[where $f_6(z)$ is a sextic function,] which is such that it can be transformed into itself by a substitution, of period two, of the form

$$z = \frac{az' + b}{cz' + d}, \quad s = A \frac{s'}{(cz' + d)^3},$$

and therefore the two integrals of the first species belonging to equation (C) can be expressed by means of elliptic integrals, if the equation

$$f_6(z) = 0$$

can be transformed into itself by such a linear substitution. The roots of the sextic are in this case in involution, and hence the condition is that the skew invariant should vanish. This is the result originally proved (in a different manner) by Prof. Gordan.]

Quartic Equations interpreted by the Parabola.

By G. HEPPEL, M.A.

[Read February 12th, 1891.]

The geometrical solution of quartic equations by means of the parabola forms the subject of a paper by Höpfe in the *Archiv der Mathematik*, Vol. LXIX., 1883, Leipzig; and also of a paper by Hofmann in the *Nouvelles Annales*, March, 1888. The object of the present paper is to attempt an extension of the method adopted in the former of these, so as to be in fact supplementary to Höpfe's investigation.

In Descartes' solution of a quartic, we derive from the quartic a certain cubic which must have at least one positive root, and if that root is found, all the roots, real or imaginary, of the quartic are at once easily deduced.

Höpfe shows how this preliminary solution may be effected by measuring the ordinates of the points of intersection of a variable circle with a fixed parabola. The following is a brief abstract of the method he employs.

Let the cubic be $y^3 + \alpha y^2 + \beta y + \gamma = 0$;

by removing the second term, and multiplying by y , we obtain a quartic, one root of which is zero, and the other three are the roots of the transformed cubic. Let this quartic be

$$y^4 + qy^3 + ry = 0;$$

and let its roots represent the ordinates of the points of intersection of

$$(x-h)^2 + (y-k)^2 - c^2 = 0,$$

and

$$y^2 = lx.$$

Then

$$y^4 + qy^3 + ry = 0$$

is identical with

$$y^4 + (l^2 - 2lh)y^3 - 2l^2ky + l^3(h^2 + k^2 - c^2) = 0;$$

whence

$$h = (l^2 - q)/2l, \quad k = -r/2l^2,$$

$$c^2 = h^2 + k^2,$$

the last of these equations not being at present made use of.

If a circle be drawn, having as centre $[h, k]$, and passing through the vertex of the parabola, it is necessary merely to measure the lengths of the ordinates to obtain at least one positive root of the original cubic before transformation.

Hence, according to the above plan, a quartic with the second term removed,

$$y^4 + qy^3 + ry + s = 0,$$

is first treated algebraically by Descartes' solution, and the derived cubic solved geometrically by Hoppe's method.

We might also consider the four roots of this quartic as being identical with the values of the ordinates of the intersections with a fixed parabola of two straight lines making equal angles with the axis of x . It is, however, evident that four points may be joined, two and two, in three ways, and therefore the determination of any one of the three pairs of lines involves the solution of a cubic equation. In the paper by Hofmann this cubic is formed, and is supposed to be solved algebraically. A pair of straight lines is then drawn, and the roots of the quartic obtained by measurement.

It is proposed in the present paper to attempt to show how a purely geometrical method may be employed which renders it unnecessary to solve a cubic as Hofmann does, or to use Descartes' algebraical treatment as Hoppe does, and that by this method,

Descartes' derived cubic may be arrived at by a geometrical road, its interpretation exhibited, and the four roots, whether real or imaginary, shown on one diagram, from which they may be measured. At the same time the whole process of Hoppe's solution of a cubic will be required as one step in the treatment of the quartic.

In order to do this, an extension of the primary machinery is necessary. Hitherto the parabola has been considered to have a latus rectum of arbitrary length, that is to say that the ratio between its length and the unit of measurement of the coefficients of the quartic has been arbitrary. Let this ratio be no longer considered arbitrary, but one to be determined in terms of the coefficients of the quartic. Then, making

$$y^4 + qy^3 + ry + s = 0$$

identical with

$$y^4 + (l^2 - 2lh)y^3 - 2l^2ky + l^3(h^3 + k^3 - c^3) = 0,$$

we get, by eliminating h, k ,

$$c^3 = \frac{(l^2 - q)^3}{4l^2} + \frac{r^3}{4l^4} - \frac{s}{l^2};$$

so that c^3 may have a variety of values according to the value assigned to l . Suppose

$$c = 0,$$

the circle becomes a point with an imaginary portion consisting of two straight lines inclined at 45° to the axis of x , and in a plane perpendicular to that of the paper. Also since the equation to determine l^3 is

$$l^6 - 2ql^4 + (q^3 - 4s)l^2 + r^3 = 0,$$

this gives at least one negative value of l^2 , thus determining an imaginary parabola, which may be interpreted as an equal parabola in a plane perpendicular to that of the paper.

Passing to this new plane, and considering it as a plane of reference, with coordinates x and y , it follows that two straight lines inclined at 45° to the axis of x may be found, provided a suitable value is given to l , such that the ordinates of the intersections of these lines with the parabola are the roots of

$$y^4 + qy^3 + ry + s = 0.$$

The intersection of

$$(x-h)^2 - (y-k)^2 = 0 \quad \text{with} \quad y^2 = lx,$$

gives $y^4 - (2hl + l^2)y^2 + 2l^2ky + l^2(h^2 - k^2) = 0$;

therefore $q = -2hl - l^2$, $r = 2l^2k$, $s = l^2(h^2 - k^2)$,

and the elimination of h and k gives

$$l^6 + 2ql^4 + (q^2 - 4s)l^2 - r^2 = 0,$$

which is Descartes' cubic, and gives one or three real values of l , such that two straight lines at 45° to the axis of x may be found fulfilling the conditions we require.

The ordinates of the four points of intersection of the pair of lines and the parabola are

$$y = \frac{l}{2} \pm \sqrt{-\frac{l^2}{4} - \frac{q}{2} - \frac{r}{2l}},$$

and

$$y = -\frac{l}{2} \pm \sqrt{-\frac{l^2}{4} - \frac{q}{2} + \frac{r}{2l}},$$

and four or two or none of these may be imaginary. The imaginary roots, if there are any, may be geometrically represented in the following manner. Let P be the point $[h, k]$. Let T' be the point of intersection of tangents at the extremities of the latus rectum, or $[-\frac{l}{4}, 0]$. Join PT' and produce it to Q , so that

$$TQ = PT.$$

Then, if through P , Q pairs of lines be drawn making angles of 45° with the axis of x , and if the line through P cuts the parabola in a point where

$$y = \pm \frac{l}{2} + z,$$

the parallel line through Q cuts it in a point where

$$y = \pm \frac{l}{2} + z\sqrt{-1};$$

so that if the intersection of the first is real, that of the second is imaginary, and *vice versa*.

Hence, considering space as divided into four compartments by the two tangents, if P is in the same compartment with the parabola, all the roots are real; if in the opposite one, all are imaginary; if in

either of the others, two are real and two are imaginary. Two will be equal if P is upon a tangent.

Before proceeding to state in order the steps to be taken in treating a quartic equation in the manner here proposed, it may be as well to notice a slight modification of Hoppe's method, which may be found to simplify the preliminary transformation. The cubic to be solved being

$$y^3 + \alpha y^2 + \beta y + \gamma = 0,$$

instead of removing the second term and multiplying by y , it may perhaps be simpler to multiply the original equation by $y - \alpha$, giving

$$y^4 + (\beta - \alpha^2) y^2 + (\gamma - \alpha\beta) y - \alpha\gamma = 0.$$

The centre of the circle is then

$$h = (l^2 + \alpha^2 - \beta) / 2l, \quad k = (\alpha\beta - \gamma) / 2l^2,$$

and the circle must pass through a point in the parabola whose ordinate is α .

In the practical application of this method, it would of course be very troublesome to draw to scale a fresh parabola for every equation requiring solution. It is quite possible to determine all the roots to a fair degree of approximation, without actually drawing a line, by means of a single parabola drawn on a surface divided into small squares, where distances between points may be measured by strips of paper, and ordinates may be read off by means of the small squares. In the solution of the derived cubic, we are at liberty to choose any convenient unit of measurement, and thus the value of l , which will be required to be found before the diagram of the roots of the quartic can be drawn, can be easily determined. It is not quite the same thing when the fixed parabola is used for the diagram with the two straight lines. It might happen, for example, that the length h or the length k might come out a thousand times that of the latus rectum. Here it would be necessary to transform the quartic equation to one having roots less than those of the original equation, in some convenient ratio. Practically, however, a parabola with a latus rectum of two inches, drawn on a surface divided into small squares with sides of a tenth of an inch in length, will be found generally suitable.

Supposing such an arrangement, the following is an example of Hoppe's solution of a cubic.

$$\text{Let the cubic be} \quad y^3 - 7y^2 + 80y - 311 = 0.$$

Here
$$h = (l^3 + \alpha^3 - \beta)/2l = \frac{1}{3}l - 31/2l,$$

$$k = (\alpha\beta - \gamma)/2l^2 = -249/2l^2.$$

Suppose the unit of measurement is four-tenths of an inch, so that

$$l = 5;$$

then $h = -\cdot 6$ units or $-2\cdot 4$ tenths, $k = -4\cdot 98$ units or $-19\cdot 92$ tenths. One point of intersection has an ordinate -7 units or -28 tenths. The measurement of the ordinate of the one other real point approximates to the true value, which is $18\cdot 08$ tenths, and the root of the cubic is $4\cdot 52$.

Example of the Diagram for a Quartic.

Let the quartic be $y^4 - 14y^3 - 23y + 68 = 0$.

The cubic for l^3 is $l^3 - 28l^2 - 76l - 529 = 0$.

This will be found by Hoppe's method to have a root

$$l^3 = 31;$$

therefore $l = 5\cdot 57$.

The unit of measurement is therefore $\frac{20}{5\cdot 57}$ tenths.

The point P is $x = -\frac{5\cdot 57}{2} + \frac{14}{11\cdot 14}$ units,

or $-10 + \frac{140}{31}$ or $-5\cdot 48$ tenths;

$$y = -\frac{23}{62} \text{ units, or } -\frac{230}{172\cdot 67}, \text{ or } -1\cdot 33 \text{ tenths.}$$

The point being neither in the parabola compartment nor in the one opposite, shows that two roots are real and two are imaginary.

The point T being $[-5, 0]$; the point Q is $[-4\cdot 52, 1\cdot 33]$. Applying a straight edge at an angle of 45° through P gives the distance of the point of intersection from the diameter through the upper extremity of the latus rectum as $4\cdot 12$ tenths, and applying it at an angle of 135° through Q gives the distance from the diameter through the lower extremity as $6\cdot 02$.

$$4\cdot 12 \times 5\cdot 57 \div 20 = 1\cdot 15, \quad 6\cdot 02 \times 5\cdot 57 \div 20 = 1\cdot 68,$$

so that the roots are

$$2.785 \pm 1.15 \quad \text{and} \quad -2.785 \pm 1.68\sqrt{-1}.$$

In conclusion, it may be interesting to prove that the criteria of the nature of the roots of the quartic equation, which depend upon the compartment in which P is situated, correspond exactly to those obtained by algebraical analysis.

From the compartment test we see that two roots will be equal if

$$\left(\frac{l}{4} + \frac{q}{2l}\right)^2 = \frac{r^2}{4l^4};$$

and all real or all imaginary if

$$\left(\frac{l}{4} + \frac{q}{2l}\right)^2 > \frac{r^2}{4l^4}.$$

The nature of the roots therefore depends upon whether

$$l^3 + 4ql^4 + 4q^2l^3 - 4r^3$$

is equal to, greater than, or less than 0; or, since

$$l^3 + 2ql^4 + (q^2 - 4s)l^2 - r^3 = 0,$$

upon whether $16s - 4ql^2 - 3l^4$ is zero, positive, or negative. Let

$$16s - 4ql^2 - 3l^4 = M.$$

Then the elimination of l^3 from

$$M = 0 \quad \text{and} \quad l^3 + 2ql^4 + (q^2 - 4s)l^2 - r^3 = 0,$$

gives $256s^3 - 128q^2s^2 + 16q^4s + 144qr^3s - 4q^3r^2 - 27r^4 = 0$,

or $\Delta = 0$,

where Δ is the discriminant of the quartic $y^4 + qy^3 + ry + s$.

Substitute, in Δ , $l^3 + 2ql^4 + (q^2 - 4s)l^2$ for r^3 ,

so that Δ is expressed as a function of q , s , and l . Then, since M and Δ vanish together, M is a factor of Δ ; and by division the other factor is found to be

$$\{4s - q^2 - 4ql^2 - 3l^4\}^2.$$

Hence M and Δ have the same sign. Therefore,

if Δ is positive the roots are all real or all imaginary ;

if Δ is negative the roots are two real and two imaginary.

Supposing the first alternative to be true, the point P will be in the parabola compartment, or the opposite one, according as $\frac{l}{2} + \frac{q}{2l}$ is less or greater than $\frac{l}{4}$, or as $l^2 + 2q$ is negative or positive. Hence, if q is positive, there is no doubt that all the roots must be imaginary. If q is negative further investigation is necessary.

Now $4s - q^2 = l^2(l^2 + 2q) - r^2/l^2$;
therefore, if $l^2 + 2q$ is negative, $4s - q^2$ is negative.

Again, $\frac{r^2}{4l^4}$ is less than $\left(\frac{l}{4} + \frac{q}{2l}\right)^2$,

since M is by hypothesis positive ; therefore

$$\frac{r^2}{l^2} \text{ is less than } \left(\frac{l^2}{2} + q\right)^2 ;$$

therefore $4s - q^2$ is greater than $l^2(l^2 + 2q) - \left(\frac{1}{2}l^2 + q\right)^2$.

$$16s - 4q^2 \text{ is greater than } (3l^2 - 2q)(l^2 + 2q) ;$$

so that if $l^2 + 2q$ is positive, $4s - q^2$ is positive.

Consequently, if Δ is positive, the roots are all imaginary, unless q is negative, and at the same time $4s - q^2$ is negative. These results exactly agree with those given in the treatise on the "Theory of Equations," by Burnside and Panton.

June 11th, 1891.

Prof. GREENHILL, F.R.S., President, in the Chair.

Prof. H. Lamb was admitted into the Society.

The following communications were made:—

Systems of Spherical Harmonics: Mr. E. W. Hobson.

On the Motion of a Liquid Ellipsoid under its own Attraction:

Dr. M. J. M. Hill.