substitution, we have

$$\begin{aligned} XZ - Y^2 &= \frac{A^4}{(\theta_1 - \theta_2)^3} \cdot -(\theta_1 - \theta_2)^3 \theta_1 \theta_2, &= -A^4 \theta_1 \theta_2, \\ XW - YZ &= \dots, &= -(\theta_1 - \theta_2)^3 (\theta_1 + \theta_2) \theta_1 \theta_2, &= -A^4 \theta_1 \theta_2 (\theta_1 + \theta_2), \\ YW - Z^2 &= \dots, &= -(\theta_1 - \theta_2)^3 \theta_1 \theta_2 \theta_1 \theta_2, &= -A^4 \theta_1 \theta_2 \theta_1 \theta_2. \end{aligned}$$

Or, what is the same thing, these are $= -A \nabla, 2B \nabla, -C \nabla$, respectively; thus the first equation is

$$\{3A^2c - 6ABb + (-AC + 4B^2)a\} \{2ABd - 3ACc + C^2a\} - (-A^2d + 3ACb - 2BCa)^2 = -A(A^2d^2 + \&c.), = -A \nabla;$$

and similarly for the other two equations. The nodal curve is thus given by the twofold system $X=0, Y=0, Z=0, W=0$.

The method may be extended to the case where, instead of the quadric equation $(A, B, C \nabla \theta, 1)^2 = 0$, we have an equation of any higher order, but the formulæ are less simple.

Note on the Equation of the Two Planes which can be drawn through Two given Points to touch a Quadric. By H. M. TAYLOR, M.A.

[Read June 10th, 1880.]

$$\text{If } U = ax^2 + by^2 + cz^2 + d + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz = 0$$

be the equation of a quadric; and, in consequence,

$$\begin{aligned} W_1 &= x(ax_1 + hy_1 + gz_1 + l) + y(hx_1 + by_1 + fz_1 + m) \\ &\quad + z(gx_1 + fy_1 + cz_1 + n) + lx_1 + my_1 + nz_1 + d = 0 \end{aligned}$$

the equation of the polar plane of the point $P_1(x_1, y_1, z_1)$; and if U_1 denote what U becomes when x_1, y_1, z_1 are written for x, y, z respectively; and $W_{1,2}$ denote what W_1 becomes when x_2, y_2, z_2 are written for x, y, z respectively; then it can be shewn that

$$\begin{vmatrix} U & W_1 & W_2 \\ W_1 & U_1 & W_{1,2} \\ W_2 & W_{1,2} & U_2 \end{vmatrix} = 0 \dots\dots\dots(1)$$

is the equation of the two planes which can be drawn through the two points $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2)$, to touch the quadric $U=0$; or, in other words, that

$$\begin{vmatrix} U_1 & W_{1,2} & W_{1,3} \\ W_{1,2} & U_2 & W_{2,3} \\ W_{1,3} & W_{2,3} & U_3 \end{vmatrix} = 0 \dots\dots\dots(2)$$

is the condition that the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ lie on a tangent plane of the quadric.

We may compare herewith the equation

$$\begin{vmatrix} U_1 & W_{1,2} \\ W_{1,2} & U_2 \end{vmatrix} = 0 \dots\dots\dots (3),$$

which is the condition that the two points P_1, P_2 lie on a tangent line of the quadric.

If we write, in the last equation, (x, y, z) (x_1, y_1, z_1) instead of (x_1, y_1, z_1) (x_2, y_2, z_2) , we have the well-known theorem that the enveloping cone drawn from the point P_1 to the quadric $U=0$ has for its equation

$$UU_1 - W_1^2 = 0 \dots\dots\dots (4).$$

If we draw an enveloping cone to this cone from the point P_2 , it will degenerate into two planes, and will in reality be the two tangent planes to the quadric $U=0$ which pass through the two points P_1, P_2 .

The equation of the enveloping cone from the point P_2 to the quadric $UU_1 - W_1^2 = 0$ will be

$$(UU_1 - W_1^2)(U_2U_1 - W_{1,2}^2) = (W_2U_1 - W_1W_{1,2})^2;$$

or, after simplification,

$$UU_1U_2 - UW_{1,2}^2 - U_1W_2^2 - U_2W_1^2 + 2W_1W_2W_{1,2} = 0 \dots\dots (5),$$

which may be written

$$\begin{vmatrix} U & W_1 & W_2 \\ W_1 & U_1 & W_{1,2} \\ W_2 & W_{1,2} & U_2 \end{vmatrix} = 0,$$

from which equation, by introducing a third suffix, we obtain the equation (2), which is the condition that a tangent plane to the quadric passes through the three points P_1, P_2, P_3 .

Note by Professor Cayley.

The equation (3) is known as a transformation of the equation

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_1 & x_2 & A & H & G & L \\ y_1 & y_2 & H & B & F & M \\ z_1 & z_2 & G & F & C & N \\ 1 & 1 & L & M & N & D \end{vmatrix} = 0,$$

where $(A, B, \dots)$ are the inverse coefficients of the system $(a, b, \dots)$. And similarly (but which is in fact more obvious), the equation (2) is

a transformation of

$$\begin{vmatrix} & & & x_1 & y_1 & z_1 & 1 \\ & & & x_2 & y_2 & z_2 & 1 \\ & & & x_3 & y_3 & z_3 & 1 \\ x_1 & x_2 & x_3 & A & H & G & L \\ y_1 & y_2 & y_3 & H & B & F & M \\ z_1 & z_2 & z_3 & G & F & C & N \\ 1 & 1 & 1 & L & M & N & D \end{vmatrix} = 0.$$

This last equation is, in fact,

$$(A, \dots \mathfrak{L} \alpha, \beta, \gamma, \delta)^2 = 0,$$

where $\alpha, \beta, \gamma, \delta$ are the coefficients of $x, y, z, 1$ in

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0;$$

viz., $\alpha x + \beta y + \gamma z + \delta = 0$ is the equation of the plane through the three points; and the condition that this may be a tangent plane thus presents itself, in the first instance, in the foregoing form

$$(A, \dots \mathfrak{L} \alpha, \beta, \gamma, \delta)^2 = 0.$$

On the Focal Conics of a Bicircular Quartic.

By HARRY HART, M.A.

[Read June 10th, 1880.]

It is a well-known theorem, that a bicircular quartic may be generated in four different ways, as the envelope of a variable circle, whose centre moves along a given conic (the focal conic of the quartic), and cuts a given circle orthogonally. It has also been shewn that the common self-reciprocal triangle of each fixed circle, and its corresponding focal conic, has for its angular points the centres of the other three fixed circles; i.e., if J, J_1, J_2, J_3 be the four fixed co-orthogonal circles, and F, F_1, F_2, F_3 the corresponding focal conics, the triangle formed by joining the centres of J_1, J_2, J_3 is the common self-reciprocal triangle of the focal conic F and the orthogonal circle J , &c.

Let O, A, B, C be the centres of J, J_1, J_2, J_3 respectively, O being that centre which lies within the acute-angled triangle formed by join-