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“Errors in Height-Scales of Barometers.”

By ROLLO APPLEYARD, M. Inst. C.E.

IN view of the increasing use of aneroid and other barometers for survey work and for aeronautical purposes, it is important to consider the limits of accuracy with which measurements of height can be made with barometers. Errors due to mechanical hysteresis of the metal in aneroid barometers and those due to thermal inequalities have already received the attention of investigators.

The merits of the aneroid barometer as a field instrument for engineers have been discussed by Major B. R. Branfill.¹ His notes are the result of observations made during several years whilst he was conducting the Madras Party of the Great Trigonometrical Survey of India, and they are essentially of a practical character. In particular he shows the importance of diurnal variations, and he makes some useful remarks in regard to methods of recording observations, and the degree of accuracy to be attained in height measurements. Instrument-errors are carefully examined by Dr. C. Chree, F.R.S.,² who deals with the lag or elastic effect, observed respectively with ascending and descending readings, and also with secular changes and the influence of temperature on the indications of aneroid barometers. In neither of these Papers, however, are the errors in height-scales discussed. It may be remarked that estimates of height deduced from barometric readings may be vitiated by local storms. In survey work, errors arising from storms can be eliminated by taking additional pairs of barometric readings; but in aeronautics such a check cannot,

¹ “Notes on the Aneroid Barometer for the use of travellers and others in determining the relative heights of places in Southern India, with tables and a list of trigonometrical heights in and about the Province of Mysore.” Bangalore, Mysore Government Press, 1871.

² “Experiments on Aneroid Barometers at Kew Observatory, and their Discussion.” Phil. Trans. Royal Society, Series A, 1898, vol. 191, pp. 441-499.

in general, be applied. The "Kew" test, applied to aneroids, is only for the instruments considered as barometers; it does not include a check upon their height-scales.

In the present Paper, therefore, it is proposed to determine the magnitude of the error likely to be introduced if the height is determined from the ordinary scale of heights attached to a barometer, instead of being calculated in each instance from the barometric readings observed at the lower and the higher stations respectively.

Height-Scale Inherent Errors.—It is shown in the Appendix that if p_0 is the barometric pressure at datum level, and p_1 the pressure at F feet above datum level, and that if it is assumed that there is an average temperature of 0° C. throughout that vertical distance, then, from equation (1)

$$F = 60,355 \times \log_{10} \frac{p_0}{p_1}.$$

Let the pressures be expressed in inches of mercury, and let p_0 be given the successive values 28, 29, 30, 31. A Table (Table I of the Appendix) can then be prepared to represent heights F corresponding with these values of p_0 , for values of p_1 throughout any required range of heights. If some definite value of p_0 is now selected, a corresponding "height-scale" can be constructed which shall read directly and accurately in feet, whenever the barometer at datum level happens to be at that selected value p_1^0 , say 30 inches. When the barometric pressure at datum-level is other than p_1^0 , the zero of the height-scale can be brought into coincidence with the new value of p_0 , on the barometric scale of pressures, and the height-scale will indicate heights, subject to a correction, for which an expression has to be found (equation (2) in Appendix). The magnitude of the correction is represented graphically in *Fig. 1*, which is drawn from the values of heights and pressures given in Table I. In that curve the firm lines correspond with true heights, such as would be read upon a true scale upon a perfect barometer, for the four cases where p_0 is selected to be 28, 29, 30, 31 respectively. When determining the divisions for his height-scale, the instrument-maker has, in general, to select a single definite value for p_0 , and for the present calculation it may now be assumed that he decided upon $p_0 = 30$. The effect of using such a scale for other values of p_0 is represented in *Fig. 1*. It is equivalent to sliding the firm line corresponding with $p = 30$ into successive positions, so that each point along it moves horizontally, and so that the lower extremity of it stops successively at 28, 29,

30, and 31, as indicated by the dotted lines. The height-scale errors are thus self-evident from the Figure, and they are considerable. The observer who uses the height-scale, instead of calculating his heights from the two barometric readings, in effect reads from the dotted lines instead of from the firm lines of the curves. For

example, when $p_0 = 31$, a height of 9,000 feet is seen to correspond with 21.991 inches on the firm line. But the observer using the height-scale graduated on the 30 basis, and shifted along to the $p_0 = 31$ mark, would read this pressure not on the firm line but on the dotted line, and he would declare the height to be 9,340 feet instead of 9,000 feet. Thermal and hysteresis errors are of an

elusive character. The height-scale error, however, is definite, and by the means here described it can always be calculated and then eliminated or avoided.

Height-Scale Instrument Errors.—The general nature and the magnitude of the inherent errors of determinations of altitude, when a single height-scale graduated to a fixed selected basis is used in the manner above discussed, are indicated in the curves in *Fig. 1*, assuming the height-scale to be engraved accurately in accordance with the relationships of pressures and heights derived from calculations upon such a selected basis. In actual barometers provided with height-scales, however, the engraving may not precisely represent those relationships. Further errors are consequently introduced, which will here be referred to as “height-scale instrument errors.”

Equations (1) and (2), in Appendix, and Table I, enable a rapid check to be applied to determine instrument errors of height-scales of existing barometers. For this purpose, the zero of the height-scale should be set to a definite mark, say 30, of the barometric scale, and keeping it at this position, the barometric readings coincident with, say, 1,000, 2,000, 3,000 . . . feet on the height-scale should be written down, as in Table II. The true barometric readings corresponding to those heights for the selected zero setting can be added from Table I. The process can if necessary be repeated, for other zero settings such as 28, 29, or 31. In Table II, which is derived from readings of an actual instrument, this additional check is shown for a setting of 31. From inspection of the Table, it is manifest that if the height-scale of this particular instrument was originally based upon 30 by the instrument-maker, the divisions were imperfectly spaced. For this instrument the graduations corresponding to the 31 mark are rather better than the 30-mark graduations near 1,000 feet, but at 9,000 feet they are less accurate than those of the 30 mark. For example, for the 30 mark the difference at 1,000 feet is $28.88 - 28.67 = 0.21$. Hence the height-scale instrument error, from equation (2), is there approximately $\delta F = -26,215 \frac{0.21}{30} = -183$ feet. For the 31 mark the difference at 1,000 feet is $29.84 - 29.88 = -0.04$, and, approximately, $\delta F = -26,215 \frac{-0.04}{31} = 34$ feet. From the reversal of the sign of the height-scale instrument error, it follows that at or near 1,000 feet the basis of the scale for this particular instrument to give accurate results at that part of the height-scale must be between 30 and 31. By a similar process the height-scale instrument errors at 9,000 feet

are seen to be, for a 30-mark setting, 177 feet, and for a 31-mark setting 241 feet, approximately. The height-scale instrument errors here determined have to be added to, or subtracted from, the inherent errors arising from the use of a fixed basis in making the height-scales.

It would appear that instrument makers could obtain better results if more care were exercised in engraving the scales. Until correctness in this work is secured, expenditure upon verniers and reading-lenses is wasteful and misleading. It would be a step in the right direction if there were denoted upon each instrument the basis upon which its height-scale is calculated. For accurate work engineers will see the wisdom of abandoning height-scales and using either equation (1), or a sheaf of curves similar to the firm lines in *Fig. 1*.

It is of course to be understood that equation (2) (Appendix) is strictly applicable only when δp is extremely small; nevertheless equation (2) is useful in obtaining a rough idea of the magnitude of instrument errors, as above. If great refinement is required in calculating instrument errors, equation (1) should be applied to each of the readings corresponding to selected pairs of pressures.

The Paper is accompanied by a diagram, from which *Fig. 1* has been prepared; also by the following Appendix.

APPENDIX.

To find an expression for the height h , determined by two barometric readings p_0 and p_1 , at the lower and higher levels respectively, at mean temperature $t^\circ \text{C.}$, and another expression to represent the error in the height measurement, assuming a small error in the "height-scale": consider a small vertical cylinder of the atmosphere, of 1 square centimetre in cross-section, and of length δh at the higher level. The small difference of pressure between the two ends of the cylinder is equal to the weight of the cylinder. Hence, if Δ_1 is the density at height h centimetres,

$$\frac{dp}{dh} \delta h = - \delta h \cdot \Delta_1 \cdot g$$

where g is the gravitation constant in centimetres per second per second. At standard pressure p , let the density be Δ , so that

$$\Delta_1 = \Delta \frac{p_1}{p};$$

therefore

$$\frac{dp}{dh} = - \Delta \frac{p_1}{p} g.$$

Integrating between $h = 0$ and $h = h$,

$$\int_{p_0}^{p_1} \frac{dp_1}{p_1} = - \Delta \frac{g}{p} \int_0^h dh,$$

the solution of which is

$$\log_e \frac{p_0}{p_1} = \Delta \frac{gh}{p};$$

or

$$h = 2 \cdot 3026 \frac{p}{\Delta g} \cdot \log_{10} \frac{p_0}{p_1}.$$

The density of dry¹ air at $t^\circ \text{C.}$ and pressure p dynes per square centimetre is

$$\Delta = \frac{p \times 1 \cdot 2759 \times 10^{-9}}{(1 + 0 \cdot 00366 \ t)}$$

and

$$g = 981;$$

therefore

$$\begin{aligned} h &= \frac{2 \cdot 3026 \times (1 + 0 \cdot 00366 \ t)}{1 \cdot 2759 \times 10^{-9} \times 981} \log_{10} \frac{p_0}{p_1} \\ &= 1,839,640 (1 + 0 \cdot 00366 \ t) \log_{10} \frac{p_0}{p_1}; \end{aligned}$$

or, if F is the height of the ascent in feet,

$$F = 60,355 (1 + 0 \cdot 00366 \ t) \log_{10} \frac{p_0}{p_1} \quad . \quad . \quad (1)$$

¹ Everett's "Units and Physical Constants," fifth edition, pp. 43-47.

This enables a height-scale to be constructed for any average temperature t between the two extreme levels. To find the error δF corresponding with a small error in the estimation of p_1 , put $t = 0$, and differentiate. As, in general, the height-scale will be accurately set, it can be assumed—except in the case of storms—that there is no error in p_0 . When differentiating equation (1), therefore, p_0 may be regarded as constant. Then—

$$\delta F = -\frac{60,355}{2 \cdot 3026} \cdot \frac{\delta p_1}{p_1} = -26,215 \frac{\delta p_1}{p_1} \quad (2)$$

The negative sign implies that when p_1 is estimated too low the altitude measurement deduced from the height-scale is too great, and vice versa.

Example.—Let the height-scale be supposed to be accurately graduated upon a 30-inch basis. It then reads accurately when the barometer at the lower level is 30 inches, and when the zero of the height-scale is correctly set to the 30 mark on the barometer scale. In equation (1) put $F = 9,000$, and $p_0 = 30$. This gives $p_1 = 21 \cdot 281$. Again, in equation (1), put $F = 9,000$ and $p_0 = 31$. This gives $p_1 = 21 \cdot 991$. The distances traversed by the pointer of the instrument in the two cases are equivalent to $30 - 21 \cdot 281 = 8 \cdot 719$ inches, and $31 - 21 \cdot 991 = 9 \cdot 009$ inches, respectively. By the use of a height-scale these differences are assumed to be equal, and there is a consequent error $\delta p = 9 \cdot 009 - 8 \cdot 719 = 0 \cdot 290$ inch. Hence, approximately, the error in the height measurement with the use of the height-scale is, from equation (2)—

$$\delta F = -26,215 \frac{0 \cdot 290}{21 \cdot 991} = -346 \text{ feet.}$$

In this case p was indicated too low; hence the true height is 8,654 feet.

Temperature Error.—If the temperature term is added to equation (1) it may be expressed as—

$$F = (1 + 0 \cdot 00366 t) \times 60,355 \log_{10} \frac{p_1}{p_2}$$

$$\text{Hence—} \quad \frac{dF}{dt} = 220 \cdot 9 \log_{10} \frac{p_1}{p_2} \quad (3)$$

In Table I the values of $\log_{10} \frac{p_1}{p_2}$ are given for successive values of F , and the error in F due to an error of 1° in t can, therefore, be written down at once from equation (3). For example, at 9,000 feet $\log_{10} \frac{p_1}{p_2} = 0 \cdot 149121$, and—

$$\frac{dF}{dt} = 220 \cdot 9 \times 0 \cdot 1491 = 32 \cdot 9 \text{ feet.}$$

It follows that the instrument error due to height-scale at 9,000 feet, which was found to be about 346 feet, is approximately equivalent to an error of 10° C. in estimating the mean temperature throughout the height.

In the following Table I the values are given from which the curves in *Fig. 1* were plotted. This Table also gives values of $\log_{10} \frac{p_1}{p_2}$ for estimating the temperature correction.

TABLE I.

True Height F.	$\frac{F}{60,355} = \log_{10} \frac{p_1}{p_2}$ from equation (1).	Ratio of Barometric Readings.		Barometric Readings at Higher Level, when Reading at Lower Level is 28, 29, 30 or 31 respectively.			
		$\frac{p_1}{p_2}$	$\frac{p_2}{p_1}$	28 $\frac{p_2}{p_1}$	29 $\frac{p_2}{p_1}$	30 $\frac{p_2}{p_1}$	31 $\frac{p_2}{p_1}$
Feet. 0	0	1.0000	1.00000	Inches. 28.00	Inches. 29.00	Inches. 30.00	Inches. 31.00
10	0.00016569	1.0004	0.99962	27.99	28.99	29.99	30.99
100	0.0016569	1.0038	0.99619	27.89	28.89	29.89	30.88
1,000	0.016569	1.0389	0.96257	26.95	27.91	28.88	29.84
2,000	0.033139	1.0793	0.92653	25.94	26.87	27.80	28.72
3,000	0.049707	1.1213	0.89185	24.97	25.86	26.76	27.65
4,000	0.066278	1.1649	0.85846	24.04	24.89	25.76	26.61
5,000	0.082845	1.2102	0.82663	23.15	23.97	24.80	25.63
6,000	0.099414	1.2572	0.79540	22.27	23.07	23.86	24.66
7,000	0.115983	1.3061	0.76563	21.44	22.20	22.97	23.73
8,000	0.132556	1.3569	0.73696	20.63	21.37	22.11	22.85
9,000	0.149121	1.4097	0.70938	19.86	20.57	21.281	21.99
10,000	0.165690	1.4645	0.68283	19.12	19.80	20.49	21.17
11,000	0.182259	1.5215	0.65727	18.40	19.06	19.82	20.38
12,000	0.198828	1.5806	0.63266	17.71	18.34	18.98	19.61
13,000	0.215397	1.6421	0.60898	17.05	17.66	18.27	18.88
14,000	0.231966	1.7060	0.58618	16.41	17.00	17.59	18.17
15,000	0.248535	1.7723	0.56424	15.80	16.36	16.93	17.49

TABLE II.—HEIGHT READINGS FROM AN ACTUAL BAROMETER.

Height-Scale Zero set at 30-Mark on Barometer-Scale.			Height-Scale Zero set at 31-Mark on Barometer-Scale.		
Instrument-Height Reading.	Instrument-Pressure Reading.	True Pressure from Equation 1.	Instrument-Height Reading.	Instrument-Pressure Reading.	True Pressure from Equation 1.
Feet.	Inches.	Inches.	Feet.	Inches.	Inches.
0	30·00	30·00	0	31·00	31·00
1,000	28·67	28·88	1,000	29·88	29·84
2,000	27·62	27·80	2,000	28·80	28·72
3,000	26·55	26·76	3,000	27·76	27·65
4,000	25·60	25·76	4,000	26·76	26·61
5,000	24·63	24·80	5,000	25·80	25·63
6,000	23·67	23·86	6,000	24·875	24·66
7,000	22·77	22·97	7,000	23·98	23·73
8,000	21·95	22·11	8,000	23·13	22·845
9,000	21·08	21·282	9,000	22·275	21·99
			10,000	21·475	21·17