LXI. On the rectification of the ellipse

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the rays to have crossed, and the point which the second Experiment determines to be anterior to their crossing; and as these extreme points approximate closely to each other, as in the first case the focus is within, and in the second without the axis, and as the retina is situate between them,—I consider my proposition established; viz. “*that the focus of the eye is upon the retina.*”

J. R. RUMBALL.

LXI. On the Rectification of the Ellipse. By the Rev. BRICE BRONWIN*.

THE well known differential expression of the length of the arc is $dx \sqrt{1 - e^2 x \sin^2 x} = d \frac{e}{2} \sqrt{\frac{2(2 - e^2)}{e^2} + 2 \cos 2x}$
 $= \frac{e}{2} dx \sqrt{p + 2 \cos z}$ by substitution.

Assume $\sqrt{p + 2 \cos z} = A + \overset{1}{A} \cos z + \overset{2}{A} \cos 2z + \&c.$ Difference both sides for z , divide by $d z$, multiply by $p + 2 \cos z$; the result will be the same as if both sides had been multiplied by $\sin z$, being the expression of $\sin z \sqrt{p + 2 \cos z}$. By equating therefore the coefficients of the like terms in both the developed expressions of this quantity, we get $\frac{5}{2} \overset{2}{A} + p \overset{1}{A} - A = 0$, $\frac{7}{2} \overset{3}{A} + 2p \overset{2}{A} + \frac{1}{2} \overset{1}{A} = 0$, $\frac{9}{2} \overset{4}{A} + 3p \overset{3}{A} + \frac{3}{2} \overset{2}{A} = 0$. The quantities $\overset{2}{A}$, $\overset{3}{A}$, &c. are therefore known when A and $\overset{1}{A}$ are known.

Resuming therefore $\sqrt{p + 2 \cos z} = A + \overset{1}{A} \cos z + \overset{2}{A} \cos 2z + \&c.$; difference both sides for p , and then multiply by $p + 2 \cos z$; the result will be the expression of $\frac{1}{2} \sqrt{p + 2 \cos z}$. Equating therefore the coefficients of like terms in both expressions of that quantity, we obtain $\frac{1}{dp} \frac{dA}{dp} + p \frac{dA}{dp} - \frac{1}{2} A = 0$, $\frac{2}{dp} \frac{dA}{dp} + p \frac{dA}{dp} + 2 \frac{dA}{dp} - \frac{1}{2} \overset{1}{A} = 0$, &c. Eliminating $\frac{dA}{dp}$ from the second of these by means of the first of the former equations, we have $p \frac{dA}{dp} + 4 \frac{dA}{dp} - \frac{3}{2} \overset{1}{A} = 0$. Eliminating $\frac{dA}{dp}$

* Communicated by the Author.

from

from this last by the first of the second set of equations we obtain $\overline{p^2-4} \cdot \frac{dA}{dp} - \frac{1}{2} p A + \frac{3}{2} \overset{1}{A} = 0$. Differencing this and eliminating $\frac{dA}{dp}$, we get finally $\overline{p^2-4} \cdot \frac{d^2A}{dp^2} + \frac{1}{4} A = 0$. Thus we can find both A and $\overset{1}{A}$.

Mr. Knight in the 3rd volume of Leybourn's Repository, has shown us how to find $\overset{1}{A}$ from A by means of an integral, and seems to intimate that it had not been done before. Perhaps therefore the conclusions here arrived at are found for the first time; and if we could integrate the above differential equation of the 2nd order in finite terms, we should have a finite expression for the quadrantal arc of an ellipse. I fear, however, this cannot be done. Still the equation is not useless; we may derive from it a multitude of different series expressive of the elliptic arc, and such as perhaps otherwise could not have been deduced.

Putting y for A and $2x$ for p to simplify, the preceding equation becomes $\overline{x^2-1} \cdot \frac{d^2y}{dx^2} + \frac{1}{4} y = 0$. Assume $y = ax^{\frac{1}{3}} + \overset{1}{a} x^{-\frac{3}{2}} + \overset{2}{a} x^{-\frac{7}{2}} + \&c.$, and substituting for y and $\frac{d^2y}{dx^2}$ in the preceding; we get $\overset{1}{a} = -\frac{1}{3 \cdot 5 + 1}$ $a = -\frac{1}{4}$, $\overset{2}{a}, \overset{3}{a} = -\frac{3 \cdot 5}{(3 \cdot 5 + 1)(7 \cdot 9 + 1)}$ $a = -\frac{3 \cdot 5}{4^2 \cdot 8^2}$ $\overset{3}{a}, \overset{4}{a} = -\frac{3 \cdot 5 \cdot 7 \cdot 9}{4^2 \cdot 8^2 \cdot 12^2} a \&c.$, where by making e nothing, or x infinite, a is found $= \sqrt{2}$. Hence if we put $t = \frac{1}{x}$ we find the length of the elliptic quadrant $= \frac{e}{2} \cdot \frac{\pi}{2} \cdot y = \frac{\pi}{2} \sqrt{1 - \frac{1}{2} e^2}$ into $1 - \frac{1}{4^2} t^2 - \frac{3 \cdot 5}{4^2 \cdot 8^2} t^4 - \frac{3 \cdot 5 \cdot 7 \cdot 9}{4^2 \cdot 8^2 \cdot 12^2} t^6 \&c.$; which is the same as it would have been found if we had expanded $\sqrt{1 - e^2 \sin^2 x}$ by the binomial theorem, and developed the powers of $\sin x$ in cosines of the multiple arcs; and is the same, under a form a little different, with the second series, page 196, vol. i. of Leybourn's Math. Repository, and there ascribed to Legendre and Euler.

Let us now consider y a function of $v = x - \sqrt{x^2 - 1} = \frac{1-c}{1+c}$, where c = semi-conjugate. Then $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$, $\frac{d^2y}{dx^2}$

$\frac{d^2 y}{d x^2} = \frac{d^2 y}{d v^2} \cdot \frac{d v^2}{d x^2} + \frac{d y}{d v} \cdot \frac{d^2 v}{d x^2} = \frac{d^2 y}{d v^2} \cdot \frac{v^2}{x^2-1} + \frac{d y}{d v} \cdot \frac{2 v}{v^2-1} \cdot \frac{1}{x^2-1}$; and the equation $\overline{x^2-1} \cdot \frac{d^2 y}{d x^2} + \frac{1}{4} y = 0$ is changed into $v^2 \cdot \overline{1-v^2} \cdot \frac{d^2 y}{d v^2} + 2 v \frac{d y}{d v} + \frac{1}{4} (1-v^2) y = 0$.

Assume $y = a v^{-\frac{1}{2}} + \frac{1}{2} a v^{\frac{3}{2}} + \frac{2}{3} a v^{\frac{7}{2}} + \&c.$, and substitute for $\frac{d^2 y}{d v^2}$, $\frac{d y}{d v}$, and y in the preceding, and we shall find $\frac{1}{2} a = \frac{1}{2^2} a$, $\frac{2}{3} a = \frac{1}{2^2 \cdot 4^2} a$, $\frac{3}{4} a = \frac{3^2}{2^2 \cdot 4^2 \cdot 6^2} a$, $\frac{4}{5} a = \frac{3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} a$, where a is found = 1. This gives the elliptic quadrant = $\frac{\pi}{2} \cdot \frac{\pi}{2} \cdot y = \frac{\pi}{2(1+v)}$ into $1 + \frac{1}{2} v^2 + \frac{1}{2^2 \cdot 4^2} v^4 + \frac{3^2}{2^2 \cdot 4^2 \cdot 6^2} v^6 + \&c.$; which is the first series in the place above referred to in Leybourn, the inventor of which is Mr. Ivory.

Next let us consider y a function of $u = \frac{x-1}{x+1} = c^2 = \text{square}$ of semi-conjugate. Then the equation will become $u(1-u)^2 \frac{d^2 y}{d u^2} - 2u(1-u) \frac{d y}{d u} + \frac{1}{4} y = 0$. Making $y = a u^m + \frac{1}{2} a u^{m+1} + \&c.$, we find $m = 1$, or $= 0$. But in the second case a also $= 0$, and the two series become the same; viz. $y = a u + \frac{1}{2} a u^2 + \frac{2}{3} a u^3 + \&c.$; whence we obtain $\frac{1}{2} a = \left(2 - \frac{1}{4}\right) \frac{a}{2}$, $\frac{2}{3} a = \left(8 - \frac{1}{4}\right) \frac{a}{6} - \frac{2a}{6}$, $\frac{3}{4} a = \left(18 - \frac{1}{4}\right) \frac{a}{12} - \frac{6a}{12}$, $\frac{4}{5} a = \left(32 - \frac{1}{4}\right) \frac{a}{20} - \frac{12a}{20}$, &c. where the law of continuance is evident, and a remains arbitrary. This series is to be multiplied by $\frac{e}{2} = \frac{1}{2} \sqrt{1-u}$; hence it becomes nothing both when $u = 0$, or $u = 1$, that is when $e = 1$ and $e = 0$. It must therefore only be regarded as a particular solution; it will, however, be useful in seeking the complete one.

Make $y = p + q \log. Cu$. Then substituting this value of y and its differentials in our equation, and making the part multiplied by $\log. Cu$ separately $= 0$, we shall have the same equation in q as we had in y ; consequently the value of y just found may be taken for that of q ; we shall have moreover

$$u(1-u)^2 \frac{d^2 p}{d u^2} - 2u(1-u) \frac{d p}{d u} + \frac{1}{4} p + 2(1-u)^2 \frac{d q}{d u} - (1-u)$$

$(1-u)^2 \frac{q}{u} - 2(1-u)q = 0$. The assumption to be made here is $p = b + b^1 u + b^2 u^2 + b^3 u^3 + \&c.$; where b will be determined in terms of a . Retaining therefore a in q as an arbitrary, we shall have two arbitraries a and C , and consequently a complete solution. And no doubt should be led to the third series in the place in Leybourn already referred to, and which is Euler's.

If we were to suppose y a function of $e = \sqrt{\frac{2}{x+1}}$, we should find $e^2 \frac{d^2 y}{d e^2} + 3e \frac{d y}{d e} + \frac{y}{1-e^2} = 0$. No doubt this would lead to a series well known.

If we were to suppose y a function of $w = \frac{1-e}{1+e}$, we should find $(w - w^2 - w^3 + w^4) \frac{d^2 y}{d w^2} - (4w - 2w^2 + 2w^3) \frac{d y}{d w} + (1+w)y = 0$. This is much like the equation between y and u , and would probably be troublesome to manage. Therefore as my paper is nearly filled, I shall not proceed with it further.

BRICE BRONWIN.

LXII. Notices respecting New Books.

A Manual of Chemistry chiefly for the Use of Pupils of Mechanics' Institutions. By ANDREW FYFE, M.D. F.R.S.E. Edinb. 12mo. pp. 340.

Experiments illustrative of Chemical Science; systematically arranged, &c. By the Author of "*The Elements of Chemical Science.*" Glasgow, 12mo. pp. 157.

IT is so little to our taste and inclination to find fault with any attempt, however humble, to extend the knowledge of chemical science, that we have long delayed noticing the first of the above-mentioned works. Upon considering, however, that it is not less a duty to expose the faults of authors than to display their merits, we have at length resolved to subject Dr. Fyfe's work to a short, but we trust not to too severe an examination.

To begin at the beginning. The author informs us that the arrangement which he has adopted "is that long ago recommended by Dr. Black." On referring, however, to the Lectures of that illustrious philosopher, it strikes us that it is impossible to discover the similarity which they are stated to possess. Dr. Black's arrangement, as far as we think it needful to quote it, is the following:—Of chemistry in general—Of heat in general—General effects of mixture—Chemical apparatus—Chemical history of bodies—Salts—Earths—Fixed air—Azote, &c. Now Dr. Fyfe's order is, Attraction—Specific gravity—Heat—Light—Atmosphere—Oxygen—Azote—Hydrogen,