ELEMENTARY MATHEMATICS.*

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This separation between pure mathematics and applied mathematics is grievous even in the domain of elementary mathematics. In witness, in the first place: The workers in physics, chemistry, and engineering need more practical mathematics; and numerous textbooks, in particular, on calculus, have recently been written from the point of view of these allied subjects. I refer to the works by Nernst and Schoenflies,¹³ Lorentz,¹⁴ Perry,¹⁵ and Mellor,¹⁶ and to a book on the very elements of mathematics now in preparation by Oliver Lodge.

In the second place, I dare say you are all familiar with the surprisingly vigorous and effective agitation with respect to the teaching of elementary mathematics which is at present in progress in England, largely under the direction of John Perry, professor of Mechanics and Mathematics in the Royal College of Science, London, and chairman of the Board of Examiners of the Board of Education in the subjects of engineering, including practical plane and solid geometry, applied mechanics, practical mathematics, in addition to more technical subjects, and in this capacity in charge of the education of some hundred thousand apprentices in English night schools. The section on Education of the British Association had its first session at the Glasgow meeting, 1901, and the session was devoted to the consideration, in connection with the section on Mathematics and Physics, of the question of the pedagogy of mathematics, and Perry opened the discussion by

* Presidential address, *On the Foundations of Mathematics*, delivered before the American Mathematical Society at its ninth annual meeting, December 29, 1902.

¹³ Nernst und Schoenflies: *Einführung in die Mathematische Behandlung der Naturwissenschaften* (Munich and Leipzig, 1895); the basis of Young and Lineberger's *Elements of Differential and Integral Calculus* (New York, 1900).

¹⁴ Lorentz: *Lehrbuch der Differential und Integralrechnung* (Leipzig),

¹⁵ Perry: *Calculus for Engineers* (Second Edition, London, E. Arnold, 1897); German translation by Fricke (Teubner, 1902). Cf. also the citations given later on.

¹⁶ Mellor: *Higher Mathematics for Students of Chemistry and Physics*, with special reference to Practical Work. (Longmans, Green & Co., 1902, pp. xxi+543)

a paper on "The Teaching of Mathematics". A strong committee under the chairmanship of Professor Forsyth of Cambridge was appointed "to report upon improvements that might be effected in the teaching of mathematics, in the first instance, in the teaching of elementary mathematics, and upon such means as they think likely to effect such improvements". The paper of Perry, with the discussion of the subject at Glasgow and additions including the report of the committee as presented to the British Association at its Belfast meeting, September, 1902, are collected in a small volume, "*Discussion on the Teaching of Mathematics*,"¹⁷ edited by Professor Perry (Macmillan; second edition, 1902).

One should consult the books of Perry, *Practical Mathematics*,¹⁸ *Applied Mechanics*,¹⁹ *Calculus for Engineers*,²⁰ and *England's Neglect of Science*,²¹ and his address²² on *The Education of Engineers*,—and furthermore the files from 1899 on—of the English journals, *Nature*, *School World*, *Journal of Education*, and *Mathematical Gazette*.

One important purpose of the English agitation is to relieve the English secondary school teachers from the burden of a too precise examination system, imposed by the great examining bodies; in particular, to relieve them from the need of retaining Euclid as the sole authority in geometry, at any rate with respect to the sequence of propositions. Similar efforts made in England about thirty years ago were unsuccessful. Apparently the forces operating since that time have just now broken forth into successful activity; for the report of the British Association committee was distinctly favorable, in a conservative sense, to the idea of reform, and already noteworthy initial changes have been made in the regulations for the secondary examinations by the examination syndicates of the universities of Oxford, Cambridge, and London.

¹⁷ Cf. also Report on the Teaching of Elementary Mathematics, issued by the Mathematical Association (G. Bell & Sons, London, 1902).

¹⁸ Published for the Board of Education by Eyre and Spottiswoode. (London, 1899).

¹⁹ D. Van Nostrand Co., New York, 1898.

²⁰ Second edition, London, E. Arnold, 1897.

²¹ T. Fisher Unwin, London, 1900.

²² In opening the discussion of the sections on Engineering and on Education at the Belfast, 1902, meeting of the British Association. Published in *Science*, Nov. 14, 1902.

The reader will find the literature of this English movement very interesting and suggestive. For instance, in a letter to *Nature* (vol. 65, p. 484; March 27, 1902) Perry mildly apologizes for having to do with the movement whose immediate results are likely to be merely slight reforms, instead of the thoroughgoing reforms called for in his pronouncements and justified by his marked success during over twenty years as a teacher of practical mathematics. He asserts that the orthodox logical sequence in mathematics is not the only possible one; that on the contrary a more logical sequence than the orthodox one (because one more possible of comprehension by the students) is based upon the notions underlying the infinitesimal calculus taken as axioms: for instance, that a map may be drawn to scale; the notions underlying the many uses of squared paper; that decimals may be dealt with as ordinary numbers. He asserts as essential that the boy should be *familiar* (by way of experiment, illustration, measurement, and by every possible means) with the ideas to which he applies his logic; and moreover that he should be thoroughly *interested* in the subject studied; and he closes with this peroration:

“Great God! I’d rather be

A pagan, suckled in a creed outworn.”

I would rather be utterly ignorant of all the wonderful literature and science of the last twenty-four centuries, even of the wonderful achievements of the last fifty years, than not to have the sense that our whole system of so-called education is as degrading to literature and philosophy as it is to English boys and men.”

As a pure mathematician, I hold as the most important suggestion of the English movement the suggestion of Perry’s, just cited,—that by emphasizing steadily the practical sides of mathematics, that is, arithmetic computations, mechanical drawing and graphical methods generally, in continuous relation with problems of physics and chemistry and engineering, it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry, and the calculus. This is accomplished on the one hand by the increase of attention and comprehension obtained by connecting the abstract mathematics with subjects which are naturally of interest to the boy, so that, for instance, all the results obtained by theoretic process are capable of

check by laboratory process, and on the other hand by a diminution of emphasis on the systematic and formal sides of the instruction in mathematics. Undoubtedly many mathematicians will feel that this decrease of emphasis will result in much if not irreparable injury to the interests of mathematics. But I am inclined to think that the mathematician with the catholic attitude of an adherent of science in general (and at any rate with respect to the problems of the pedagogy of elementary mathematics there is no other rational attitude) will see that the boy will be learning to make practical use in his scientific investigations, to be sure, in a naive and elementary way, of the finest mathematical tools which the centuries have forged, that under skilful guidance he will learn to be interested not merely in the achievements of the tools but in the theory of the tools themselves, and that thus he will ultimately have a feeling towards his mathematics extremely different from that which is now met with only too frequently,—a feeling that mathematics is indeed itself a fundamental reality of the domain of thought, and not merely a matter of symbols and arbitrary rules and conventions.

THE AMERICAN MATHEMATICAL SOCIETY.

The American Mathematical Society has, naturally, interested itself chiefly in promoting the interests of research in mathematics. It has, however, recognized that those interests are closely bound up with the interests of education in mathematics. I refer in particular to the valuable work done by the committee appointed, with the authorization of the Council, by the Chicago Section of the Society, to represent mathematics in connection with Dr. Nightingale's committee of 1899 of the National Educational Association in the formulation of standard curricula for high schools and academies, and to the fact that two committees are now at work, one appointed in December, 1901, by the Chicago Section, to formulate the desirable conditions for the granting, by institutions of the Mississippi valley, of the degree of Master of Arts for work in mathematics, and the other appointed by the Society at its last summer meeting to coöperate with similar committees of the National Educational Association and of the Society for the Promotion of Engineering Education, in formulating

standard definitions of requirements in mathematical subjects for admission to colleges and technological schools; and furthermore I refer to the fact that (although not formally) the Society has made a valuable contribution to the interests of secondary education in that the College Entrance Examination Board has as its Secretary the principal founder of the Society. I have accordingly felt at liberty to bring to the attention of the Society these matters of the pedagogy of elementary mathematics, and I do so with the firm conviction that it would be possible for the Society, by giving still more attention to these matters, to further most effectively the highest interests of mathematics in this country.

A VISION.

AN INVITATION.

The pure mathematicians are invited to determine how mathematics is regarded by the world at large, including their colleagues of other science departments and the students of elementary mathematics, and to ask themselves whether by modification of methods and attitude they may not win for it the very high position in general esteem and appreciative interest which it assuredly deserves.

This general invitation and the preceding summary view invoke this vision of the future of elementary mathematics in this country.

THE PEDAGOGY OF ELEMENTARY MATHEMATICS.

We survey the pedagogy of elementary mathematics in the primary schools, in the secondary schools, and in the junior colleges (the lower collegiate years). It is, however, understood that there is a movement for the enlargement of the strong secondary schools, by the addition of the two years of junior college work and by the absorption of the last two or three grades of the primary schools, into institutions more of the type of the German gymnasium and the French lycée;²⁸ in favor of this movement there are strong arguments, and among them this, that in such institutions, especially if closely related to strong colleges or universities, the mathematical reforms may the more easily be carried out.

²⁸ As to the mathematics of these institutions, one may consult the book on "The Teaching of Mathematics in the Higher Schools of Prussia." (New York: Longmans, Green & Co., 1900) by Professor Young, and the article (Bulletin Amer. Math. Soc. (2) vol. 6, p. 225) by Professor Pierpont.

The fundamental problem is that of *the unification of pure and applied mathematics*. If we recognize the branching implied by the very terms "pure", "applied", we have to do with a special case of *the correlation of different subjects* of the curriculum, a central problem in the domain of pedagogy from the time of Herbart on. In this case the fundamental solution is to be found rather by way of indirection,—by arranging the curriculum so that throughout the domain of elementary mathematics the branching be not recognized.

THE PRIMARY SCHOOLS.

Would it not be possible for the children in the grades to be trained in power of observation and experiment and reflection and deduction, so that always their mathematics should be directly connected with matters of thoroughly concrete character? The response is immediate that this is being done today in the kindergartens and in the better elementary schools. I understand that serious difficulties arise with children of from nine to twelve years of age, who are no longer contented with the simple concrete methods of earlier years and who, nevertheless, are unable to appreciate the more abstract methods of the later years. These difficulties, some say, are to be met by allowing the mathematics to enter only implicitly in connection with the other subjects of the curriculum. But rather the material and methods of the mathematics should be enriched and vitalized. In particular, the grade teachers must make wiser use of the foundations furnished by the kindergarten. The drawing and the paper folding should lead on directly to systematic study of intuitional geometry, with simple exercises in geometrical reasoning.²⁴ The children are to be taught to represent, according to the usual conventions, various familiar and interesting phenomena. The geometry must be closely connected with the numerical and literal arithmetic. The cross-grooved tables of the kindergarten furnish an especially important type of connection, viz., a conventional graphical depiction of any phenomenon in which one magnitude depends upon another. These tables and the similar cross-section blackboards and paper must enter largely into all the mathematics of the

²⁴ Here I refer to the very suggestive paper of Benchara Branford, entitled "Measurement and Simple Surveying. An Experiment in the Teaching of Elementary Geometry" to a small class of beginners of about ten years of age (Journal of Education London—the first part appearing in the number for August, 1899.)

grades, and to study the properties of the phenomena in the pictures; to know, for example, what concrete meaning attaches to the fact that a graph curve at a certain point is going down or going up or is horizontal. Thus the problems of percentage,—interest, etc.,—have their depiction in straight line or broken line graphs.

(To be continued.)

WHAT HIGH SCHOOL PHYSICS SHOULD REQUIRE FROM ALGEBRA AND GEOMETRY.

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“Education,” says Lyman Abbott, “should furnish the student with two things for his work in life; facts or information and power, or ability to use these facts.”

The study of algebra and geometry has furnished the student with an abundance of facts and information, concerning algebraic expressions and geometrical propositions, but it has failed almost utterly to supply him with the power to apply these facts to the solution of the problems found in elementary physics.

It is not the intention to enter into a general discussion in this paper as to why the study of these subjects has failed to furnish this power, but only to outline some of the facts of algebra and geometry to show where more power is needed.

These requirements with a brief statement as to why they exist may be stated as follows:

1st. Ability to solve an equation no matter what letters may be used, *i. e.*, $x = \frac{1}{2} y z^2$ or $s = \frac{1}{2} a t^2$ or $v = \frac{1}{2} m t^2$ all of which are forms of the same equation.

One of the most common and troublesome difficulties is that a student can work all sorts of expressions between x, y, z, a, b, c , but when given the same expressions between s, a, t, v , etc., he stumbles at once. Of course, he ought to see that there is no difference between the expressions but the fact remains that for some reason he fails more or less completely to grasp the relation. This is probably due to the fact that the subject and terms used