XXXVII. On the number of fractions contained in any "Farey series" of which the limiting number is given

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We may here observe (though this is a different point) that the expressions "time-integral of force," for impulsion, and "space-integral of force," for potential energy, have in some cases, from frequent use, reacted upon the users of them; so that we find impulsion called the "total force in a finite time," and potential energy the "sum of the tensions," "force" and "tensions" being force and forces proper. The competent writers now alluded to have inadvertently allowed themselves to use expressions precisely analogous to the statement that the area of a plane curve is the sum of the ordinates. It can be only in some transcendental sense that a force or pressure, supposed constant for simplicity, which has existed statically for a given time, may be said to be the sum of the applications of force in each second of that time; and this is equally true if the force have been existing kinetically.

What we may call the mathematical analogy between energy, or Fs , and impulsion, or Ft , is complete, notwithstanding the metaphysical considerations involved in the difference between their factors s and t . Each is of two dimensions; while force, or F , is of one. F being measurable in pounds, Fs is measurable in foot-pounds, and Ft in second-pounds. Each is equally disparate from F ; though very dissimilarly so, since they are disparate from each other. As Fs is the power of performing work, so Ft is the power of producing momentum. Neither had a name until the present century. It was this that gave room to the remarkable controversy concerning the true measure of the "moving force" of a body in motion which went on for over forty years in the last century, "to the great scandal of science" as Montucla says. Each needs a distinctive appropriated name, as much as the other. One, viz. Ft , has not yet acquired a universally recognized proper name; but the sooner it does so the better for the learner, and also for the science of dynamics, which then need not be guilty of the rather unscientific proceeding of occasionally giving more than one meaning to "Force."

Your obedient servant,

M. H. CLOSE.

XXXVII. *On the Number of Fractions contained in any "Farey Series" of which the Limiting Number is given.* By J. J. SYLVESTER*.

A *FAREY series* ("suite de Farey") is a system of all the unequal vulgar fractions arranged in order of magnitude, the numerator and denominator of which do not exceed a given number.

The first scientific notice of these series appeared in the

* Communicated by the Author.

Philosophical Magazine, vol. xlvii. (1816) pp. 385, 386. In 1879 Mr. Glaisher published in the Philosophical Magazine (pp. 321–336) a paper on the same subject containing a proof of their known properties, an important extension of the subject to series in which the numerators and denominators are subject to distinct limits, and a bibliography of Mr. Goodwyn's tables of such series. Finally, in 1881 Sir George Airy contributed a paper also to the Philosophical Magazine of that year, in which he refers to a table calculated by him "some years ago," and printed in the Selected Papers of the Transactions of the Institution of Civil Engineers, which is in fact a Farey table with the logarithms of the fractions appended to each of them. Previous tables had only given the decimal values of such fractions. The drift of this paper is to point out a caution which is necessary to observe in the use of such tables, and which limits their practical utility: this arises from the fact of the differences receiving a very large augmentation in the immediate neighbourhood of the fractions which are a small aliquot part of unity—a fact which may be inferred *à priori* from the well-known law discovered by Farey applicable to those differences, but to which the author of the paper makes no allusion.

In addition to the tables of Farey series by Goodwyn, Wucherer, an anonymous author mentioned in the Babbage Catalogue, and Gauss, referred to by Mr. Glaisher in his Report to the Bradford Meeting of the British Association (1873), may be mentioned one contained in Herzer's *Tabellen* (Basle, 1864) with the limit 57, and another in Hrabak's *Tabellen-Werk* (Leipsic, 1876), in which the limit is taken at 50.

The writers on the theory are:—Cauchy (as mentioned by Mr. Glaisher), who inserted a communication relating to it in the *Bulletin des Sciences par la Société Philomathique de Paris*, republished in his *Exercices de Mathématiques*; Mr. Glaisher himself (*loc. cit.*); M. Halphen, in a recent volume of the Proceedings of the Mathematical Society of France; and M. Lucas, in the next following volume of the same collection. I am indebted to my friend and associate Dr. Story for these later references.

For theoretical purposes it is desirable to count $\frac{1}{2}$ as one of the fractions in a Farey series. The number of such fractions for the limit j then becomes identical with the sum of the *totients* of all the natural numbers up to j inclusive—a totient to x (which I always denote by τx) meaning the number of numbers less than x and prime to it. Such sum, *i. e.*

$\sum_{x=0}^j \tau x$, I denote by Tj . My attention was called to the subject by this number Tj expressing the number of terms in a

function whose residue (in Cauchy's sense) is the generating function to any given simple denominator (see footnote to American Journal of Mathematics, vol. v. p. 123); and I became curious to know something about the value of T_j . I had no difficulty in finding a functional equation which serves to determine its limits (see Johns Hopkins University Circular, Jan. and Feb. 1883). The most simple form of that equation (omitted to be given in the Circular) is

$$T_j + T_{\frac{j}{2}} + T_{\frac{j}{3}} + T_{\frac{j}{4}} + T_{\frac{j}{5}} + \dots = \frac{j^2 + j}{2},$$

(where, when x is a fraction, T_x is to be understood to mean T_j , j being the integer next below x); and from this it is not difficult to deduce by strict demonstration that T_j/j^2 , when j increases indefinitely, approximates indefinitely near to $3/\pi^2$.

I have subsequently found that if ux be used to denote the sum of all the numbers inferior and prime to x , and $U_j = \sum_{x=1}^{j-1} ux$,

$$U_j + 2U_{\frac{j}{2}} + 3U_{\frac{j}{3}} + 4U_{\frac{j}{4}} + \dots = \frac{j(j+1)(j+2)}{3}$$

(where U_x , when x is a fraction, means the U of the integer next inferior to x). From this equation it is also possible to prove that U_j/j^3 , when j becomes indefinitely great, approximates to $1/\pi^2$. U_j , it may be well to notice, is the sum of all the numerators of the fractions in a Farey series whose limit is j , just as T_j is the number of these fractions.

In the annexed Table the value of τx (the totient), of T_x (the sum-totient), and of $3/\pi^2 \cdot x^2$ is calculated for all the values of x from 1 to 500; and the remarkable fact is brought to light that T_x is always greater than $3/\pi^2 \cdot x^2$ (the number opposite to it), and less than $3/\pi^2 \cdot (x+1)^2$, the number which comes after the former one in the same table.

I have calculated in my head the first few values of U_x , and find (if I have made no mistake) that it obeys an analogous law, viz. is always intermediate between $1/\pi^2 \cdot x^3$ and $1/\pi^2 \cdot (x+1)^3$.

It may also be noticed that when n is a prime number, T_n is always nearer, and usually very much nearer, to the superior than to the inferior limit—as might have been anticipated from the circumstance that, when this is the case, in passing from $n-1$ to n the T receives an augmentation of $n-1$, whereas its average augmentation is only $\frac{3}{\pi^2}(2n-1)$.

In like manner and for a similar reason, when n contains several small factors T_n is nearer to the inferior than to the superior limit. For instance, when $n=210$, $T_n=13414$ and $3/\pi^2 \cdot n^2=1340479$.

TABLE of Totients, of Sum-totients, and of $3/\pi^2$ into the Squares of all the Numbers from 1 to 500 inclusive.

$$\left[\frac{3}{\pi^2} = .30396355 \right].$$

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
1	1	1	.30	49	42	754	729.82	97	96	2902	2860.00
2	1	2	1.22	50	20	774	759.91	98	42	2944	2919.27
3	2	4	2.74	51	32	806	790.61	99	60	3004	2979.15
4	2	6	4.86	52	24	830	821.92	100	40	3044	3039.64
5	4	10	7.60	53	52	882	853.83	101	100	3144	3100.73
6	2	12	10.94	54	18	900	886.36	102	32	3176	3162.44
7	6	18	14.90	55	40	940	919.49	103	102	3278	3224.75
8	4	22	19.46	56	24	964	953.23	104	48	3326	3287.67
9	6	28	24.62	57	36	1000	987.58	105	48	3374	3351.20
10	4	32	30.40	58	28	1028	1022.54	106	52	3426	3415.34
11	10	42	36.78	59	58	1086	1058.10	107	106	3532	3480.08
12	4	46	43.77	60	16	1102	1094.27	108	36	3568	3545.44
13	12	58	51.37	61	60	1162	1131.05	109	108	3676	3611.40
14	6	64	59.58	62	30	1192	1168.44	110	40	3716	3677.96
15	8	72	68.39	63	36	1228	1206.43	111	72	3788	3745.14
16	8	80	77.81	64	32	1260	1245.03	112	48	3836	3812.92
17	16	96	87.84	65	48	1308	1284.25	113	112	3948	3881.31
18	6	102	98.48	66	20	1328	1324.07	114	36	3984	3950.31
19	18	120	109.73	67	66	1394	1364.49	115	88	4072	4019.92
20	8	128	121.58	68	32	1426	1405.53	116	56	4128	4090.14
21	12	140	134.05	69	44	1470	1447.17	117	72	4200	4160.96
22	10	150	147.12	70	24	1494	1489.42	118	58	4258	4232.39
23	22	172	160.79	71	70	1564	1532.28	119	96	4354	4304.43
24	8	180	175.08	72	24	1588	1575.75	120	32	4386	4377.08
25	20	200	189.98	73	72	1660	1619.82	121	110	4496	4450.33
26	12	212	205.48	74	36	1696	1664.51	122	60	4556	4524.19
27	18	230	221.59	75	40	1736	1709.80	123	80	4636	4598.66
28	12	242	238.31	76	36	1772	1755.69	124	60	4696	4673.74
29	28	270	255.63	77	60	1832	1802.20	125	100	4796	4749.43
30	8	278	273.56	78	24	1856	1849.31	126	36	4832	4825.72
31	30	308	292.11	79	78	1934	1897.04	127	126	4958	4902.63
32	16	324	311.26	80	32	1966	1945.37	128	64	5022	4980.14
33	20	344	333.01	81	54	2020	1994.31	129	84	5106	5058.26
34	16	360	351.38	82	40	2060	2043.85	130	48	5154	5136.98
35	24	384	372.35	83	82	2142	2094.01	131	130	5284	5216.32
36	12	396	393.93	84	24	2166	2144.77	132	40	5324	5296.26
37	36	432	416.12	85	64	2230	2196.14	133	108	5432	5376.81
38	18	450	438.92	86	42	2272	2248.12	134	66	5498	5457.97
39	24	474	462.32	87	56	2328	2300.70	135	72	5570	5539.74
40	16	490	486.34	88	40	2368	2353.90	136	64	5634	5622.11
41	40	530	510.96	89	88	2456	2407.70	137	136	5770	5705.09
42	12	542	536.19	90	24	2480	2462.10	138	44	5814	5788.68
43	42	584	562.02	91	72	2552	2517.12	139	138	5952	5872.88
44	20	604	588.47	92	44	2596	2572.75	140	48	6000	5957.69
45	24	628	615.52	93	60	2656	2628.98	141	92	6092	6043.10
46	22	650	643.19	94	46	2702	2685.82	142	70	6162	6129.12
47	46	696	671.45	95	72	2774	2743.27	143	120	6282	6215.75
48	16	712	700.33	96	32	2806	2801.33	144	48	6330	6302.99

Table (continued).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
145	112	6442	6390.83	198	60	11954	11916.59	251	250	19274	19150.01
146	72	6514	6479.29	199	198	12152	12037.26	252	72	19346	19302.90
147	84	6598	6568.35	200	80	12232	12158.54	253	220	19566	19456.40
148	72	6670	6658.02	201	132	12364	12280.43	254	126	19692	19610.51
149	148	6818	6748.29	202	100	12464	12402.93	255	128	19820	19765.23
150	40	6858	6839.18	203	168	12632	12526.03	256	128	19948	19920.56
151	150	7008	6930.67	204	64	12696	12649.75	257	256	20204	20076.49
152	72	7080	7022.77	205	160	12856	12774.07	258	84	20288	20233.03
153	96	7176	7115.48	206	102	12958	12899.00	259	216	20504	20390.18
154	60	7236	7208.80	207	132	13090	13024.54	260	96	20600	20547.94
155	120	7356	7302.72	208	96	13186	13150.68	261	168	20768	20706.30
156	48	7404	7397.26	209	180	13366	13277.43	262	130	20898	20865.28
157	156	7560	7492.40	210	48	13414	13404.79	263	262	21160	21024.86
158	78	7638	7588.15	211	210	13624	13532.76	264	80	21240	21185.05
159	104	7742	7684.51	212	104	13728	13661.34	265	208	21448	21345.84
160	64	7806	7781.47	213	140	13868	13790.52	266	108	21556	21507.25
161	132	7938	7879.04	214	106	13974	13920.32	267	176	21732	21669.26
162	54	7992	7977.22	215	168	14142	14050.72	268	132	21864	21831.88
163	162	8154	8076.01	216	72	14214	14181.73	269	268	22132	21995.11
164	80	8234	8175.41	217	180	14394	14313.34	270	72	22204	22158.95
165	80	8314	8275.41	218	108	14502	14445.57	271	270	22474	22323.39
166	82	8396	8376.02	219	144	14646	14578.40	272	128	22602	22488.44
167	166	8562	8477.24	220	80	14726	14711.84	273	144	22746	22654.10
168	48	8610	8579.07	221	192	14918	14845.89	274	136	22882	22820.37
169	156	8766	8681.50	222	72	14990	14980.54	275	200	23082	22987.25
170	64	8830	8784.55	223	222	15212	15115.81	276	88	23170	23154.73
171	108	8938	8888.20	224	96	15308	15251.68	277	276	23446	23322.82
172	84	9022	8992.46	225	120	15428	15388.16	278	138	23584	23491.52
173	172	9194	9097.33	226	112	15540	15525.25	279	180	23764	23660.83
174	56	9250	9202.80	227	226	15766	15662.94	280	96	23860	23830.75
175	120	9370	9308.88	228	72	15838	15801.24	281	280	24140	24001.27
176	80	9450	9415.57	229	228	16066	15940.15	282	92	24232	24172.40
177	116	9566	9522.87	230	88	16154	16079.67	283	282	24514	24344.14
178	88	9654	9630.78	231	120	16274	16219.80	284	140	24654	24516.49
179	178	9832	9739.29	232	112	16386	16360.53	285	144	24798	24689.44
180	48	9880	9848.42	233	232	16618	16501.87	286	120	24918	24863.00
181	180	10060	9958.15	234	72	16690	16643.82	287	240	25158	25037.18
182	72	10132	10068.49	235	184	16874	16786.38	288	96	25254	25211.96
183	120	10252	10179.44	236	116	16990	16929.55	289	272	25526	25387.34
184	88	10340	10290.99	237	156	17146	17073.32	290	112	25638	25563.34
185	144	10484	10403.15	238	96	17242	17217.70	291	192	25830	25739.94
186	60	10544	10515.92	239	238	17480	17362.70	292	144	25974	25917.15
187	160	10704	10629.30	240	64	17544	17508.30	293	292	26266	26094.97
188	92	10796	10743.29	241	240	17784	17654.51	294	84	26350	26273.40
189	108	10904	10857.88	242	110	17894	17801.32	295	232	26582	26452.43
190	72	10976	10973.09	243	162	18056	17948.74	296	144	26726	26632.07
191	190	11166	11088.90	244	120	18176	18096.77	297	180	26906	26812.32
192	64	11230	11205.31	245	168	18344	18245.41	298	148	27054	26993.18
193	192	11422	11322.34	246	80	18424	18394.66	299	264	27318	27174.65
194	96	11518	11439.97	247	216	18640	18544.51	300	80	27398	27356.72
195	96	11614	11558.21	248	120	18760	18694.97	301	252	27650	27539.40
196	84	11698	11677.06	249	164	18924	18846.04	302	150	27800	27722.69
197	196	11894	11796.52	250	100	19024	18997.72	303	200	28000	28006.59

Table (continued).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
304	144	28144	28091.10	357	192	38822	38739.85	410	160	51210	51096.27
305	240	28384	28276.21	358	178	39000	38957.18	411	272	51482	51345.83
306	96	28480	28461.93	359	358	39358	39175.13	412	204	51686	51595.99
307	306	28786	28648.26	360	96	39454	39393.68	413	348	52034	51846.76
308	120	28906	28835.20	361	342	39796	39612.83	414	132	52166	52098.14
309	204	29110	29022.75	362	180	39976	39832.60	415	328	52494	52350.12
310	120	29230	29210.90	363	220	40196	40052.97	416	192	52686	52602.72
311	310	29540	29399.66	364	144	40340	40273.95	417	276	52962	52855.92
312	96	29636	29589.03	365	288	40628	40495.54	418	180	53142	53109.73
313	312	29948	29779.01	366	120	40748	40717.74	419	418	53560	53364.15
314	156	30104	29969.59	367	366	41114	40940.55	420	96	53656	53619.17
315	144	30248	30160.79	368	176	41290	41163.96	421	420	54076	53874.80
316	156	30404	30352.59	369	240	41530	41387.98	422	210	54286	54131.04
317	316	30720	30545.00	370	144	41674	41612.61	423	276	54562	54387.89
318	104	30824	30738.01	371	312	41986	41837.85	424	208	54770	54645.35
319	280	31104	30931.64	372	120	42106	42063.69	425	320	55090	54903.42
320	128	31232	31125.87	373	372	42478	42290.15	426	140	55230	55162.09
321	212	31444	31320.71	374	160	42638	42517.21	427	360	55590	55421.39
322	132	31576	31516.16	375	200	42838	42744.87	428	212	55802	55681.26
323	288	31864	31712.22	376	184	43022	42973.15	429	240	56042	55941.76
324	108	31972	31908.88	377	336	43358	43202.04	430	168	56210	56202.86
325	240	32212	32106.15	378	108	43466	43431.53	431	430	56640	56464.57
326	162	32374	32304.03	379	378	43844	43661.63	432	144	56784	56726.89
327	216	32590	32502.52	380	144	43988	43892.34	433	432	57216	56989.82
328	160	32750	32701.62	381	252	44240	44123.65	434	180	57396	57253.36
329	276	33026	32901.32	382	190	44430	44355.58	435	224	57620	57517.50
330	80	33106	33101.63	383	382	44812	44588.11	436	216	57836	57782.26
331	330	33436	33302.55	384	128	44940	44821.25	437	396	58232	58047.62
332	164	33600	33504.08	385	240	45180	45055.00	438	144	58376	58313.58
333	216	33816	33706.22	386	192	45372	45289.35	439	438	58814	58580.16
334	166	33982	33908.96	387	252	45624	45524.32	440	160	58974	58847.34
335	264	34246	34112.31	388	192	45816	45759.89	441	252	59226	59115.14
336	96	34342	34316.27	389	388	46204	45996.07	442	192	59418	59383.54
337	336	34678	34520.84	390	96	46300	46232.86	443	442	59860	59652.54
338	156	34834	34726.01	391	352	46652	46470.25	444	144	60004	59922.16
339	224	35058	34931.80	392	168	46820	46708.25	445	352	60356	60192.38
340	128	35186	35138.19	393	260	47080	46946.87	446	222	60578	60463.22
341	300	35486	35345.19	394	196	47276	47186.09	447	296	60874	60734.66
342	108	35594	35552.80	395	312	47588	47425.91	448	192	61066	61006.70
343	294	35888	35761.01	396	120	47708	47666.35	449	448	61514	61279.36
344	168	36056	35969.83	397	396	48104	47907.39	450	120	61634	61552.62
345	176	36232	36179.26	398	198	48302	48149.04	451	400	62034	61826.49
346	172	36404	36389.30	399	216	48518	48391.30	452	224	62258	62100.97
347	346	36750	36599.95	400	160	48678	48634.17	453	300	62558	62376.06
348	112	36862	36811.21	401	400	49078	48879.64	454	226	62784	62651.75
349	348	37210	37023.07	402	132	49210	49121.73	455	288	63072	62928.05
350	120	37330	37235.54	403	360	49570	49366.42	456	144	63216	63204.97
351	216	37546	37448.61	404	200	49770	49611.72	457	456	63672	63482.48
352	160	37706	37662.30	405	216	49986	49857.62	458	228	63900	63760.61
353	352	38058	37876.59	406	168	50154	50104.14	459	288	64188	64039.35
354	116	38174	38091.50	407	360	50514	50351.26	460	176	64364	64318.69
355	280	38454	38307.01	408	128	50642	50598.99	461	460	64824	64598.64
356	176	38630	38523.12	409	408	51050	50847.33	462	120	64944	64879.20

Table (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
463	462	65406	65160.36	476	192	68964	68870.85	489	324	72872	72684.07
464	224	65630	65442.14	477	312	69276	69160.52	490	168	73040	72981.65
465	240	65870	65724.52	478	238	69514	69450.81	491	490	73530	73279.84
466	232	66102	66007.51	479	478	69992	69741.70	492	160	73690	73578.63
467	466	66568	66291.11	480	128	70120	70033.20	493	448	74138	73878.04
468	144	66712	66575.31	481	432	70552	70325.31	494	216	74354	74178.05
469	396	67108	66860.13	482	240	70792	70618.03	495	240	74594	74478.67
470	184	67292	67145.55	483	264	71056	70911.35	496	240	74834	74779.90
471	312	67604	67431.58	484	220	71276	71205.29	497	420	75254	75081.73
472	232	67836	67718.22	485	384	71660	71499.83	498	164	75418	75384.18
473	420	68256	68005.46	486	162	71822	71794.98	499	498	75916	75687.23
474	156	68412	68293.32	487	486	72308	72090.73	500	200	76116	75990.89
475	360	68772	68581.78	488	240	72548	72387.10				

XXXVIII. *On Permanent Magnetism.*By R. H. M. BOSANQUET, *St. John's College, Oxford.**To the Editors of the Philosophical Magazine and Journal.*

GENTLEMEN,

I HAVE succeeded in completely determining, to a moderate approximation, all the quantities involved in the experiment mentioned in my paper in the March number of the *Philosophical Magazine*. I propose to give a short summary of the results, reserving a more complete account for a future occasion.

The external or air resistances were determined by magnetizing, in the bifilar, arrangements of soft iron similar to the steel. The resistance of the soft iron is small, and is allowed for by Rowland's table. The total resistance is the quotient of the magnetomotive force, ($4\pi Cn$), by the magnetic induction $\mathfrak{B}$.

A set of 18 pieces of soft iron similar to the pieces of the compound magnet gave for the total resistance of each piece .454 centim.

$\mathfrak{B}=116$ gives from Rowland's table μ =about 400, whence $\frac{1.58}{400}=.004$ is to be subtracted, and we have for the air resistance .450 centimetre, or .47 R. (R is not quite the same in the iron and steel.)

A long piece of soft iron, similar to the whole magnet joined up, gave

Total resistance = .339 centim.