ADDITIONAL NOTE ON TWO PROBLEMS IN THE ANALYTIC THEORY OF NUMBERS

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1. This note is a short supplement to my paper "The average order of the arithmetical functions $P(n)$ and $\Delta(n)$", published in Vol. 15 of the *Proceedings*.* I did not point out explicitly the relation of the results which I proved there to the well known, but very difficult, theorems expressed by the equations

$$P(n) = O(n^{\frac{1}{3}+\epsilon}), \qquad \Delta(n) = O(n^{\frac{1}{3}+\epsilon}). \tag{1}$$

These theorems may be deduced very simply, and by two different methods, from the theorems proved in my paper; and each deduction has points of interest. I state them in terms of $P(n)$ and the associated functions.

2. In § 2.6, I proved that†

$$\int_1^x \{U_\alpha(\tau)\}^2\, d\tau = O(x). \tag{2.61}$$

But it is easy to see that my argument really proves more than this, viz.,

$$\int_{x'}^x \{U_\alpha(\tau)\}^2\, d\tau = O(x-x'), \tag{2.61'}$$

provided only that $x-x' > K\sqrt{x}$. In fact the argument used at the top of p. 208 gives

$$\int_{x'}^x \{U_\alpha(w^2\}^2\, dw = O(x-x'),$$

provided only that $x-x' > K$. If now we write $w^2 = \tau$, $x^2 = \xi$,

* Pp. 192–213.

† I retain the notation of my paper, and the numbering of formulæ quoted from it.

$x'^2 = \xi'$ (as on p. 207), and

$$W(\xi, \xi') = \int_{\xi'}^{\xi} \frac{\{U_a(\tau)\}^2}{\sqrt{\tau}}\, d\tau = O(\sqrt{\xi}-\sqrt{\xi'}),$$

we have $\displaystyle\int_{\xi'}^{\xi} \{U_a(\tau)\}^2\, d\tau = \int_{\xi'}^{\xi} \sqrt{\tau}\, \frac{d}{d\tau} W(\tau, \xi')\, d\tau$

$$< \sqrt{\xi}\, W(\xi, \xi') < (\sqrt{\xi}+\sqrt{\xi'})\, W(\xi, \xi') = O(\xi-\xi'):$$

a result equivalent to (2 . 61'), since $\sqrt{\xi}-\sqrt{\xi'} > K$ corresponds to

$$\xi-\xi' > K\sqrt{\xi}.$$

From (2 . 61') follows, as at the beginning of § 2 . 7,

$$\text{(2 . 71')} \qquad \frac{1}{x-x'} \int_{x'}^{x} \{S_a(\tau)\}^2\, d\tau = O(x^{\frac{1}{2}+a}).$$

3. Suppose now that $\quad |S_a(x)| > x^{\beta},$

where $0 < \beta < \frac{1}{2}+a$, for a sequence of values of x, say $x_1, x_2, x_3, \ldots$, surpassing all limit. When x increases by unity, the change in $S_a(x)$ is of order $O(x^{a+\epsilon})$.* Hence

$$|S_a(x)| > Kx^{\beta}$$

throughout an interval extending on either side of x_i to a distance greater than $Kx_i^{\beta-a-\epsilon}$ We have therefore

$$\text{(2 . 71'')} \qquad \int_{x'}^{x_i} \{S_a(\tau)\}^2\, d\tau > Kx_i^{3\beta-a-\epsilon},$$

if $x_i-x_i' > Kx_i^{\beta-a-\epsilon}$. This last inequality will certainly be satisfied if $x_i-x_i' > K\sqrt{x_i}$, since $\beta-a < \frac{1}{2}$. Thus we may take $x_i-x_i' = \sqrt{x_i}$ and use (2 . 71'') in conjunction with (2 . 71'); and plainly this gives

$$x^{3\beta-a-\epsilon} = O(x^{1+a}),$$

or

$$\beta \leqslant \tfrac{1}{3}+\tfrac{2}{3}a+\tfrac{1}{3}\epsilon.$$

We have therefore $\quad S_a(x) = O(x^{\frac{1}{3}+\frac{2}{3}a+\epsilon}),$

for every pair of positive values of a and ϵ; and therefore, by the theorem quoted in the footnote to p. 200,

$$S(x) = O(x^{\frac{1}{3}+\frac{2}{3}a+\epsilon}).$$

* This follows at once from the equation which immediately precedes (2 . 82), if we observe that $S_a(x)$ is a continuous function of x.

Since a and ϵ are arbitrarily small, this is equivalent to the first of the equations (1).

It would seem that it should be possible to prove that

$$\frac{1}{x-x'}\int_{x'}^{x}\{\mathrm{P}(\tau)\}^2\,d\tau = O(x^{\frac{1}{2}+\epsilon}), \tag{2}$$

if only $x-x' > K\sqrt{x}$, and so to prove that the average order of $\mathrm{P}(n)$ is $O(n^{\frac{1}{4}+\epsilon})$ in a sense more precise than that in which it is proved to be so in my former paper. I have, however, not been able to complete the proof.

4. We can also deduce the first of the equations (1) from the formula

$$S_a(x) = O(x^{\frac{1}{4}+\frac{1}{2}a}) \quad (a > \tfrac{1}{2}), \tag{2.42'}$$

an equation equivalent to (2.42) of my former paper. This deduction depends on a general arithmetic theorem proved by K. Ananda Rau.

If $a_n = O(n^{\epsilon})$ *and*

$$S_\kappa(x) = \sum_{n\leqslant x}(x-n)^\kappa a_n = O(x^{\gamma+\epsilon}),$$

where $0 < \kappa < 1$ *and* $\gamma > 0$, *for every positive value of* ϵ, *then*

$$S(x) = \sum_{n\leqslant x} a_n = O\left(x^{\frac{\gamma}{1+\kappa}+\epsilon}\right)$$

for every positive value of ϵ.

This theorem, which I state in the form in which I wish to apply it, is a particular case of more general theorems proved by Ananda Rau, the aim of which is to extend, to arbitrary (non-integral) orders of integration and summation, a number of theorems proved by Mr. Littlewood and myself in a paper published in Vol. 11 of the *Proceedings*.* Dr. Marcel Riesz first succeeded in extending some of our principal theorems in this manner, and Ananda Rau has since investigated the generalisations more systematically. I shall not insert a proof here, as I hope that Ananda Rau will soon be able to publish proofs of the more general theorems.

In the present case we have

$$a_n = r_n, \quad \kappa = a > \tfrac{1}{2}, \quad \gamma = \tfrac{1}{4}+\tfrac{1}{2}a;$$

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* "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.*, Ser. 2, Vol. 11, 1912, pp. 411–478.

and the conclusion is that

$$S(x) = O\left\{x^{\frac{1+2a}{4(1+a)}+\epsilon}\right\}.$$

Since a may be as near to $\frac{1}{2}$ as we please, and $(1+2a)/4(1+a) = \frac{1}{3}$ when $a = \frac{1}{2}$, we have

$$S(x) = O(x^{\frac{1}{3}+\epsilon}),$$

which is equivalent to the first of the equations (1).

The *best* known " O " results concerning $P(n)$ and $\Delta(n)$ are

$$P(n) = O(n^{\frac{1}{3}}), \quad \Delta(n) = O(n^{\frac{1}{3}}\log n).$$

These results may, as has been shown by Landau,* be deduced from the expressions of

$$\sum_{n \leqslant x} (x-n)^k r(n), \quad \Sigma (x-n)^k d(n),$$

where k is a positive integer, as series of Bessel functions. But the deduction is of a much less direct character than that outlined above.

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* "Über die Gitterpunkte in einen Kreise (II)", *Göttinger Nachrichten*, 1915: "Über Dirichlet's Teilerproblem", *Münchener Sitzungsberichte*, 1915, pp. 317–328.