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1894 Proc. Phys. Soc. London 13 145

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XV. *On Electromagnetic Induction in Plane, Cylindrical, and Spherical Current-Sheets, and its Representation by Moving Trails of Images.* By G. H. BRYAN, M.A.*

PART I.—GENERAL EQUATIONS—PLANE SHEETS.

Introduction.

1. It is well known that if a very thin, indefinitely extended plane sheet of metal of finite conductivity is placed in a varying magnetic field due to the presence of moving magnetic poles in its neighbourhood, induction-currents are set up in the sheet and the field of force due to these currents may be represented by a moving trail of images.

In the present paper the surface-conditions which hold at the surface of a plane, cylindrical, spherical, or other conducting sheet of uniform small thickness are deduced directly from the fundamental laws of electromagnetic induction. By working directly with the scalar magnetic potential, and avoiding the introduction of the vector potential and the quantity which Maxwell denotes by P , the investigations are much simplified. Moreover, in at least one comparatively recently published paper the boundary conditions satisfied by the vector potential at the surface of separation of two different media have been erroneously stated, and for this reason it is advantageous to employ a method which obviates the difficulty.

The results will be employed to show how the field due to the presence of a magnetic pole of varying intensity in the neighbourhood of a plane, cylindrical, or spherical current-sheet may be represented by means of a moving trail of images. By the principle of superposition the effect of any number of poles of varying intensity can be deduced, and the corresponding expressions for the field can thus be obtained when the variable inducing system of magnets is of the most general possible character, as, for example, one (or more) magnetic poles moving about in any manner whatever.

In this way a synthetic solution of the problem of induction

* Read May 11, 1894.

in current sheets is obtained. The phenomena of induction in spherical and other current sheets have been treated at considerable length from an analytical point of view by Larmor, Lamb, and Niven, and the last-named writer has made some attempt, in the case of a sphere, to interpret the results by means of images; but the present investigation seemed desirable for many reasons, and it is hoped that it will overcome some of the difficulties, and elucidate some of the obscurities which present themselves in most treatments of this interesting application of the principles of electromagnetism.

Fundamental Assumptions.

2. The laws of electromagnetic induction assert that in bodies at rest

I. *The total current across any enclosed portion of a surface which always contains the same particles is equal to $1/4\pi$ of the line-integral of magnetic force round the curve bounding the surface.*

II. *The rate of decrease of the surface-integral of magnetic induction across any enclosed surface which always contains the same particles is equal to the line-integral of electromotive force round the curve bounding the surface.*

In applying these laws to an infinite dielectric separated into two portions by a thin conducting sheet, it is usually assumed that the disturbance produced by the inducing system is not a very rapidly alternating one, so that displacement currents in the dielectric have no appreciable magnetic effect*. With this assumption, the magnetic force in the dielectric will always be derivable from a potential which will only depend on the inducing system and the currents in the sheet. In other words, the state of the dielectric will be given by an "equilibrium theory."

It is also assumed that the induction-currents at any point distribute themselves uniformly throughout the thickness of the sheet. This requires that the disturbance shall not be a very rapidly alternating one, and also that the thickness of the sheet shall be very small compared with the other linear

* Watson and Burbury, 'Mathematical Theory of Electricity and Magnetism,' ii. § 405.

dimensions of the system (such as the distances of the moving poles, the radius of the sheet if spherical, &c.).

Surface Conditions at a Plane-Current Sheet.

3. Let the plane of the sheet be taken as the plane of x, y , let the thickness of the sheet be c , and specific conductivity C .

Let Ω_1, Ω_2 be the magnetic potentials on the negative and positive side of the sheet respectively, ϕ the current function in the sheet at any point. Apply Law I. to the circuit formed by going along the positive side of the sheet from the origin to any point and returning along the negative side from that point to the origin. Since the thickness of the sheet is small, the terms contributed to the line-integral of the magnetic force by the passage of the circuit from one side of the sheet to the other may be neglected and we obtain *

$$\Omega_2 - \Omega_1 = 4\pi\phi + \text{constant.} \quad . \quad . \quad . \quad (1)$$

Now apply Law II. to any circuit s drawn in the plane of the sheet. Let S_1, S_2 be two surfaces drawn in the dielectric infinitely near to the positive and negative faces of the sheet respectively and both bounded by the curve s , and let P, Q be the components of electromotive force at any point. Then, assuming the magnetic permeability of the dielectric to be unity, we have

$$\int (Pdx + Qdy) \text{ round } s = \frac{d}{dt} \iint \frac{d\Omega_1}{dz} dS_1 = \frac{d}{dt} \iint \frac{d\Omega_2}{dz} dS_2,$$

and since corresponding elements of the near surfaces S_1, S_2 are equal, it is evident that

$$\frac{d\Omega_1}{dz} = \frac{d\Omega_2}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

at the surface, so that the normal component of magnetic induction is continuous, as it should be †.

* Maxwell, 'Electricity and Magnetism,' ii. § 658.

† The magnetic permeability of the sheet itself will not affect the conditions of the problem unless it is required to proceed to a higher order of approximation by taking into account first powers of the thickness (or unless the sheet is formed of soft iron whose magnetic permeability may be large).

Again, the equations of conduction give

$$CcP = \frac{d\phi}{dy}, \quad CcQ = -\frac{d\phi}{dx}.$$

Hence

$$\begin{aligned} \frac{d}{dt} \iint \frac{d\Omega_1}{dz} dS &= \frac{d}{dt} \iint \frac{d\Omega_2}{dz} dS = \frac{1}{Cc} \int \left(\frac{d\phi}{dy} \frac{dx}{ds} - \frac{d\phi}{dx} \frac{dy}{ds} \right) ds \\ &= -\frac{1}{Cc} \int \left(\frac{d\phi}{dy} \frac{dy}{dn} + \frac{d\phi}{dx} \frac{dx}{dn} \right) ds = -\frac{1}{Cc} \int \frac{d\phi}{dn} ds \\ &= -\frac{1}{Cc} \iint \left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} \right) dS, \end{aligned}$$

where dn is the element of the outward-drawn normal to s and the surface-integral taken over the area S of the sheet bounded by the curve s .

Hence at the surface of the sheet

$$\begin{aligned} \frac{d}{dt} \frac{d\Omega_1}{dz} &= \frac{d}{dt} \frac{d\Omega_2}{dz} \\ &= -\frac{1}{Cc} \left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} \right) \\ &= -\frac{1}{4\pi Cc} \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) (\Omega_2 - \Omega_1) \quad \text{by (1)} \\ &= \frac{1}{4\pi Cc} \frac{d^2}{dz^2} (\Omega_2 - \Omega_1) \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

(remembering that $\nabla^2\Omega_1=0$ and $\nabla^2\Omega_2=0$).

4. Now let Ω_0 be the magnetic potential due to the external or inducing system, Ω' and Ω'' the magnetic potentials on the two sides due to the sheet itself, so that

$$\Omega_1 = \Omega_0 + \Omega', \quad \Omega_2 = \Omega_0 + \Omega''. \quad . \quad . \quad . \quad (4)$$

Then Ω_0 and its differential coefficients with respect to z are continuous in crossing the sheet, while from the symmetry of the field due to the sheet

$$\Omega'(x, y, z) = -\Omega''(x, y, -z), \quad . \quad . \quad (5)$$

so that

$$\frac{d^n \Omega'}{dz^n} = (-1)^{n+1} \frac{d^n \Omega''}{dz^n}.$$

Therefore the surface condition (3) gives

$$\frac{d}{dt} \frac{d\Omega'}{dz} + \frac{d}{dt} \frac{d\Omega_0}{dz} = - \frac{1}{2\pi Cc} \frac{d^2\Omega'}{dz^2}, \quad \dots \quad (6)$$

$$\frac{d}{dt} \frac{d\Omega''}{dz} + \frac{d}{dt} \frac{d\Omega_0}{dz} = + \frac{1}{2\pi Cc} \frac{d^2\Omega''}{dz^2} \quad \dots \quad (7)$$

as the surface conditions satisfied by the potential at the two sides of the sheet.

Surface Conditions at a Cylindrical Sheet.

5. Using cylindrical coordinates, let $r=a$ be the equation of the middle surface of a cylindrical current-sheet, and let its thickness c be so small that the corresponding elements of its bounding surfaces and middle surface, viz.

$$(a - \frac{1}{2}c) d\theta dz, (a + \frac{1}{2}c) d\theta dz, \text{ and } a d\theta dz,$$

are to be regarded as equal.

If Ω_1, Ω_2 are the magnetic potentials inside and outside the sheet, we shall still have as for a plane sheet

$$\Omega_2 - \Omega_1 = 4\pi\phi + \text{constant}; \quad \dots \quad (1)$$

and by application of the second law to the small circuit bounding the surface-element $a d\theta dz$, we have at the sheet

$$\begin{aligned} \frac{d}{dt} \frac{d\Omega_1}{dr} &= \frac{d}{dt} \frac{d\Omega_2}{dr} = - \frac{1}{Cc} \left\{ r d\theta \frac{d\phi}{rd\theta} + \frac{d^2\phi}{dz^2} \right\} \\ &= - \frac{1}{4\pi Cc} \left\{ \frac{1}{r^2} \frac{d^2}{d\theta^2} + \frac{d^2}{dz^2} \right\} (\Omega_2 - \Omega_1) \\ &= \frac{1}{4\pi Cc} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) (\Omega_2 - \Omega_1), \end{aligned}$$

or

$$\frac{d}{dt} \left(r \frac{d\Omega_1}{dr} \right) = \frac{d}{dt} \left(r \frac{d\Omega_2}{dr} \right) = \frac{1}{4\pi Cc} \frac{d}{dr} r \frac{d}{dr} (\Omega_2 - \Omega_1). \quad \dots \quad (8)$$

Surface Conditions at a Spherical Sheet.

6. In the case of a spherical sheet of small thickness, we obtain in like manner

$$\frac{d}{dt} \frac{d\Omega_1}{dr} = \frac{d}{dt} \frac{d\Omega_2}{dr} = \frac{1}{4\pi Cc} \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) (\Omega_2 - \Omega_1),$$

or

$$\frac{d}{dt} \left(r^2 \frac{d\Omega_1}{dr} \right) = \frac{d}{dt} \left(r^2 \frac{d\Omega_2}{dr} \right) = \frac{1}{4\pi Cc} \frac{d}{dr} r^2 \frac{d}{dr} (\Omega_2 - \Omega_1). \quad (9)$$

Extension to Curved Sheets of other Forms.

7. When the sheet is of any form other than those above considered, it is necessary to assume a certain law of variation of the thickness in order to put the boundary conditions into a form similar to those given above*.

Let us use orthogonal coordinates α , β , γ ; let the line-element be given by

$$ds^2 = \frac{d\alpha^2}{h_1^2} + \frac{d\beta^2}{h_2^2} + \frac{d\gamma^2}{h_3^2},$$

and let the equation of the current sheet be

$$\gamma = \text{constant}.$$

Equation (1) still holds, and by applying Law II. to the circuit enclosed on the sheet by the curves α , $\alpha + \delta\alpha$, β and $\beta + \delta\beta$, we have the surface-condition

$$\begin{aligned} \frac{1}{h_1 h_2} \frac{d}{dt} h_3 \frac{d\Omega_1}{d\gamma} &= \frac{1}{h_1 h_2} \frac{d}{dt} h_3 \frac{d\Omega_2}{d\gamma} \\ &= - \frac{d}{d\alpha} \left\{ \frac{h_1}{C h_2} \frac{d\phi}{d\alpha} \right\} - \frac{d}{d\beta} \left\{ \frac{h_2}{C h_1} \frac{d\phi}{d\beta} \right\} \\ &= - \frac{1}{4\pi} \left\{ \frac{d}{d\alpha} \left(\frac{h_1}{C h_2} \frac{d}{d\alpha} \right) + \frac{d}{d\beta} \left(\frac{h_2}{C h_1} \frac{d}{d\beta} \right) \right\} (\Omega_2 - \Omega_1). \quad (10) \end{aligned}$$

But Laplace's equation which is satisfied by Ω_1 Ω_2 may be written

$$\frac{d}{d\alpha} \left(\frac{h_1}{h_2 h_3} \frac{d\Omega}{d\alpha} \right) + \frac{d}{d\beta} \left(\frac{h_2}{h_3 h_1} \frac{d\Omega}{d\beta} \right) + \frac{d}{d\gamma} \left(\frac{h_3}{h_1 h_2} \frac{d\Omega}{d\gamma} \right) = 0.$$

Hence it is possible to eliminate differential coefficients with respect to α , β from the surface-condition if, and only if,

$$C c \propto h_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence the thickness of the sheet at any point must be inversely proportional to the distance between two neighbouring surfaces of the family $\gamma = \text{constant}$ at that point. In the case of an ellipsoidal sheet this leads to the well-known condition that the thickness must be proportional to the perpen-

* Compare Watson and Burbury, §426.

dicular on the tangent-plane, or that the sheet must be a shell bounded by similar and similarly situated ellipsoids.

Writing

$$R = \frac{h_3}{2\pi C\bar{c}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the surface-condition now becomes

$$\frac{d}{dt} \left(\frac{h_3}{h_1 h_2} \frac{d\Omega_1}{d\gamma} \right) = \frac{d}{dt} \left(\frac{h_3}{h_1 h_2} \frac{d\Omega_2}{d\gamma} \right) = \frac{R}{2} \frac{d}{d\gamma} \left\{ \frac{h_3}{h_1 h_2} \frac{d}{d\gamma} (\Omega_2 - \Omega_1) \right\} \quad (13)$$

Synthetic Determination of the Images in a Plane Sheet.

8. In order to illustrate the synthetic method, consider its application to the well-known case of a plane-current sheet. Let the induced currents be excited by a magnetic pole placed at the fixed point (x_0, y_0, z_0) on the positive side of the sheet whose strength is $f(t)$ an arbitrary function of the time. Then

$$\Omega_0 = \frac{f(t)}{\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\}^{\frac{1}{2}}}, \quad . \quad . \quad (14)$$

and the surface-condition satisfied by Ω' the potential on the negative side is

$$\frac{d^2 \Omega'}{dz^2} + R \frac{d^2 \Omega}{dz^2} = - \frac{d^2 \Omega_0}{dz^2}, \quad . \quad . \quad . \quad . \quad (15)$$

where

$$R = \frac{1}{2\pi C\bar{c}}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Integrating (15), the value obtained for Ω' is found to be the potential due to magnetic poles or images on the positive side of the sheet, and these are outside the region over which the solution is required to hold, as they should be. We have

$$\begin{aligned} \left(\frac{d}{dt} + R \frac{d}{dz} \right) \Omega' &= - \frac{f'(t)}{\sqrt{\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\}}}, \\ \therefore \Omega' &= - \int_{-\infty}^t e^{-R(t-\tau)} \frac{d}{dz} \frac{f'(\tau)}{\sqrt{\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\}}} d\tau \\ &= - \int_{-\infty}^t \frac{f'(\tau)}{\sqrt{\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0-Rt+R\tau)^2\}}} d\tau \\ &= - \frac{f(t)}{\sqrt{\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\}}} \\ &\quad + \int_{-\infty}^t R f(\tau) \frac{d}{dz} \frac{1}{\sqrt{\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0-Rt+R\tau)^2\}}} d\tau. \quad (17) \end{aligned}$$

The first term represents the effect of a pole equal and opposite to the given pole at the point (x_0, y_0, z_0) , while the second term shows that the effect of the action of the pole between the times τ and $\tau + \delta\tau$ is represented at time t by a magnet of strength $Rf(\tau)\delta\tau$ situated at the point $(x_0, y_0, z_0 + R(t - \tau))$. The potential at the positive side is found from the relation (5), or

$$\Omega''(x, y, z) = -\Omega'(x, y, -z).$$

When the pole is in motion the results thus arrived at enable us to plot out the moving images contributed during every time element $\delta\tau$ of the motion in the well-known way, or they may be obtained as follows :—

9. If, instead of a single magnetic pole, we are dealing with a magnetic distribution on the positive side of the sheet such that the volume-density of magnetism of the point (x_0, y_0, z_0) is $F(x_0, y_0, z_0, t)$, we must write $F(x_0, y_0, z_0, t) dx_0 dy_0 dz_0$ for $f(t)$ in the expressions for Ω_0 and Ω' . Hence we have

$$\Omega_0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{F(x_0, y_0, z_0, t) dx_0 dy_0 dz_0}{\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\}^{\frac{1}{2}}}, \quad \dots \quad (18)$$

$$\Omega' = -\Omega_0 - \int_{-\infty}^t d\tau \int_0^{\infty} dz_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{RF(x_0, y_0, z_0, \tau)}{z - z_0 - R(t - \tau)} \frac{dx_0 dy_0}{\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0 - R(t - \tau))^2\}^{\frac{1}{2}}}. \quad (19)$$

If now the inducing system consists of a single pole of strength $f(t)$ which moves about *in any manner* so that its coordinates at any time t are $\xi(t), \eta(t), \zeta(t)$, functions of t , we have

$$F(x_0, y_0, z_0, \tau) = 0,$$

except when $x_0 = \xi(\tau), y_0 = \eta(\tau), z_0 = \zeta(\tau)$, and then

$$F(x_0, y_0, z_0, \tau) dx_0 dy_0 dz_0 = f(\tau).$$

Therefore

$$\Omega_0 = \frac{f(t)}{\{(x-\xi(t))^2 + (y-\eta(t))^2 + (z-\zeta(t))^2\}^{\frac{1}{2}}}, \quad \dots \quad (20)$$

$$\Omega' = -\Omega_0 - \int_{-\infty}^t Rf(\tau) \frac{z - \zeta(\tau) - R(t - \tau)}{\{(x-\xi(\tau))^2 + (y-\eta(\tau))^2 + (z-\zeta(\tau) - R(t - \tau))^2\}^{\frac{1}{2}}} d\tau. \quad (21)$$

This is the expression for the magnetic potential due to the induced currents on the negative side of the sheet, *i. e.* on the opposite side to the moving pole; and it is to be observed that z is taken to be negative on this side.

The quantity Ω'' , which represents the potential due to the induced currents on the positive side (where z is positive), is given by

$$\Omega'' = \frac{f(t)}{\{(x - \xi(t))^2 + (y - \eta(t))^2 + (z + \zeta(t))^2\}^{\frac{3}{2}}} - \int_{-\infty}^t R f(\tau) \frac{z + \zeta(\tau) + R(t - \tau)}{\{(x - \xi(\tau))^2 + (y - \eta(\tau))^2 + (z + \zeta(\tau) + R(t - \tau))^2\}^{\frac{3}{2}}} d\tau \quad (22)$$

and the complete expression for the potential due to the induced currents and the moving pole itself is, by (4),

$$\Omega_2 = \Omega_0 + \Omega''.$$

10. The components of magnetic force at (x, y, z) due to the induced currents alone are respectively

$$-\frac{d\Omega''}{dx}, \quad -\frac{d\Omega''}{dy}, \quad -\frac{d\Omega''}{dz};$$

and the components of the force on the pole itself at any time t are found by multiplying these expressions by the strength of the pole, *i. e.* by $f(t)$, and putting $x = \xi(t)$, $y = \eta(t)$, $z = \zeta(t)$ after the differentiations have been performed.

In particular, if the pole moves along a straight line perpendicular to the plate, and we take this line as axis of z , the force on the moving pole itself

$$= f(t) \times \left\{ \frac{f(t)}{\{2\zeta(t)\}^2} - 2R \int_{-\infty}^t \frac{f(\tau) d\tau}{\{\zeta(t) + \zeta(\tau) + R(t - \tau)\}^3} \right\}. \quad (23)$$

The well-known case in which a pole is suddenly generated at the time $t=0$, and its strength remains constant and equal to m after that time, is deduced by making

$$\begin{aligned} f(t) &= 0 & \text{from } t = -\infty \text{ to } t = 0, \\ f(t) &= m & \text{from } t = 0 \text{ to } t = +\infty. \end{aligned}$$

We therefore have to write m for $f(t)$ in the above expressions and to reduce the inferior limit of integration with respect to τ from $-\infty$ to 0.

The advantage of starting with a fixed instead of a moving pole is more evident in the case of a cylindrical or spherical sheet, especially in the latter, since we are thus enabled to use zonal spherical harmonics only and the analysis is consequently much simplified.

DISCUSSION.

Prof. MINCHIN showed that equation (1) of the paper ($\Omega_2 - \Omega_1 = 4\pi\phi + \text{constant}$, where Ω_2 and Ω_1 are the magnetic potentials at the two sides of the sheet, and ϕ the current function) could be deduced by purely mathematical reasoning, instead of being based on the laws of electromagnetic induction. Moreover it was true of any function whatever, and did not depend on ϕ being the current function. Equation (2),

$$\left(\frac{d\Omega_2}{dz} = \frac{d\Omega_1}{dz}\right),$$

followed immediately from the fact that the magnetic force was continuous. The latter part of the paper might be simplified by integrating the linear partial differential equation (15)

$$\left(\frac{d^2\Omega_1}{dz\,dt} - R\frac{d^2\Omega_1}{dz^2} = -\frac{d_2\Omega_0}{dz\,dt}\right)$$

in the ordinary way, for the form was one for which the auxiliary equations are well known.

Dr. BURTON, in reply, said he thought Mr. Bryan's reason for developing the equations from the laws of electromagnetic induction was to give his work a physical rather than a mathematical basis.