{\partial \mu_4}{\partial \lambda} - \frac{\mu_4}{\lambda^2} \left(\frac{1}{r_4} - \frac{1}{u_4} \right). \quad (31)$$

Making $\mu_0 = \mu_2 = \mu_4 = 1$ and substituting the above values in (15), we have

$$\begin{aligned} \frac{\partial^2 N}{\partial \lambda \partial s} = \frac{1}{\lambda^2} \left\{ \lambda \left[\left(\frac{1}{r_1} - \frac{1}{r_2} \right) \frac{\partial \mu_1}{\partial \lambda} + \left(\frac{1}{r_3} - \frac{1}{r_4} \right) \frac{\partial \mu_3}{\partial \lambda} \right] \right. \\ \left. + \left[(\mu_1 - 1) \left(\frac{1}{r_2} - \frac{1}{r_1} \right) + (\mu_3 - 1) \left(\frac{1}{r_4} - \frac{1}{r_3} \right) + \frac{1}{u_4} - \frac{1}{u_0} \right] \right\} = 0. \end{aligned} \quad (32)$$

From (24) and (25) we see that we must have

$$\left(\frac{1}{r_1} - \frac{1}{r_2} \right) \frac{\partial \mu_1}{\partial \lambda} + \left(\frac{1}{r_3} - \frac{1}{r_4} \right) \frac{\partial \mu_3}{\partial \lambda} = 0 \quad (33)$$

and

$$\frac{1}{u_4} - \frac{1}{u_0} = \frac{1}{f_1} + \frac{1}{f_3} = (\mu_1 - 1) \left(\frac{1}{r_1} - \frac{1}{r_2} \right) + (\mu_3 - 1) \left(\frac{1}{r_3} - \frac{1}{r_4} \right). \quad (34)$$

Substituting

$$\left(\frac{1}{r_1} - \frac{1}{r_2} \right) = \frac{1}{f_1(\mu_1 - 1)}$$

in (33), we have also

$$\frac{1}{f_1(\mu_1 - 1)} \frac{\partial \mu_1}{\partial \lambda} + \frac{1}{f_3(\mu_3 - 1)} \frac{\partial \mu_3}{\partial \lambda} = 0. \quad (35)$$

From these equations we may calculate the values of $r_1 \cdot \cdot \cdot r_4$ which will unite the wave-lengths λ and $\lambda + \delta\lambda$ in the conjugate points u_0 and u_4 on the axis.

Substituting in (33) the value of the curvatures from (20), we have

$$\frac{\lambda}{\mu_1} \frac{\partial \mu_1}{\partial \lambda} \frac{\partial N_1}{\partial s} + \frac{\lambda}{\mu_3} \frac{\partial \mu_3}{\partial \lambda} \frac{\partial N_3}{\partial s} = 0. \quad (36)$$

This shows that $\partial N_1 / \partial s$ must be of opposite sign to $\partial N_3 / \partial s$ for normal dispersion. Hence the variation in thickness of the one lens must be opposite to that of the other to obtain achromatism.

The value of $(\partial \mu_1 / \partial \lambda) / (\partial \mu_3 / \partial \lambda)$ from (33) may be readily obtained by means of spectral bands. We have from (5)

$$\begin{aligned} \frac{\partial N_0}{\partial \lambda} &= \frac{d_0 \mu_0}{\lambda^2} \\ \frac{\partial N_1}{\partial \lambda} &= \frac{d_1 \mu_1}{\lambda^2} + \frac{d_1}{\lambda} \frac{\partial \mu_1}{\partial \lambda} \\ \frac{\partial N_3}{\partial \lambda} &= \frac{d_3 \mu_3}{\lambda^2} + \frac{d_3}{\lambda} \frac{\partial \mu_3}{\partial \lambda}. \end{aligned} \quad (37)$$

If we make $N_0 = N_1 = N_3$, we have

$$\frac{d_3}{d_1} \frac{\frac{\partial N_1}{\partial \lambda} - \frac{\partial N_0}{\partial \lambda}}{\frac{\partial N_3}{\partial \lambda} - \frac{\partial N_0}{\partial \lambda}} = \frac{\frac{\partial \mu_1}{\partial \lambda}}{\frac{\partial \mu_3}{\partial \lambda}}. \quad (38)$$

Thus if we have air-bands in juxtaposition with the bands of the first medium, and then with those of the second medium, and adjust until they have the same order for one wave-length, λ_c , say, the ratio of the respective increments in order, over the air-bands for another wave-length, will be the ratio of the increments in indices sought.

In order that m wave-lengths may have a common focus, m lenses would be required, and we should find in a similar way the following $m - 1$ equations,

$$\begin{aligned} \left(\frac{1}{r_1} - \frac{1}{r_2}\right) \frac{\partial \mu_1}{\partial \lambda_1} + \left(\frac{1}{r_3} - \frac{1}{r_4}\right) \frac{\partial \mu_3}{\partial \lambda_1} + \cdots + \left(\frac{1}{r_{2m-1}} - \frac{1}{r_{2m}}\right) \frac{\partial \mu_{2m-1}}{\partial \lambda_1} &= 0 \\ \vdots \\ \left(\frac{1}{r_1} - \frac{1}{r_2}\right) \frac{\partial \mu_1}{\partial \lambda_{m-1}} + \left(\frac{1}{r_3} - \frac{1}{r_4}\right) \frac{\partial \mu_3}{\partial \lambda_{m-1}} + \cdots + \left(\frac{1}{r_{2m-1}} - \frac{1}{r_{2m}}\right) \frac{\partial \mu_{2m-1}}{\partial \lambda_{m-1}} &= 0 \end{aligned} \quad (39)$$

or

$$\begin{aligned} \frac{1}{f_1(\mu_1 - 1)} \frac{\partial \mu_1}{\partial \lambda_1} + \frac{1}{f_3(\mu_3 - 1)} \frac{\partial \mu_3}{\partial \lambda_1} + \cdots + \frac{1}{f_{2m-1}(\mu_{2m-1} - 1)} \frac{\partial \mu_{2m-1}}{\partial \lambda_1} &= 0 \\ \vdots \\ \frac{1}{f_1(\mu_1 - 1)} \frac{\partial \mu_1}{\partial \lambda_{m-1}} + \frac{1}{f_3(\mu_3 - 1)} \frac{\partial \mu_3}{\partial \lambda_{m-1}} + \cdots + \frac{1}{f_{2m-1}(\mu_{2m-1} - 1)} \frac{\partial \mu_{2m-1}}{\partial \lambda_{m-1}} &= 0 \end{aligned} \quad (40)$$

and the focal equation

$$\begin{aligned} \frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_3} + \cdots + \frac{1}{f_{2m-1}} &= (\mu_1 - 1) \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \\ &+ (\mu_3 - 1) \left(\frac{1}{r_3} - \frac{1}{r_4} \right) + \cdots + (\mu_{2m-1} - 1) \left(\frac{1}{r_{2m-1}} - \frac{1}{r_{2m}} \right). \end{aligned} \quad (41)$$

Here $\partial \lambda_1, \partial \lambda_2, \dots, \partial \lambda_{m-1}$ are the increments in wave-lengths referred to some specified wave-length, λ_D , say; and $\partial \mu_1, \partial \mu_3, \dots, \partial \mu_{2m-1}$, the corresponding increments in index in each lens.

The above equations show that the order of sequence of the lenses is arbitrary. This method of satisfying the equation of achromatism (15), by achromatizing for different wave-lengths, gives rise to so-called secondary spectra.

We may, however, satisfy this equation of condition in a different way, which can now be realized, in part at least, experimentally,

with less residual chromatism than the old method of using a greater number of lenses.

Instead of using m lenses of different indices, we may combine two having $m + 1$ color coincidences. We shall then have the m equations,

$$\begin{aligned}\frac{\partial^2 N}{\partial \lambda_1 \partial s} &= \frac{\partial}{\partial \lambda_1} \left(\frac{\partial N_0}{\partial s} + \frac{\partial N_1}{\partial s} + \frac{\partial N_2}{\partial s} + \frac{\partial N_3}{\partial s} + \frac{\partial N_4}{\partial s} \right) = 0 \\ \frac{\partial^2 N}{\partial \lambda_2 \partial s} &= \frac{\partial}{\partial \lambda_2} \left(\frac{\partial N_0}{\partial s} + \frac{\partial N_1}{\partial s} + \frac{\partial N_2}{\partial s} + \frac{\partial N_3}{\partial s} + \frac{\partial N_4}{\partial s} \right) = 0 \\ &\vdots \\ \frac{\partial^2 N}{\partial \lambda_m \partial s} &= \frac{\partial}{\partial \lambda_m} \left(\frac{\partial N_0}{\partial s} + \frac{\partial N_1}{\partial s} + \frac{\partial N_2}{\partial s} + \frac{\partial N_3}{\partial s} + \frac{\partial N_4}{\partial s} \right) = 0\end{aligned}\quad (42)$$

Proceeding in the same way as with (32), we should thus have the m equations

$$\begin{aligned}\left(\frac{1}{r_1} - \frac{1}{r_2} \right) \frac{\partial \mu_1}{\partial \lambda_1} + \left(\frac{1}{r_3} - \frac{1}{r_4} \right) \frac{\partial \mu_3}{\partial \lambda_1} &= 0 \\ \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \frac{\partial \mu_1}{\partial \lambda_2} + \left(\frac{1}{r_3} - \frac{1}{r_4} \right) \frac{\partial \mu_3}{\partial \lambda_1} &= 0 \\ &\vdots \\ \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \frac{\partial \mu_1}{\partial \lambda_m} + \left(\frac{1}{r_3} - \frac{1}{r_4} \right) \frac{\partial \mu_3}{\partial \lambda_m} &= 0\end{aligned}\quad (43)$$

and the focal equation

$$\frac{1}{f_1} + \frac{1}{f_3} = (\mu_1 - 1) \left(\frac{1}{r_1} - \frac{1}{r_2} \right) + \mu_3 - 1 \left(\frac{1}{r_3} - \frac{1}{r_4} \right). \quad (34)$$

We have seen how, by means of (37) and (38), to achromatize for two wave-lengths by means of two lenses. We may by means of (43) achromatize for $m + 1$ wave-lengths by means of *two* lenses. Suppose $m = 2$, we have then to find the corresponding equations similar to (38) in terms of their orders. We have from (5)

$$\begin{aligned}\frac{\partial N_0}{\partial \lambda_1} &= \frac{d_0 \mu_0}{\lambda^2} \\ \frac{\partial N_1}{\partial \lambda_1} &= \frac{d_1 \mu_1}{\lambda^2} + \frac{d_1}{\lambda} \frac{\partial \mu_1}{\partial \lambda_1} \\ \frac{\partial N_1}{\partial \lambda_2} &= \frac{d_1 \mu_1}{\lambda^2} + \frac{d_1}{\lambda} \frac{\partial \mu_1}{\partial \lambda_2}\end{aligned}\quad (44)$$

$$\frac{\partial N_3}{\partial \lambda_1} = \frac{d_3 \mu_3}{\lambda^2} + \frac{d_3}{\lambda} \frac{\partial \mu_3}{\partial \lambda_1}$$

$$\frac{\partial N_3}{\partial \lambda_2} = \frac{d_3 \mu_3}{\lambda^2} + \frac{d_3}{\lambda} \frac{\partial \mu_3}{\partial \lambda_2}$$

where $\partial \lambda_1$ and $\partial \lambda_2$ are the increments in wave-length, referred to some wave-length of reference λ_D , of any two other wave-lengths, λ_C and λ_F , say.

If we make $N_0 = N_1 = N_3$, we shall have

$$\frac{\frac{\partial N_1}{\partial \lambda_1} - \frac{\partial N_0}{\partial \lambda_1}}{\frac{\partial N_1}{\partial \lambda_2} - \frac{\partial N_0}{\partial \lambda_2}} = \frac{\frac{\partial \mu_1}{\partial \lambda_1}}{\frac{\partial \mu_1}{\partial \lambda_2}} \quad (45)$$

$$\frac{\frac{\partial N_3}{\partial \lambda_1} - \frac{\partial N_0}{\partial \lambda_1}}{\frac{\partial N_3}{\partial \lambda_2} - \frac{\partial N_0}{\partial \lambda_2}} = \frac{\frac{\partial \mu_3}{\partial \lambda_1}}{\frac{\partial \mu_3}{\partial \lambda_2}}. \quad (46)$$

But (43) requires that

$$\frac{\frac{\partial \mu_1}{\partial \lambda_1}}{\frac{\partial \mu_1}{\partial \lambda_2}} = \frac{\frac{\partial \mu_3}{\partial \lambda_1}}{\frac{\partial \mu_3}{\partial \lambda_2}} \quad (47)$$

and hence we must have

$$\frac{\frac{\partial N_1}{\partial \lambda_1} - \frac{\partial N_0}{\partial \lambda_1}}{\frac{\partial N_3}{\partial \lambda_1} - \frac{\partial N_0}{\partial \lambda_1}} = \frac{\frac{\partial N_1}{\partial \lambda_2} - \frac{\partial N_0}{\partial \lambda_2}}{\frac{\partial N_3}{\partial \lambda_2} - \frac{\partial N_0}{\partial \lambda_2}}. \quad (48)$$

Equations (34) and (38) are the conditions for achromatism of two colors, while (48) is the further condition for achromatism of three colors by means of two lenses. Thus, suppose we have our three sets of spectral bands in the field of view, the air-bands being in the middle, we set for coincidence at the *D*-line say, making the sets of the same order, which can readily be accomplished by varying the thicknesses successively and counting the bands, as described above. The eye may then note the gain relatively to the air-bands of each

of the other sets of bands on each side of the D -line. If these two ratios should become equal at two points on each side of the D -line, we know that we shall have chromatic compensation for all three colors. Should we fail to find this condition satisfied, we may shift the region of coincidence with the air-bands to some other point and examine for a closer approximation of the ratios, and so on. Very close approximations to such ratios are now possible with certain combinations of the silicate and borate glasses, such as those obtainable from the Jena factory. Compare, for example, factory Nos. 0.152 and S.7, whose partial dispersions for $A' - D$, $D - F$, $F - G'$ are .628, .708, .582 and .628, .708 and .583 respectively. The facility with which comparisons may be made with different experimental castings in obtaining new series and pairs of glasses is evident from the above discussion.

The treatment for media with plane surfaces, *i. e.*, achromatic prisms, is similar to that for curved surfaces or lenses. Thus in the case of two thin prisms, we have to satisfy equations (14) and (15). We obtain, as before,

$$\begin{aligned}\frac{\partial N_0}{\partial s} &= \frac{\mu_0}{\lambda} \frac{\partial l_0}{\partial s} = \frac{\mu_0}{\lambda} \tan i = \frac{\mu_0}{\lambda} i \\ \frac{\partial N_1}{\partial s} &= \frac{\mu_1}{\lambda} \frac{\partial l}{\partial s} = -\frac{\mu_1}{\lambda} \tan a = -\frac{\mu_1}{\lambda} a_1 \\ \frac{\partial N}{\partial s} &= \frac{\mu}{\lambda} \frac{\partial l}{\partial s} = \frac{\mu}{\lambda} \tan i' = \frac{\mu}{\lambda} i'.\end{aligned}\tag{49}$$

where i is the angle of incidence and a that of the prism. From (14) we have

$$\frac{\partial N}{\partial s} = \frac{\partial N_0}{\partial s} + \frac{\partial N_1}{\partial s} + \frac{\partial N}{\partial s} = \frac{\mu_0}{\lambda} i - \frac{\mu_1}{\lambda} a_1 + \frac{\mu}{\lambda} i' = 0.$$

Making $\mu_0 = \mu = 1$, we have, calling D the deviation,

$$i + i' - \mu_1 a_1 = 0\tag{50}$$

or, since $D = i + i' - a$, we must have

$$D = (\mu_1 - 1) a_1.\tag{51}$$

Similarly for two prisms we should find

$$i_1 + i_1' - \mu_1 a_1 + i_3 + i_3' - \mu_3 a_3 = 0, \quad (53)$$

or

$$D = (\mu_1 - 1) a_1 + (\mu_3 - 1) a_3. \quad (54)$$

Differentiating (49), we have

$$\begin{aligned} \frac{\partial^2 N_0}{\partial \lambda \partial s} &= \frac{i_1}{\lambda} \frac{\partial \mu_0}{\partial \lambda} - \frac{\mu_0 i_1}{\lambda^2} \\ \frac{\partial^2 N_1}{\partial \lambda \partial s} &= -\frac{a_1}{\lambda} \frac{\partial \mu_1}{\partial \lambda} + \frac{\mu_1 a_1}{\lambda^2} \\ \frac{\partial^2 N_2}{\partial \lambda \partial s} &= \frac{i_1'}{\lambda} \frac{\partial \mu_2}{\partial \lambda} - \frac{\mu_2 i_1'}{\lambda^2} \\ \frac{\partial^2 N_2}{\partial \lambda \partial s} &= \frac{i_3}{\lambda} \frac{\partial \mu_2}{\partial \lambda} - \frac{\mu_2 i_3}{\lambda^2} \\ \frac{\partial^2 N}{\partial \lambda \partial s} &= -\frac{a_3}{\lambda} \frac{\partial \mu_3}{\partial \lambda} + \frac{\mu_3 a_3}{\lambda^2} \\ \frac{\partial^2 N}{\partial \lambda \partial s} &= \frac{i_3'}{\lambda} \frac{\partial \mu_4}{\partial \lambda} - \frac{\mu_4 i_3'}{\lambda^2}. \end{aligned} \quad (55)$$

If we make $\mu_0 = \mu_2 = \mu_4 = 1$, we still have

$$\begin{aligned} \frac{\partial^2 N}{\partial \lambda \partial s} &= -\frac{1}{\lambda^2} \left\{ \lambda \left[a_1 \frac{\partial \mu_1}{\partial \lambda} + a_3 \frac{\partial \mu_3}{\partial \lambda} \right] + [i_1 + i_1' + i_3 \right. \\ &\quad \left. + i_3' - \mu_1 a_1 - \mu_3 a_3] \right\} = 0 \end{aligned} \quad (56)$$

or from (53)

$$a_1 \frac{\partial \mu_1}{\partial \lambda} + a_3 \frac{\partial \mu_3}{\partial \lambda} = 0. \quad (57)$$

It is evident from this equation that the angles a_1 and a_3 must have opposite signs. We may proceed to a higher order of achromatism by using a greater number of prisms, just as with lenses, obtaining equations similar to (39) and (40). Or we may reduce the residual chromatism in a pair of prisms by satisfying the equation (48) in addition to (53) and (57).