On Elliptic and Hyper-Elliptic Systems of Differential Equations and their Rational and Integral Algebraic Integrals, with a Discussion of the Periodicity of Elliptic and Hyper-Elliptic Functions. By W. R. WESTROPP ROBERTS, M.A. Received and communicated 4th April, 1895. Received, in revised form, 28th August, 1895.

I do not by any means claim originality or novelty for some of the results given in this paper, but only for the method by means of which they are obtained. The subject of elliptic and the higher Abelian integrals still receives such attention from, and occupies so largely the time of, our distinguished mathematicians that I do not hesitate to put forward a method which enables us immediately to write down all the *rational* and *integral* algebraic integrals of the system of differential equations known as Abelian, and embraces them in the unity of the larger theory of covariants. In what follows in these pages, it will appear that, being given any one of these rational and integral algebraic integrals, all the remaining ones may be obtained by an operative process alone.

Further, it will be shown, by integration of what I term the "fundamental equation," an irrational algebraic integral will be obtained, which will yield us a whole series of equations of a similar nature by the application of the same operative process.

Finally, I treat of Abelian functions, defining them, and giving a complete proof of the nature of their periodicity.

I now give a short summary of the contents of my paper.

Articles 1-4 treat of general theorems connected with functions of the differences of various sets of variables, and discuss two operators δ and Δ .

5 contains a definition of $I(z)$.

6. Investigation of the conditions which must be fulfilled in order that a binary quantic of degree $2n$ may be a perfect square. Derivation of all these conditions from a "square-matrix."

7. Abelian system of differential equations.

8-9. Development of our theory and definition of the function $F(z)$, which introduces $m-1$ arbitrary constants.

10-16. Further development and determination of all *rational* and

integral algebraic integrals by showing that $F(z)$ must be a perfect square.

17-18. Integral in the case of $m = 2$. Cayley's form,

19-20. Deduction of well-known equation

$$\frac{dz_1}{\sqrt{A_0 f(z_1)}} + \frac{dz_2}{\sqrt{A_0 f(z_2)}} = \frac{-d\lambda}{\sqrt{-4\lambda^3 + I\lambda - J}}.$$

21-22. Value of $\frac{z_1^4 dz_1}{\sqrt{A_0 f(z_1)}} + \frac{z_2^4 dz_2}{\sqrt{A_0 f(z_2)}}$.

23. Application to the case $m = 3$, and deduction of the rational algebraic integrals, which are, as far as I am aware, now given for the first time.

24. Case of $m = 4$.

25. General theory and integration of the "fundamental equation."

26-27. The T -function.

28. Transformation theory.

29-30. Determination of p_m in terms of $t - t^0$.

31-33. Determination of $\Sigma I_r(z) - \Sigma I_r(\zeta)$.

34. Determination of $\Sigma I_{-r}(z) - \Sigma I_{-r}(\zeta)$.

35. Discussion of $\Sigma L(z, n) - \Sigma L(\zeta, n)$.

36. General theory.

37. Proof that $\Sigma I_r(z) - \Sigma_r I(\zeta)$ can be found by an operative process alone.

38-41. Definition of Abelian functions and their periodicity.

42. The number of periods.

43. Interesting algebraic equation, when $z_m = a_1$ and $\zeta_m = a_2$.

1. Before entering on the discussion of the theory of the transcendents known as Abelian, we shall first investigate some algebraic theorems having an important bearing on the subject.

Let the roots of the equation

$$\phi(z) \equiv z^m + p_1 z^{m-1} + p_2 z^{m-2} + \dots + p_m = 0$$

be denoted by $z_1, z_2, \dots, z_m$, and let us suppose we have a relation such as the following

$$c_1 z^{m-1} + c_2 z^{m-2} + \dots + c_m = 0,$$

connecting m quantities $c_1, c_2, \dots, c_m$ with z , where z is a root of

$$\phi(z) = 0;$$

then it is plain that, if this relation is satisfied for each root of

$$\phi(z) = 0,$$

it being understood that $\phi(z) = 0$

has no equal roots, we must have

$$c_1 = 0, \quad c_2 = 0, \quad \dots \quad c_m = 0 \quad \dots \dots \dots (1).$$

Now, since $z^m = -\{p_1 z^{m-1} + p_2 z^{m-2} + \dots + p_m\}$,

when z is a root of $\phi(z) = 0$,

we can express z^r , r being greater than $m-1$, in terms of $z^{m-1}, z^{m-2}, \dots, z$ and the quantities $p_1, p_2, \dots, p_m$; so that, if we had a relation of the form

$$D_0 z^r + D_1 z^{r-1} + \dots + D_r = 0$$

satisfied for each root of the equation

$$\phi(z) = 0,$$

we could reduce this case to the one first discussed, by substituting for all powers of z higher than the $m-1^{\text{th}}$ their values in terms of $z^{m-1}, z^{m-2}, \dots, z$ and the quantities $p_1, p_2, \dots, p_m$, and then, equating the coefficients of $z^{m-1}, z^{m-2}, \dots, z, z^0$ each to zero, arrive at m equations connecting the quantities $D_0, D_1, \dots, D_r$ with the quantities $p_1, p_2, \dots, p_m$.

As an example, suppose

$$\phi(z) \equiv z^3 + p_1 z + p_2,$$

and that we had $D_0 z^3 + D_1 z^2 + D_2 z + D_3 = 0$

for both roots of $\phi(z) = 0$.

We have $z^3 = -(p_1 z + p_2)$;

consequently $z^3 = p_1 (p_1 z + p_2) - p_2 z$

$$= (p_1^2 - p_2) z + p_1 p_2;$$

introducing then these values of z^3 and z^3 into the expression

$$D_0 z^3 + D_1 z^2 + D_2 z + D_3,$$

we find $\{D_0(p_1^2 - p_2) - D_1 p_1 + D_2\} z + D_0 p_1 p_2 - D_1 p_2 + D_3$,

and this being, by hypothesis, zero for both roots of

$$\phi(z) = 0,$$

we must have

$$\left. \begin{aligned} D_0(p_1^2 - p_2) - D_1 p_1 + D_2 &= 0 \\ D_0 p_1 p_2 - D_1 p_2 + D_3 &= 0 \end{aligned} \right\} \dots\dots\dots (2).$$

2. Again, being given that an expression such as

$$V \equiv B_1 z^{m-1} + B_2 z^{m-2} + \dots B_m$$

is a function of the differences of the quantities

$$z_1, z_2, \dots z_m; \quad a_1, a_2, \dots a_{2m}; \quad \zeta_1, \zeta_2, \dots \zeta_m,$$

where z stands for any root of

$$\psi(z) = 0,$$

and the coefficients $B_1, B_2, \dots B_m$ are *symmetric* functions of the above quantities, then it is clear that, when V is operated on by

$$\Sigma \frac{d}{dz} + \Sigma \frac{d}{da} + \Sigma \frac{d}{d\zeta},$$

it must vanish identically where

$$\Sigma \frac{d}{dz} \equiv \frac{d}{dz_1} + \frac{d}{dz_2} + \dots + \frac{d}{dz_m},$$

$$\Sigma \frac{d}{da} = \frac{d}{da_1} + \frac{d}{da_2} + \dots + \frac{d}{da_{2m}},$$

$$\Sigma \frac{d}{d\zeta} = \frac{d}{d\zeta_1} + \frac{d}{d\zeta_2} + \dots + \frac{d}{d\zeta_m}.$$

Professor Cayley has applied the name *facients* to the quantities to which the expression "degree" refers, or in regard to which a function is considered a quantic, the term quantic being used to denote the entire subject of rational and integral functions.

Now, although most of the algebraic expressions we shall have occasion to deal with are neither rational nor integral, we shall

retain the name "facients" to denote the various sets of quantities which enter into them.

If we now call the operation

$$\Sigma \frac{d}{dz} + \Sigma \frac{d}{du} + \Sigma \frac{d}{d\zeta} \equiv -\delta,$$

and operate on the expression V with δ , we shall find

$$\begin{aligned} \delta V \equiv z^{m-1} \delta B_1 + z^{m-2} \{ \delta B_2 - (m-1) B_1 \} + \dots \\ \dots + z \{ \delta B_{m-1} - 2B_{m-2} \} + \{ \delta B_m - B_{m-1} \} = 0, \end{aligned}$$

and, since δV must vanish for each root of

$$\phi(z) = 0,$$

the coefficient of each power of z in the above expression must vanish by Art. 1, giving us

$$\left. \begin{aligned} B_{m-1} - \delta B_m &= 0, \\ 2B_{m-2} - \delta B_{m-1} &= 0, \\ 3B_{m-3} - \delta B_{m-2} &= 0, \\ \dots \quad \dots \quad \dots \quad \dots & \\ \delta B_1 &= 0, \end{aligned} \right\}$$

or

$$\left. \begin{aligned} B_{m-1} &= \delta B_m \\ B_{m-2} &= \frac{\delta^2 B_m}{1.2} \\ B_{m-3} &= \frac{\delta^3 B_m}{1.2.3} \\ \dots \quad \dots \quad \dots & \\ \delta B_1 &= 0 \end{aligned} \right\} \dots \dots \dots (1).$$

From these results we learn that, if an expression such as

$$V \equiv B_1 z^{m-1} + B_2 z^{m-2} + \dots + B_m,$$

where z is a root of $\phi(z) = 0$,

is a function of the differences of the facients

$$z_1, z_2, \dots, z_m; \quad a_1, a_2, \dots, a_{2m}; \quad \zeta_1, \zeta_2, \dots, \zeta_m,$$

the coefficient of z^{m-1} is a function of the differences of these facients,

and all the coefficients $B_1, B_2, \dots B_{m-1}$, can be derived by successive operations of δ on B_m , to which function we shall apply the term *source*, a name which has already been applied to that term from which covariants of a quantic are derived.

The function V must then be capable of being written in the form

$$V \equiv B_m + z\delta B_m + \frac{z^2}{1 \cdot 2} \delta^2 B_m + \dots + \frac{z^{m-1} \delta^{m-1} B_m}{(m-1)!} \dots\dots\dots (2),$$

or, symbolically, $V \equiv (1+z\delta)^{m-1} B_m \dots\dots\dots (3).$

To illustrate our theory, let us take $m=3$, and consider the expression

$$z^3 (H_1 + H_2 + H_3) - z \{ H_1 (z_3 + z_2) + H_2 (z_3 + z_1) + H_3 (z_1 + z_2) \} \\ + H_1 z_2 z_3 + H_2 z_3 z_1 + H_3 z_1 z_2,$$

in which H_1, H_2, H_3 are functions of the differences of the facients, and consequently vanish when operated on by δ .

Now this expression is evidently a function of the differences of the facients, whenever z is a root of

$$\phi(z) = 0,$$

but it is also clearly a function of the differences of the facients and a new quantity z , no matter what z may be.

And this is easily seen to be true in general, that, if any expression

$$V \equiv B_1 z^{m-1} + B_2 z^{m-2} + \dots + B_m$$

is a function of the differences of the facients $z_1, z_2, \dots z_m$ and other sets of facients in each case, that z is a root of

$$\phi(z) = 0,$$

then the same expression is a function of the differences of the same facients and a new quantity z , no matter what z may be. For, if the above expression is a function of the differences of z and the various facients $z_1, z_2, \dots z_m$; $a_1, a_2, \dots a_m$; $\zeta_1, \zeta_2, \dots \zeta_m$ it must, when operated on by $-\frac{d}{dz} + \delta$, vanish identically, and we shall be led to the very same equations connecting the functions $B_1, B_2, \dots B_m$ and their derivation from B_m as a source, by equating to zero the coefficients of

$$z^{m-1}, z^{m-2}, \dots z', z^0, \dots$$

in the result which is to vanish identically.

In general, then, V can be written in the form

$$V \equiv (1 + z\delta)^{m-1} R_m.$$

3. It will be found convenient in what follows to introduce new sets of facients in order to render homogeneous certain algebraic expressions we shall have to deal with.

We shall write

$$z_1 = \frac{x_1}{y_1}, \quad z_2 = \frac{x_2}{y_2}, \quad \dots \quad z_m = \frac{x_m}{y_m};$$

$$a_1 = \frac{a_1}{b_1}, \quad a_2 = \frac{a_2}{b_2}, \quad \dots \quad a_{2m} = \frac{a_{2m}}{b_{2m}};$$

$$\zeta_1 = \frac{\xi_1}{\eta_1}, \quad \zeta_2 = \frac{\xi_2}{\eta_2}, \quad \dots \quad \zeta_m = \frac{\xi_m}{\eta_m};$$

the introduced facients or symbols being ultimately replaced by unity in each case. With this change in our notation, the operator we have called δ becomes altered as follows :

$$-\delta \equiv \Sigma y \frac{d}{dx} + \Sigma b \frac{d}{du} + \Sigma \eta \frac{d}{d\xi}.$$

It is clear we have also an operator which affects the introduced facients, which we shall denote by Δ , and write

$$-\Delta \equiv \Sigma x \frac{d}{dy} + \Sigma a \frac{d}{db} + \Sigma \xi \frac{d}{d\eta}.$$

Any function of the differences of the old facients will vanish when operated on by δ , and any invariant will vanish when operated on by δ or Δ .

4. Let us now write

$$\begin{aligned} f(z) &\equiv z^{2m} + P_1 z^{2m-1} + P_2 z^{2m-2} + \dots + P_{2m}, \\ &\equiv (z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_{2m}), \end{aligned}$$

calling $\alpha_1, \alpha_2, \dots, \alpha_{2m}$, the $2m$ roots of

$$f(z) = 0.$$

If we now put $z = \frac{x}{y}$ and $a_r = \frac{a_r}{b_r}$,

r being an integer which may have any value from $2m$ to unity, and introduce these values into the above identity, we find

$$f(z) y^{2m} b_1 b_2 \dots b_{2m} \equiv (xb_1 - ya_1)(xb_2 - ya_2) \dots (xb_{2m} - ya_{2m}), \\ \equiv U, \text{ say,}$$

and we write U with binomial coefficients, so that

$$U \equiv A_0 x^{2m} + 2mA_1 x^{2m-1}y + \dots A_{2m} y^{2m};$$

we have finally, then, $b_1 b_2 \dots b_{2m}$ being equal to A_0 ,

$$f(z) \equiv \frac{U}{A_0 y^{2m}},$$

and also, by differentiation,

$$dz = \frac{y dx - x dy}{y^2}.$$

5. Let us now write

$$I_r(z) \equiv \int^* \frac{z^r dz}{\sqrt{A_0 f(z)}} \equiv \int^{s/y} \frac{x^r y^{m-r-2} (y dx - x dy)}{\sqrt{U}},$$

r being any positive or negative integer. We find easily, by operating with δ ,

$$\delta I_r(z) = -r I_{r-1}(z),$$

and again, by operating with Δ ,

$$\Delta I_r(z) = -(m-2-r) I_{r+1}(z).$$

We shall also write

$$L(z, n) \equiv \int^* \frac{dz}{(z-n) \sqrt{A_0 f(z)}},$$

and we readily find

$$\delta L(z, n) = \int^* \frac{dz}{(z-n)^2 \sqrt{A_0 f(z)}}.$$

6. We shall now investigate the conditions which must be fulfilled in order that a binary quantic of degree $2n$ may be a perfect square.

These conditions are of the utmost importance to us, as they enable

us to write down the rational and integral algebraic integrals of the Abelian system of differential equations, which we shall treat of presently.

If a binary quantic contains a square factor, it is clear one condition among its coefficients is necessary, and if it contain two square factors, two conditions are necessary, and if the quantic of degree $2n$ has n square factors, or, what is the same thing, becomes a perfect square, n conditions must be fulfilled.

Let U be our quantic, which we shall suppose written with binomial coefficients, and which, by hypothesis, is the square of a quantic u , of the degree n , which we also write with binomial coefficients.

We write then

$$U \equiv A_0 z^{2n} + 2nA_1 z^{2n-1} + \dots + A_{2n} \equiv u^2 \equiv (a_0 z^n + na_1 z^{n-1} + \dots + a_n)^2.$$

Now we have, of course,

$$A_{2n} = a_n^2,$$

and it is easily seen that the same operator δ which converts A_r , r being an integer which may have any value from zero to $2n$, into A_{r-1} , so that

$$\delta A_r = rA_{r-1},$$

will also convert a_i into a_{i-1} , agreeably to the same law of derivation, so that

$$\delta a_i = sa_{i-1},$$

s being an integer having any value from zero to n .

Now the source of the Hessian of U is

$$A_{2n-2}A_{2n} - A_{2n-1}^2,$$

and that of the Hessian of u is

$$a_{n-2}a_n - a_{n-1}^2,$$

and, as we easily find that

$$A_{2n} = a_n^2,$$

$$A_{2n-1} = a_n a_{n-1},$$

$$(2n-1)A_{2n-2} = na_{n-1}^2 + (n-1)a_n a_{n-2},$$

it follows that

$$(2n-1) A_{2n-2} A_{2n} - A_{2n-1}^2 = (n-1) a_n^2 \{a_{n-2} a_n - a_{n-1}^2\},$$

$$(2n-1) H = (n-1) A_{2n} h,$$

where we have called the source of $H(U)$, H , and that of $h(u)$, h ; $H(U)$ and $h(u)$ denoting the Hessians of the quantics U and u respectively.

We can now operate $4n-4$ times on the equation

$$(2n-1) H = (n-1) A_{2n} h,$$

thus obtaining $4n-4$ equations which, added to the one above written, give us in all $4n-3$ equations connecting the $2n-3$ unknown quantities h , δh , $\delta^2 h$, ... $\delta^{2n-4} h$.

The various results of elimination of these quantities may be most clearly exhibited in the following manner.

Form the matrix written below, which I call the "square-matrix" for the quantic U , consisting of $2n-2$ columns, and $4n-3$ rows, the law of its formation being obvious on inspection,

$$\left\| \begin{array}{cccccc} A_{2n} & 0 & 0 & 0 & \dots & H \\ \delta A_{2n} & A_{2n} & 0 & 0 & \dots & \delta H \\ \frac{\delta^2 A_{2n}}{1.2} & \delta A_{2n} & A_{2n} & 0 & \dots & \frac{\delta^2 H}{1.2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \frac{\delta^{4n-4} H}{(4n-4)!} \end{array} \right\| ;$$

then the conditions that U may be a perfect square will be obtained by equating to zero all the determinants which can be derived from this matrix, and consisting of $2n-2$ columns of $2n-2$ elements taken in the order they stand in any $2n-2$ of the $4n-3$ rows of this matrix.

This "square-matrix" is full of interest, and an investigation of it leads us to many curious theorems which, however, I cannot here examine, but will only mention one of its most striking properties, namely, that, if *any one* of the algebraic equations resulting from it be given, all the remaining equations can be found by repeated operation of δ on the given equation, and on that which results from it by substituting for the suffixes their complementary values.

8. So far we have followed Jacobi in the introduction of a new variable t , leading to the well known equation written above, but we now proceed to obtain the algebraic integrals of the differential system in a totally different manner. If we differentiate the equation

$$\phi(z) = 0,$$

we obtain
$$\frac{d\phi(z)}{dz} \left(\frac{dz}{dt} \right) + \frac{d\phi(z)}{dt} = 0 \dots\dots\dots(1),$$

where
$$\frac{d\phi(z)}{dz} = \phi'(z),$$

and
$$\frac{d\phi(z)}{dt} = p_1 z^{m-1} + p_2 z^{m-2} + \dots p_m,$$

the dot indicating differentiation with regard to t . Introducing into equation (1) the value of $\frac{dz}{dt}$ given in the last article, we find

$$\frac{d\phi(z)}{dt} + \sqrt{A_0 f(z)} = 0,$$

from which we obtain

$$\left\{ \frac{d\phi(z)}{dt} \right\}^2 = A_0 f(z) \dots\dots\dots(2),$$

an equation which is true for all values of z which make $\phi(z)$ become zero.

9. Let us now take $m-1$ new quantities $\lambda_m, \lambda_{m-1}, \dots \lambda_2$, and let us determine them so that the equation

$$\left\{ \frac{d\phi(z)}{dt} \right\}^2 = A_0 \{ f(z) + [\phi(z)]^2 - 2\phi(z) L(z) \} \equiv F(z)$$

may be *identically satisfied* for all possible values of z , where

$$L(z) \equiv z^m + \frac{P_1}{z} z^{m-1} + \lambda_2 z^{m-2} + \lambda_3 z^{m-3} + \dots + \lambda_m.$$

Now the left-hand side of the above identity is obviously of the degree $2m-2$ in z , and it will be found that the right-hand side is likewise of the degree $2m-2$, since the coefficients of both z^{2m} and z^{2m-1} vanish; we have consequently $2m-1$ equations connecting the p 's, p 's and λ 's with the quantities $P_1, P_2, \dots P_{2m}$. But it is obvious

that, if we eliminate, from these $2m-1$ equations, the λ 's, we arrive at the typical equation of the last article

$$\left\{ \frac{d\phi(z)}{dt} \right\}^2 = A_0 f(z),$$

where z is now a root of $\phi(z) = 0$,

which determines the z 's and consequently the p 's in terms of t , their values being agreeable to this system of differential equations, contained in the equations

$$\sum \frac{z^i dz}{\sqrt{A_0 f(z)}} = 0, \quad \sum \frac{z^{m-1} dz}{\sqrt{A_0 f(z)}} = dt;$$

we have therefore $m-1$ equations to determine the $m-1$ λ 's in terms of the p 's and $\dot{p}$'s; and finally in terms of t .

The problem is then a possible one.

10. If we now consider the equation

$$\frac{d\phi(z)}{dt} + \sqrt{A_0 f(z)} = 0,$$

and operate on it with δ , we obtain, since $\delta A_0 f(z)$ is zero,

$$\delta \frac{d\phi(z)}{dt} = 0,$$

$$\text{or} \quad z^{m-1} \delta \dot{p}_1 + z^{m-2} \{ \delta \dot{p}_2 - (m-1) \dot{p}_1 \} + z^{m-3} \{ \delta \dot{p}_3 - (m-2) \dot{p}_2 \} + \dots \\ \dots + \{ \delta \dot{p}_m - \dot{p}_{m-1} \} = 0,$$

whenever z is a root of $\phi(z) = 0$;

hence by Article 1, the coefficients of the several powers of z must vanish, and we find

$$\left. \begin{aligned} \delta \dot{p}_m &= \dot{p}_{m-1} \\ \delta \dot{p}_{m-1} &= 2\dot{p}_{m-2} \\ \delta \dot{p}_{m-2} &= 3\dot{p}_{m-3} \\ \dots &\dots \dots \dots \\ \delta \dot{p}_3 &= (m-1) \dot{p}_1 \\ \delta \dot{p}_1 &= 0 \end{aligned} \right\} \dots \dots \dots (1).$$

Again, it follows from Article 2 that, since

$$z^{m-1} \dot{p}_1 + z^{m-2} \dot{p}_2 + \dots + \dot{p}_m$$

is a function of the differences of the facients when z is a root of

$$\phi(z) = 0,$$

it must also be a function of the differences of the various facients and a new quantity z , no matter what z may be, and moreover derivable from a source p_m . Hence it follows that, if we write

$$\left\{ \frac{d\phi(z)}{dt} \right\}^2 \equiv F(z) \dots\dots\dots(2),$$

$F(z)$ must vanish when operated on by δ , and must be itself derivable from a source which is easily seen to be

$$A_0(p_m^2 - 2\lambda_m p_m + P_{2m}),$$

from which it follows that the λ 's are obedient to the same law of derivation from a source λ_m . This being the case, we must have

$$\left. \begin{aligned} \delta\lambda_r &= (m+1-r)\lambda_{r-1} \\ \delta\lambda_2 &= (m-1)P_1 \end{aligned} \right\} \dots\dots\dots(3),$$

r being an integer which may have any value from 3 to m .

We now see that the $2m-1$ equations obtained by equating the coefficients of like powers of z in the identity

$$\left\{ \frac{d\phi(z)}{dt} \right\}^2 \equiv F(z)$$

can all be obtained by successive applications of the operator δ on the equation

$$p_m^2 = A_0(p_m^2 - 2\lambda_m p_m + P_{2m}) \dots\dots\dots(4).$$

This equation I call the *fundamental equation* in the theory of elliptic or hyper-elliptic functions, since from it flow all the Protean forms which the algebraic equivalents of the transcendental system are capable of assuming. We also find easily

$$p_1^2 = A_0 \{ (p_1 - P_1)^2 + P_2 + P_1^2 - 2\lambda_2 \},$$

an equation we shall make use of presently.

11. We shall now write the identity

$$\dot{\phi}(z)^2 \equiv F(z),$$

in the form $\dot{\phi}(z)^2 = A_0 V(z) + A_0 \{ \phi(z) - L(z) \}^2 \dots\dots\dots(1),$

where $V(z) \equiv f(z) - L(z)^2 \equiv V_0 z^{2m-2} + V_1 z^{2m-1} + \dots + V_{2m-2}$
 $\equiv V_0 (z^{m-1} + R_1 z^{m-1} + \dots + R_{m-1})(z^{m-1} + s_1 z^{m-1} + \dots + s_{m-1})$
 $\equiv V_0 R(z) S(z),$

and we shall denote the roots of

$$R(z) = 0 \quad \text{and} \quad S(z) = 0$$

by $r_1, r_2, \dots, r_{m-1}; s_1, s_2, \dots, s_{m-1}$, respectively.

If we now differentiate the identity

$$\dot{\phi}(z)^2 = A_0 \{f(z) - 2L(z)\phi(z) + \phi(z)^2\}$$

with regard to t , we readily obtain

$$A_0 L(z) \phi(z) \equiv \{A_0 [\phi(z) - L(z)] - \ddot{\phi}(z)\} \dot{\phi}(z).$$

Now the left-hand side of the identity vanishes for

$$z = z_1, \quad z = z_2, \quad \dots \quad z = z_m,$$

while $\dot{\phi}(z)$ does not vanish for such values of z . Consequently, if $L(z)$ be not zero, we should have

$$A_0 [\phi(z) - L(z)] - \ddot{\phi}(z),$$

a function of the $m-1$ th degree vanishing for m values of z , which is impossible; hence it follows that $L(z)$ must vanish identically, that is to say, that $\lambda_2, \lambda_3, \dots, \lambda_{m-2}$ must be regarded as constants, and independent of t . We must then have

$$\ddot{\phi}(z) \equiv A_0 [\phi(z) - L(z)] \dots \dots \dots (2),$$

from which we infer $\ddot{p}_r = A_0 (p_r - \lambda_r) \dots \dots \dots (3),$

r being an integer which may have any value from $r = 1$ to $r = m$.

12. Being given, then, the system of differential equations contained in the following typical equation

$$\Sigma \frac{z^i dz}{\sqrt{A_0 f(z)}} = 0,$$

holding for all integer values of i from $i = 0$ to $i = m-2$, we see that, if we determine m quantities $\lambda_2, \lambda_3, \dots, \lambda_m$ by the $m-1$ conditions that must be satisfied in order that $F(z)$, as previously defined, may be a perfect square, these quantities will be constants, and that con-

sequently these $m-1$ conditions constitute $m-1$ rational and integral algebraic integrals, involving $m-1$ arbitrary constants, of the above system of differential equations. Hence we have the following canon for the determination of the *rational* and *integral* forms of the differential system

$$\Sigma \frac{z' dz}{\sqrt{A_0 f(z)}} = 0.$$

First form the function $F(z)$, thus introducing $m-1$ arbitrary constants $\lambda_2, \lambda_3, \dots \lambda_m$; then construct the square-matrix of $F(z)$, considered as a binary quantic of the $2m-2^{\text{th}}$ degree, this matrix consisting of $2m-4$ columns and $4m-7$ rows; then every determinant which can be derived from this matrix, and consisting of any $2m-4$ of the $4m-7$ rows of the matrix of $2m-4$ elements, will, when equated to zero, be an algebraic integral of the system of differential equations.

13. If we now write

$$\dot{\phi}(z)^2 - A_0 \{\phi(z) - L(z)\}^2 \equiv A_0 V_0 R(z) S(z),$$

we may write

$$\dot{\phi}(z) + \sqrt{A_0} \{\phi(z) - L(z)\} \equiv \theta \sqrt{A_0} R(z) \dots \dots \dots (1),$$

$$\phi(z) - \sqrt{A_0} \{\phi(z) - L(z)\} \equiv \frac{V_0 \sqrt{A_0}}{\theta} S(z) \dots \dots \dots (2),$$

θ being some function of t .

To determine θ we shall differentiate (1) with regard to t , when we shall find, since the λ 's and consequently $r_1, r_2, \dots r_{m-1}$ are constants,

$$\ddot{\phi}(z) + \sqrt{A_0} \dot{\phi}(z) = \dot{\theta} \sqrt{A_0} R(z).$$

But from (2) of the last article, we have

$$\ddot{\phi}(z) = A_0 \{\phi(z) - L(z)\},$$

and, introducing this value into the above equation, we obtain

$$\dot{\theta} = \sqrt{A_0} \theta,$$

$$\text{or, by integration,} \quad \theta = \theta_0 e^{\sqrt{A_0}(t-t_0)} \dots \dots \dots (3),$$

where θ_0 is the value of θ when $t = t_0$, we have also, by equating the coefficients of z^{m-1} in (1),

$$\theta \sqrt{A_0} = \dot{p}_1 + \sqrt{A_0} (p_1 - P_1) \dots \dots \dots (4).$$

14. From (1) and (2) of Article 13, we readily deduce

$$2\sqrt{A_0}\{\phi(z) - L(z)\} = \theta\sqrt{A_0}R(z) - \frac{V_0\sqrt{A_0}S(z)}{\theta},$$

or $2\theta\phi(z) \equiv \theta^2R(z) + 2\theta L(z) - V_0S(z) \dots\dots\dots(1),$

and, if we multiply each side of this identity by $R(z)$, we obtain, calling

$$R(z)\phi(z) \equiv \Psi(z),$$

$$\begin{aligned} 2\theta\Psi(z) &\equiv \{\theta R(z) + L(z)\}^2 - L(z)^2 - V_0R(z)S(z) \\ &\equiv \{\theta R(z) + L(z)\}^2 - f(z) \dots\dots\dots(2). \end{aligned}$$

We see then that the roots of the expression

$$\{\theta R(z) + L(z)\}^2 - f(z),$$

when equated to zero, are

$$z_1, z_2, \dots z_m; \quad r_1, r_2, \dots r_{m-1};$$

and, if we differentiate

$$\{\theta R(z) + L(z)\}^2 - f(z) = 0,$$

on the hypothesis that the z 's and the r 's as well as θ are variable, we obtain

$$2\theta\Psi'(z)dz + 2\{\theta R(z) + L(z)\}(\theta dR + R d\theta) = 0,$$

z being a root of $\phi(z)R(z) = 0.$

Now, when $\Psi(z) = 0,$

we must have $\theta R(z) + L(z) = h\sqrt{f(z)} \dots\dots\dots(3),$

where $h^2 = 1;$

introducing this value into the above equation, we easily obtain

$$\frac{dz}{\sqrt{f(z)}} + h\left\{\frac{\theta dR(z) + R(z)d\theta}{\theta\Psi'(z)}\right\} = 0 \dots\dots\dots(4),$$

the symbol d affecting only the coefficients in $R(z).$

Now, the above equation holds for each root of

$$\Psi(z) = 0;$$

consequently, if we sum for the z 's and the r 's, that is, for the roots of

$$\Psi(z) = 0,$$

we obtain, by the well-known theory of partial fractions, $\Psi(z)$ being of the $2m-2^{\text{th}}$ degree in z , $R(z)$ of the $m-1^{\text{th}}$, and $dR(z)$ of the $m-2^{\text{th}}$,

$$\sum \frac{dz}{\sqrt{f(z)}} \pm \sum \frac{dr}{\sqrt{f(r)}} = 0,$$

and if we multiply (4) by z^r , where r has any integer value from $r = 1$ to $r = m-1$, and sum, we obtain results which are contained in the typical equation

$$\sum \frac{z^i dz}{\sqrt{A_0 f(z)}} \pm \sum \frac{r^i dr}{\sqrt{A_0 f(r)}} = 0 \dots\dots\dots(5),$$

where i has any integer value from $i = 0$ to $i = m-2$; and the following

$$\sum \frac{z^{m-1} dz}{\sqrt{A_0 f(z)}} \pm \sum \frac{r^{m-1} dr}{\sqrt{A_0 f(r)}} + \frac{h d\theta}{\sqrt{A_0 \theta}} = 0 \dots\dots\dots(6),$$

where we must take $h = -1$ to give results in accordance with what precedes.

15. We shall now investigate the relations connecting the $2m-2$ roots of

$$V(z) = 0,$$

which are independent of any values the λ 's may assume.

We have $V(z) \equiv f(z) - L(z)^2$;

and if we differentiate, regarding the roots of

$$V(z) = 0$$

and the λ 's as variable, we find

$$V'(z) dz - 2 L(z) dL(z) = 0;$$

and, replacing $L(z)$ by its value $\sqrt{f(z)}$, which it possesses when z is a root of

$$V(z) = 0,$$

we readily obtain $\frac{dz}{\sqrt{A_0 f(z)}} - 2 \frac{dL(z)}{V'(z) \sqrt{A_0}} = 0 \dots\dots\dots(1),$

and, multiplying this typical equation by z^k , where k has any integer value from $k = 0$ to $k = m-1$, and summing, we obtain

$$\sum \frac{r^k dr}{\sqrt{A_0 f(r)}} + \sum \frac{s^k ds}{\sqrt{A_0 f(s)}} = 0 \dots\dots\dots(2);$$

holding for all integer values of k from $k = 0$ to $k = m-2$; also

$$\pm \Sigma \frac{r^{m-1} dr}{\sqrt{A_0 f(r)}} \pm \delta \Sigma \frac{s^{m-1} ds}{\sqrt{A_0 f(s)}} = \frac{2d\lambda_2}{V_0 \sqrt{A_0}} \dots\dots\dots (3).$$

16. If we were to connect the z 's with the s 's by a process of reasoning precisely similar to that employed in Article 14, we must have

$$\Sigma \frac{z^i dz}{\sqrt{A_0 f(z)}} \pm \Sigma \frac{s^i ds}{\sqrt{A_0 f(s)}} = 0 \dots\dots\dots (1),$$

holding for all integer values of i from $i = 0$ to $i = m-2$, also

$$\Sigma \frac{z^{m-1} dz}{\sqrt{A_0 f(z)}} \pm \Sigma \frac{s^{m-1} ds}{\sqrt{A_0 f(s)}} = \frac{2d\lambda_2}{V_0 \sqrt{A_0}} + \frac{d\theta}{\theta \sqrt{A_0}} \dots\dots\dots (2),$$

which gives results which are agreeable to the results obtained in the last two articles.

17. If we now write

$$\left. \begin{array}{l} \pi_m = p_m - \lambda_m \\ \pi_{m-1} = p_{m-1} - \lambda_{m-1} \\ \dots \quad \dots \quad \dots \quad \dots \\ \pi_1 = p_1 - P_1 \end{array} \right\} \dots\dots\dots (1),$$

it is clear that $\pi_i = p_i$, i being any integer from $i = 1$ to $i = m$; the fundamental equation becomes

$$\pi_m^2 = A_0 (\pi_m^2 + V_{2m-2}),$$

and from this equation, as before indicated, we can obtain, by successive operations of δ , the systems

$$\left. \begin{array}{l} \pi_m^2 = A_0 (\pi_m^2 + V_{2m-2}), \\ 2\pi_m \pi_{m-1} = A_0 (2\pi_m \pi_{m-1} + V_{2m-2}), \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \pi_1^2 = A_0 (\pi_1^2 + V_0). \end{array} \right\}$$

18. I now discuss the case in which $m = 2$, or the case of elliptic integrals. Here we have

$$f(z) \equiv z^4 + P_1 z^3 + P_2 z^2 + P_3 z + P_4,$$

and there are two variables z_1 and z_2 connected by the equation

$$\frac{dz_1}{\sqrt{A_0 f(z_1)}} + \frac{dz_2}{\sqrt{A_0 f(z_2)}} = 0.$$

The source of $F(z)$ being $A_0(\pi_2^2 + V_2)$, the function itself is evidently

$$F(z) \equiv A_0 \{ (\pi_2^2 + V_2) + z(2\pi_1\pi_2 + V_1) + (\pi_1^2 + V_0) \},$$

and, since $F(z)$ must be a square, the rational and integral algebraic integral is at once seen to be

$$(2\pi_1\pi_2 + V_1)^2 = 4(\pi_2^2 + V_2)(\pi_1^2 + V_0) \dots\dots\dots(1),$$

$$\text{or} \quad V_0\pi_2^2 - V_1\pi_1\pi_2 + V_2\pi_1^2 = \frac{V_1^2 - 4V_0V_2}{4} = \Lambda_2, \text{ say } \dots\dots\dots(2),$$

where

$$V_2 = P_4 - \lambda_2^2,$$

$$V_1 = P_3 - \lambda_2 P_1,$$

$$V_0 = P_2 - P_4^2 - 2\lambda_2,$$

it being remembered that $\delta\lambda_2$, by Article 10, is $\frac{m-1}{2}P_1$; or in this case $\frac{1}{2}P_1$.

This result will be found identical with that given by Cayley in his *Elementary Treatise on Elliptic Functions*, provided we write

$$2\lambda_2 = P_2 - c.$$

The reader will perceive that the above integral (2) is obtained directly by the application of our theory, and not derived from an irrational form of the integral which, when rationalized, contains the irrelevant factor $(z_1 - z_2)^2$, of which it must be cleared before the result given above can be obtained.

19. We have also, by Article 14,

$$\frac{dz_1}{\sqrt{A_0 f(z_1)}} + \frac{dz_2}{\sqrt{A_0 f(z_2)}} - \frac{dr}{\sqrt{A_0 f(r)}} = 0 \dots\dots\dots(1),$$

where we consider λ_2 variable, r being a root of the equation

$$V_0 r^2 + V_1 r + V_2 = 0,$$

from which we find

$$(V_0 r + V_1)^2 = \frac{V_1^2 - 4V_0V_2}{4} = \Lambda_2,$$

or, taking the square root,

$$V_0 r + V_1 = \sqrt{\Lambda_2}.$$

We have also $V(z) \equiv f(z) - [L(z)]^2 \dots \dots \dots (2);$

hence, differentiating this identity with regard to λ_2 , we find

$$\frac{dV(z)}{d\lambda_2} \equiv -2L(z) \frac{dL(z)}{d\lambda_2} = -2L(z),$$

since

$$\frac{dL(z)}{d\lambda_2} = 1.$$

If we now differentiate the equation

$$V_0 r^3 + V_1 r + V_2 = 0,$$

we obtain

$$2(V_0 r + V_1) dr + \frac{dV(r)}{d\lambda_2} d\lambda_2 = 0,$$

and consequently, by what we have proved above, we find

$$2\sqrt{\Lambda_2} dr - 2L(r) d\lambda_2 = 0;$$

but, since

$$L(r) = \sqrt{f(r)},$$

we have

$$\frac{dr}{\sqrt{f(r)}} = \frac{d\lambda_2}{\sqrt{\Lambda_2}} \dots \dots \dots (3),$$

and, finally,

$$\frac{dz_1}{\sqrt{A_0 f(z_1)}} + \frac{dz_2}{\sqrt{A_0 f(z_2)}} = -\frac{d\lambda_2}{\sqrt{A_0 \Lambda_2}} \dots \dots \dots (4),$$

20. If we now introduce binomial coefficients into the equations discussed in the last two articles, and write

$$A_0 \lambda_1 = A_2 + 2\lambda,$$

we find

$$\left. \begin{aligned} A_0^2 V_2 &= \{A_0 A_4 - A_3^2 - 4\lambda^2 - 4\lambda A_2\} \\ A_0^2 V_1 &= 4\{A_0 A_3 - A_1 A_2 - 2\lambda A_1\} \\ A_0^2 V_0 &= 4\{A_0 A_2 - A_1^2 - \lambda A_0\} \\ \delta\lambda &= 0 \end{aligned} \right\} \dots \dots \dots (1).$$

Again, if we write

$$p_1 = \frac{a_2}{a_0}, \quad p_1 = \frac{2a_1}{a_0},$$

we find

$$\left. \begin{aligned} A_0 a_0 \pi_2 &= A_0 a_2 - A_2 a_0 - 2\lambda a_0 \\ A_0 a_0 \pi_1 &= 2(A_0 a_1 - A_1 a_0) \end{aligned} \right\} \dots \dots \dots (2).$$

and also we have

$$A_0^4 \Lambda_2 = 4A_0 \{ -4\lambda^3 + I\lambda - J \} \equiv 4A_0 \Lambda, \text{ say,}$$

whence
$$\Lambda_2 = \frac{4\Lambda}{A_0^3},$$

I and J being the invariants of the quartic.

The algebraic equation of Article 18 then becomes, when arranged in powers of λ ,

$$\begin{aligned} 4\lambda^3 (a_1^2 - a_0 a_2) + \lambda \{ A_0 a_2^2 - 4A_1 a_1 a_2 + 2A_2 (2a_1^2 + a_0 a_2) - 4A_3 a_0 a_1 + a_0^2 A_4 \} \\ + a_2^2 (A_1^2 - A_0 A_2) + a_1^2 (A_2^2 - A_0 A_4) + a_0^2 (A_3^2 - A_2 A_4) \\ + 2a_0 a_2 (A_2^2 - A_1 A_3) + 2a_0 a_1 (A_1 A_4 - A_2 A_3) = 0 \dots\dots\dots(3), \end{aligned}$$

an equation which vanishes identically when operated on by δ . At the same time the transcendental equation assumes the form

$$\frac{y_1 dx_1 - x_1 dy_1}{\sqrt{U(x_1, y_1)}} + \frac{y_2 dx_2 - x_2 dy_2}{\sqrt{U(x_2, y_2)}} = \frac{-d\lambda}{\sqrt{-4\lambda^3 + I\lambda - J}} \dots\dots\dots(4),$$

where we have put
$$z_1 = \frac{x_1}{y_1}, \quad z_2 = \frac{x_2}{y_2}.$$

21. Again, from Article 14, we have

$$\frac{z_1 dz_1}{\sqrt{A_0 f(z_1)}} + \frac{z_2 dz_2}{\sqrt{A_0 f(z_2)}} \pm \frac{r dr}{\sqrt{A_0 f(r)}} - \frac{d\theta}{\sqrt{A_0 \theta}} = 0;$$

now
$$\frac{dr}{\sqrt{A_0 f(r)}} = \frac{d\lambda}{\sqrt{\Lambda}}$$

by what precedes, and

$$V_0 r + V_1 = \sqrt{\Lambda_2} = \frac{2\sqrt{\Lambda}}{A_0 \sqrt{A_0}};$$

consequently
$$\frac{r dr}{\sqrt{A_0 f(r)}} = \left\{ \frac{V_0 r + V_1 - V_1}{V_0} \right\} \frac{d\lambda}{\sqrt{\Lambda}} \dots\dots\dots(1),$$

which, after a few reductions, we find to equal

$$-\frac{1}{\sqrt{A_0}} d \log (H - A_0 \lambda) - \frac{V_1}{2V_0} \frac{d\lambda}{\sqrt{\Lambda}},$$

where
$$H = A_0 A_2 - A_1^2.$$

We have also

$$\sqrt{A_0 \theta} = p_1 + \sqrt{A_0} \pi_1 = \sqrt{A_0} (\pi_1 + \sqrt{\pi_1^2 + V_0}),$$

so that we easily arrive at the formula

$$\begin{aligned} & \frac{x_1}{y_1} \left\{ \frac{y_1 dx_1 - x_1 dy_1}{\sqrt{U(x_1, y_1)}} \right\} + \frac{x_2}{y_2} \left\{ \frac{y_2 dx_2 - x_2 dy_2}{\sqrt{U(x_2, y_2)}} \right\} \\ & \pm \frac{1}{\sqrt{A_0}} d \log \left\{ \frac{(A_0 a_1 - A_1 a_0) + \sqrt{(A_0 a_1 - A_1 a_0)^2 + a_0^2 (H - A_0 \lambda)}}{a_0 \sqrt{H - A_0 \lambda}} \right\} \\ & - \frac{1}{2} \left\{ \frac{A_0 A_2 - A_1 A_2 - 2\lambda A_1}{H - A_0 \lambda} \right\} \frac{d\lambda}{\sqrt{\Lambda}} = 0 \dots\dots\dots (2). \end{aligned}$$

22. From the form of the equation (4) in Article 20, and from other considerations, it is easy to see that λ is an absolute invariant of the facients; and that consequently $\Delta\lambda$ vanishes identically.

If we now operate on equation (2) of the last article, bearing in mind that

$$\Delta I_r(z) = r I_{r+1}(z),$$

we obtain a formula which gives us the value of

$$\frac{x_1}{y_1^2} \left\{ \frac{y_1 dx_1 - x_1 dy_1}{\sqrt{U(x_1, y_1)}} \right\} + \frac{x_2}{y_2^2} \left\{ \frac{y_2 dx_2 - x_2 dy_2}{\sqrt{U(x_2, y_2)}} \right\},$$

in terms of $a_0, a_1, a_2, A_0, A_1, A_2, A_3, A_4$ and λ , and from which, by a second application of Δ , we can obtain a formula for $\Sigma dI_3(z)$.

Again, in equation (2) above referred to, if we interchange x and y and (replace the suffixes of the A 's and a 's by their complementary values) λ remains unaltered, and we find

$$\begin{aligned} & \frac{y_1}{x_1} \left\{ \frac{y_1 dx_1 - x_1 dy_1}{\sqrt{U(x_1, y_1)}} \right\} + \frac{y_2}{x_2} \left\{ \frac{y_2 dx_2 - x_2 dy_2}{\sqrt{U(x_2, y_2)}} \right\} \\ & = \frac{1}{\sqrt{A_4}} d \log \left\{ \frac{(A_4 a_1 - A_3 a_2) + \sqrt{(A_4 a_1 - A_3 a_2)^2 + a_2^2 (A_4 A_2 - A_3^2 - A_4 \lambda)}}{a_2 \sqrt{A_4 A_2 - A_3^2 - A_4 \lambda}} \right\} \\ & \quad - \frac{1}{2} \left\{ \frac{A_4 A_1 - A_3 A_2 - 2\lambda A_3}{A_4 A_2 - A_3^2 - A_4 \lambda} \right\} \frac{d\lambda}{\sqrt{\Lambda}}. \end{aligned}$$

If we wish to obtain $\Sigma dI_{-2}(z)$, we can operate on the above formula with δ , and so, having obtained $\Sigma dI_{-3}(z)$, we can, by repeated applications of the operator δ , obtain $\Sigma dI_{-r}(z)$, r being any positive integer.

23. We shall now discuss the case in which $m = 3$. Here we have three quantities z_1, z_2, z_3 , and two equations, which we may write

$$\left. \begin{aligned} \Sigma dI_0(z) &= 0 \\ \Sigma dI_1(z) &= 0 \end{aligned} \right\} \dots\dots\dots(1),$$

and two arbitrary constants λ_3 and λ_2 connected as follows :

$$\partial\lambda_3 = \lambda_2, \quad \partial\lambda_2 = P_1.$$

The source of the function $F(z)$ is

$$A_0(\pi_3^2 + V_4),$$

and the coefficients of $F(z)$, supposing it written in the form

$$F(z) \equiv F_4 + zF_3 + z^2F_2 + \dots z^4F_0,$$

are found, by application of ∂ , to be

$$\left. \begin{aligned} F'_4 &= A_0(\pi_3^2 + V_4) \\ F'_3 &= A_0(2\pi_3\pi_2 + V_3) \\ F'_2 &= A_0(\pi_2^2 + 2\pi_3\pi_1 + V_2) \\ F'_1 &= A_0(2\pi_2\pi_1 + V_1) \\ F'_0 &= A_0(\pi_1^2 + V_0) \end{aligned} \right\} \dots\dots\dots(2),$$

it being remembered that

$$\partial\pi_3 = \pi_2, \quad \partial\pi_2 = 2\pi_1, \quad \partial\pi_1 = 0.$$

The square-matrix is then at once seen to be

$$\left\| \begin{array}{l} F'_4 \quad 8F_4F_2 - 3F_3^2 \\ F'_3 \quad 4(6F_4F_1 - F_3F_2) \\ F'_2 \quad 2(3F_3F_1 + 24F_4F_0 - 2F_2^2) \\ F'_1 \quad 4(6F_3F_0 - F_2F_1) \\ F'_0 \quad 8F_2F_0 - 3F_1^2 \end{array} \right\| \dots\dots\dots(3),$$

giving us the well known conditions that a binary quartic should be a square.

From these we deduce

$$\frac{F'_4}{F'_0} = \frac{F_2^2}{F_1^2} \dots\dots\dots(4),$$

and, if we now operate on this equation with δ , we obtain

$$F_3 F_1^2 + 8F_4 F_1 F_0 - 4F_0 F_3 F_2 = 0 \dots \dots \dots (5).$$

We have consequently the two following rational and integral algebraic equations as equivalents of the transcendental system (1) of this article :

$$\left. \begin{aligned} \frac{\pi_3^2 + V_4}{\pi_1^2 + V_0} &= \left\{ \frac{2\pi_3 \pi_2 + V_3}{2\pi_2 \pi_1 + V_1} \right\}^2, \\ (2\pi_3 \pi_2 + V_3)(2\pi_2 \pi_1 + V_1)^2 + 8(\pi_3^2 + V_4)(\pi_1^2 + V_0)(2\pi_2 \pi_1 + V_1) \\ &- 4(\pi_1^2 + V_0)(2\pi_3 \pi_2 + V_3)(\pi_3^2 + 2\pi_3 \pi_1 + V_2) = 0 \end{aligned} \right\} \dots (6).$$

I merely give these two equations as an example of the method, an immense number being easily obtained by combining the various results flowing from the square-matrix.

We have also the transcendental forms

$$\left. \begin{aligned} \Sigma dI_0(z) + \frac{dr_1}{\sqrt{A_0 f(r_1)}} + \frac{dr_2}{\sqrt{A_0 f(r_2)}} &= 0 \\ \Sigma dI_1(z) + \frac{r_1 dr_1}{\sqrt{A_0 f(r_1)}} + \frac{r_2 dr_2}{\sqrt{A_0 f(r_2)}} &= 0 \end{aligned} \right\} \dots \dots \dots (7),$$

r_1 and r_2 being a pair of roots of the equation

$$V(z) = 0.$$

In this case I have not succeeded in expressing in terms of λ_3 and λ_2 the sums

$$\frac{dr_1}{\sqrt{A_0 f(r_1)}} + \frac{dr_2}{\sqrt{A_0 f(r_2)}},$$

$$\frac{r_1 dr_1}{\sqrt{A_0 f(r_1)}} + \frac{r_2 dr_2}{\sqrt{A_0 f(r_2)}}.$$

By a process of reasoning similar to that employed in Articles 21 and 22, we can obtain $\Sigma dI_r(z)$ in terms of r_1 and r_2 , the p 's, and λ_3 and λ_2 , r being any positive or negative integer.

24. For the case in which $n = 4$, we have four quantities z_1, z_2, z_3, z_4 , and three equations, viz.,

$$\Sigma dI_0(z) = 0,$$

$$\Sigma dI_1(z) = 0,$$

$$\Sigma dI_3(z) = 0,$$

and three arbitrary constants $\lambda_4, \lambda_3, \lambda_2$ connected as follows :

$$\delta\lambda_4 = \lambda_3, \quad \delta\lambda_3 = 2\lambda_2, \quad \delta\lambda_2 = \frac{3P_1}{2}.$$

Now the source of $F(z)$ is

$$A_0(\pi_4^2 + V_0),$$

and the coefficients of $F(z)$ found by repeated applications of the operator δ are found as follows :—

$$F_6 = A_0(\pi_4^2 + V_0),$$

$$F_5 = A_0(2\pi_4\pi_3 + V_0),$$

$$F_4 = A_0(2\pi_4\pi_2 + \pi_3^2 + V_0),$$

$$F_3 = A_0(2\pi_2\pi_3 + 2\pi_4\pi_1 + V_0),$$

$$F_2 = A_0(2\pi_3\pi_1 + \pi_2^2 + V_0),$$

$$F_1 = A_0(2\pi_2\pi_1 + V_0),$$

$$F_0 = A_0(\pi_1^2 + V_0).$$

If we now form, by the general rule, the square-matrix for $F(z)$, we find it to be

$$\begin{vmatrix} F_6 & 0 & 0 & H_8 \\ F_5 & F_0 & 0 & H_7 \\ F_4 & F_5 & F_0 & H_6 \\ F_3 & F_4 & F_5 & H_5 \\ F_2 & F_3 & F_4 & H_4 \\ F_1 & F_2 & F_3 & H_3 \\ F_0 & F_1 & F_2 & H_2 \\ 0 & F_0 & F_1 & H_1 \\ 0 & 0 & F_0 & H_0 \end{vmatrix},$$

where $H_0, H_1, \dots, H_8$ are the coefficients of the Hessian of $F(z)$ written without binomial coefficients.

We thus find for the equation the weight of which is the greatest

the following form :

$$\begin{vmatrix} F_0 & 0 & 0 & 9F_0F_4-5F_5^2 \\ F_5 & F_0 & 0 & 27F_0F_3-11F_5F_4 \\ F_4 & F_5 & F_0 & 54F_0F_2-11F_4^2-3F_5F_3 \\ F_3 & F_4 & F_5 & 90F_0F_1+12F_5F_2-24F_4F_3 \end{vmatrix} = 0 \dots\dots\dots(1),$$

and from this equation by successive operations of δ all the rest can be obtained by the rules already indicated in Article 6.

A simpler form can, however, easily be found by forming a determinant from the first two and last two rows of the square-matrix, giving us

$$F_6^2(F_1H_0-F_0H_1) = F_0^2(F_5H_3-F_3H_7),$$

$$\text{or} \quad \frac{F_0^2}{F_6^2} = \frac{20F_0F_5F_4-5F_5^3-27F_6^2F_3}{20F_2^2F_1F_0-5F_1^3-27F_0^2F_3} \dots\dots\dots(2).$$

We have also

$$\begin{aligned} \pm \Sigma dI_0(z) + \frac{dr_1}{\sqrt{A_0f(r_1)}} + \frac{dr_2}{\sqrt{A_0f(r_2)}} + \frac{dr_3}{\sqrt{A_0f(r_3)}} &= 0, \\ \pm \Sigma dI_1(z) + \frac{r_1dr_1}{\sqrt{A_0f(r_1)}} + \frac{r_2dr_2}{\sqrt{A_0f(r_2)}} + \frac{r_3dr_3}{\sqrt{A_0f(r_3)}} &= 0, \\ \pm \Sigma dI_2(z) + \frac{r_1^2dr_1}{\sqrt{A_0f(r_1)}} + \frac{r_2^2dr_2}{\sqrt{A_0f(r_2)}} + \frac{r_3^2dr_3}{\sqrt{A_0f(r_3)}} &= 0, \end{aligned}$$

r_1, r_2, r_3 being three roots of $V(z) = 0$.

25. We now proceed to investigate some forms and properties of the algebraic integrals when the arbitrary constants are simultaneous values of the z 's, when

$$t-t^0 = 0.$$

Let $\zeta_1, \zeta_2, \dots \zeta_m$ be the corresponding values of $z_1, z_2, \dots z_m$, when $t-t^0$ is zero; we shall have then

$$\Sigma I_i(z) - \Sigma I_i(\zeta) = 0 \dots\dots\dots(1)$$

for all integer values of i from $i = 0$ to $i = m-2$, and in addition we have the equation

$$\Sigma I_{m-1}(z) - \Sigma I_{m-1}(\zeta) = t-t^0 \dots\dots\dots(2).$$

It easily follows from these equations that $t-t^0$ is a function of the differences of the facients involved in these formulæ, namely the z 's, the ζ 's, and the α 's, for we find

$$\delta(t-t_0) = 0.$$

We shall now integrate the fundamental equation, writing it in the form

$$\dot{\pi}_m^2 = A_0(\pi_m^2 + V_{2m-2});$$

we have then

$$\frac{d\pi_m}{\sqrt{\pi_m^2 + V_{2m-2}}} = \sqrt{A_0} dt,$$

and by integration find

$$\sqrt{A_0}(t-t^0) = \log \left\{ \frac{\pi_m + \sqrt{\pi_m^2 + V_{2m-2}}}{c} \right\} \dots\dots\dots(3),$$

c being a constant.

If we now suppose $\zeta_1, \zeta_2, \dots \zeta_m$ to be the roots of the equation

$$\zeta^m + q_1\zeta^{m-1} + q_2\zeta^{m-2} + \dots q_m = 0,$$

it is clear that, when $t-t^0 = 0$, $p_i = q_i$,

and also

$$\dot{p}_i = \dot{q}_i,$$

the q 's being differentiated with regard to t^0 .

Again, if we write

$$\kappa_i = q_i - \lambda_i,$$

we must have

$$\pi_i = \kappa_i,$$

when

$$t-t^0 = 0,$$

as well as

$$\dot{\pi}_i = \dot{\kappa}_i,$$

from which it follows that the constant

$$c = \kappa_m + \sqrt{\kappa_m^2 + V_{2m-2}};$$

introducing this value of c into equation (3), we obtain

$$\sqrt{A_0}(t-t_0) = \log \left\{ \frac{\pi_m + \sqrt{\pi_m^2 + V_{2m-2}}}{\kappa_m + \sqrt{\kappa_m^2 + V_{2m-2}}} \right\} \dots\dots\dots(4),$$

a formula we may also write in the more convenient form

$$\sqrt{A_0}(t-t_0) = \log \left\{ \frac{\sqrt{A_0} \pi_m + \dot{\pi}_m}{\sqrt{A_0} \kappa_m + \dot{\kappa}_m} \right\} \dots\dots\dots(5).$$

26. From the value of $t-t^0$ given in the last article, we find

$$e^{\sqrt{A_0}(t-t^0)} = \frac{\sqrt{A_0} \pi_m + \dot{\pi}_m}{\sqrt{A_0} \kappa_m + \dot{\kappa}_m} \dots\dots\dots (1),$$

from which equation we obtain

$$\frac{e^{\sqrt{A_0}(t-t^0)} - 1}{e^{\sqrt{A_0}(t-t^0)} + 1} = T, \text{ say, } = \frac{\sqrt{A_0}(\pi_m - \kappa_m) + \dot{\pi}_m - \dot{\kappa}_m}{\sqrt{A_0}(\pi_m + \kappa_m) + \dot{\pi}_m + \dot{\kappa}_m}.$$

Now, since the κ 's are simultaneous values of the π 's, when $t-t^0$ is zero, we have

$$\dot{\kappa}_m^2 = A_0(\kappa_m^2 + V_{2m-1}),$$

and consequently $\dot{\pi}_m^2 - \dot{\kappa}_m^2 = A_0(\pi_m^2 - \kappa_m^2),$

or
$$\frac{\dot{\pi}_m - \dot{\kappa}_m}{\sqrt{A_0}(\pi_m - \kappa_m)} = \frac{\sqrt{A_0}(\pi_m + \kappa_m)}{\pi_m + \kappa_m},$$

from which it easily follows that

$$T = \frac{\sqrt{A_0}(\pi_m - \kappa_m)}{\pi_m + \kappa_m} = \frac{\dot{\pi}_m - \dot{\kappa}_m}{\sqrt{A_0}(\pi_m + \kappa_m)}.$$

This function of $t-t^0$ which we have called T plays an important part in the theory of Abelian functions and integrals, and obviously vanishes when

$$t-t^0 = 0,$$

and when
$$t-t^0 = \frac{2i\pi}{\sqrt{A_0}},$$

becoming infinite for
$$t-t^0 = \frac{i\pi}{\sqrt{A_0}},$$

but for no real values of $t-t^0$.

27. By Article 7, we have

$$\frac{dz}{dt} = \frac{\sqrt{A_0 f(z)}}{\phi'(z)};$$

consequently $\dot{p}_i$ and $\dot{q}_i$ are expressible in terms of the z 's and the ζ 's respectively, and therefore we are enabled to express T in terms of the z 's and ζ 's, by means of the formula

$$T = \sqrt{A_0} \left\{ \frac{p_m - q_m}{\dot{p}_m + \dot{q}_m} \right\} \dots\dots\dots (1),$$

which the reader will observe does not contain the λ 's. By operating on this formula with δ , bearing in mind that

$$\delta T = 0,$$

since T is a function of the differences of the facients $z_1, z_2, \dots z_m$; $\zeta_1, \zeta_2, \dots \zeta_m$; $a_1, a_2, \dots a_{2m}$, we arrive at the important system of equations

$$T = \frac{\sqrt{A_0(p_m - q_m)}}{\dot{p}_m + \dot{q}_m} = \frac{\sqrt{A_0(p_{m-1} - q_{m-1})}}{\dot{p}_{m-1} + \dot{q}_{m-1}} = \dots = \frac{\sqrt{A_0(p_1 - q_1)}}{\dot{p}_1 + \dot{q}_1} \dots (2),$$

which affords us a ready means of obtaining a number of algebraic integrals involving simultaneous values of the variables $z_1, z_2, \dots z_m$, which are algebraic equivalents of the transcendental system

$$\Sigma I_i(z) - \Sigma I_i(\zeta) = 0.$$

We have also the system

$$\begin{aligned} T &= \frac{\dot{p}_m - \dot{q}_m}{\sqrt{A_0(p_m + q_m - 2\lambda_m)}} = \frac{\dot{p}_{m-1} - \dot{q}_{m-1}}{\sqrt{A_0(p_{m-1} + q_{m-1} - 2\lambda_{m-1})}} = \dots \\ &= \frac{\dot{p}_1 - \dot{q}_1}{\sqrt{A_0(p_1 + q_1 - P_1)}} \dots \dots \dots (3). \end{aligned}$$

This latter system enables us to express the λ 's linearly in terms of the z 's and ζ 's.

28. We shall now derive a new and important system of equations from the value of T given in the last article, viz.,

$$T = \frac{\sqrt{A_0(p_m - q_m)}}{\dot{p}_m + \dot{q}_m}.$$

If we diminish all the facients in the above expression by ζ_1 , say, ζ_1 being the value of z_1 when

$$t - t^0 = 0,$$

and call

$$(z_1 - \zeta_1)(z_2 - \zeta_1) \dots (z_m - \zeta_1) = u(\zeta_1),$$

or, simply u if we have no occasion to put in evidence the element ζ_1 , it is clear that

$$\left. \begin{aligned} p_m &\text{ becomes } (-1)^m u(\zeta_1) \\ \dot{p}_m &\text{ ,, } (-1)^m \frac{du(\zeta_1)}{dt} \\ q_m &\text{ ,, zero} \\ \dot{q}_m &\text{ ,, } -\sqrt{A_0} f(\zeta_1) \end{aligned} \right\} \dots \dots \dots (1),$$

while T remains unaltered; we then find, introducing these values into the above formula, having first multiplied by $\dot{p}_m + \dot{q}_m$,

$$T \left\{ \frac{du(\zeta_1)}{dt} + (-1)^{m-1} \sqrt{A_0 f(\zeta_1)} \right\} = \sqrt{A_0} u(\zeta_1) \dots\dots\dots (2),$$

it being understood, of course, that differentiation with respect to t does not affect ζ_1 .

By substituting in succession for ζ_1 the remaining $m-1$ values of ζ , we obtain $m-1$ other equations, giving us, in all, m equations of which the following is typical:

$$T \left\{ d \frac{u(\zeta)}{dt} + (-1)^{m-1} \sqrt{A_0 f(\zeta)} \right\} = \sqrt{A_0} u(\zeta) \dots\dots\dots (3).$$

Now, if we call

$$(\zeta_1 - z)(\zeta_2 - z) \dots (\zeta_m - z) \equiv v(z),$$

z standing for any root of $\phi(z) = 0$,

and interchange corresponding z 's and ζ 's, it is clear that $t - t^0$ changes sign, and consequently T , while $u(\zeta)$ becomes $v(z)$, and therefore from the above equation we are led to the following:

$$T \left\{ \frac{dv(z)}{dt_0} + (-1)^{m-1} \sqrt{A_0 f(z)} \right\} = -\sqrt{A_0} v(z) \dots\dots\dots (4),$$

from which it appears that, if one of the z 's becomes equal to one of the ζ 's, we must have $T = 0$, and consequently $p_i = q_i$.

29. These equations are of considerable value, as we shall show by what follows; in the first place, they enable us to obtain readily an algebraic equation which contains only *two* of the quantities $z_1, z_2, \dots, z_m$. This equation we obtain by eliminating T between two equations of the type (4) in the last article, and find

$$\begin{aligned} & \left\{ \frac{dv(z_1)}{dt_0} + (-1)^{m-1} \sqrt{A_0 f(z_1)} \right\} v(z_2) \\ &= \left\{ \frac{dv(z_2)}{dt_0} + (-1)^{m-1} \sqrt{A_0 f(z_2)} \right\} v(z_1), \end{aligned}$$

into which equation only enter z_1, z_2 , and the ζ 's.

30. By comparison of the coefficients of z^{m-1} in (1) and (2) of Article 13, we find

$$\dot{p}_m + \sqrt{A_0} (p_m - \lambda_m) = \theta \sqrt{A_0} R_{m-1} \dots \dots \dots (1),$$

$$\dot{p}_m - \sqrt{A_0} (p_m - \lambda_m) = \frac{V_0 \sqrt{A_0} S_{m-1}}{\theta} \dots \dots \dots (2),$$

from which we obtain

$$2 (p_m - \lambda_m) = \theta R_{m-1} - \frac{V_0 S_{m-1}}{\theta} \dots \dots \dots (3),$$

$$2 \frac{\dot{p}_m}{\sqrt{A_0}} = \theta R_{m-1} + \frac{V_0 S_{m-1}}{\theta} \dots \dots \dots (4)$$

We have also, from (3) of Article 11,

$$\frac{\ddot{p}_m}{A_0} = p_m - \lambda_m \dots \dots \dots (5).$$

If we now put $t = t^0$, θ becomes θ^0 , and p_m , $\dot{p}_m$, and $\ddot{p}_m$ go into q_m , $\dot{q}_m$, and $\ddot{q}_m$, respectively, and we have

$$\left. \begin{aligned} 2 (q_m - \lambda_m) &= \theta^0 R_{m-1} - \frac{V_0 S_{m-1}}{\theta^0} \\ 2 \frac{\dot{q}_m}{\sqrt{A_0}} &= \theta^0 R_{m-1} + \frac{V_0 S_{m-1}}{\theta^0} \\ 2 \frac{\ddot{q}_m}{A_0} &= \theta^0 R_{m-1} - \frac{V_0 S_{m-1}}{\theta^0} \end{aligned} \right\} \dots \dots \dots (6),$$

from which we find

$$\left. \begin{aligned} R_{m-1} &= \left(\frac{\dot{q}_m}{\sqrt{A_0}} + \frac{\ddot{q}_m}{A_0} \right) \frac{1}{\theta^0} \\ V_0 S_{m-1} &= \left(\frac{\dot{q}_m}{\sqrt{A_0}} - \frac{\ddot{q}_m}{A_0} \right) \theta^0 \end{aligned} \right\} \dots \dots \dots (7),$$

and, introducing these values into the equation

$$2 (p_m - q_m) = (\theta - \theta^0) R_{m-1} + \frac{(\theta - \theta^0)}{\theta \theta^0} V_0 S_{m-1},$$

we find easily

$$2 (p_m - q_m) = \left\{ \frac{\theta^2 - \theta^{02}}{\theta \theta^0} \right\} \frac{\dot{q}_m}{\sqrt{A_0}} + \frac{(\theta - \theta^0)^2}{\theta \theta^0} \frac{\ddot{q}_m}{A_0} \dots \dots \dots (8),$$

or, since

$$T = \frac{\theta - \theta^0}{\theta + \theta^0},$$

$$(p_m - q_m)(1 - T^2) = 2T \frac{\dot{q}_m}{\sqrt{A_0}} + 2T^2 \frac{\ddot{q}_m}{A_0} \dots\dots\dots (9).$$

We can operate on this equation with δ and readily deduce

$$(p_i - q_i)(1 - T^2) = 2T \frac{\dot{q}_i}{\sqrt{A_0}} + 2T^2 \frac{\ddot{q}_i}{A_0} \dots\dots\dots (10),$$

which holds for any integer value of i from $i = 1$ to $i = m$.

The form of the typical equation (10) shows us that we can, by elimination of $1 - T^2$, T and T^2 , obtain $m - 2$ independent linear equations connecting the p 's with the constants $\zeta_1, \zeta_2, \dots, \zeta_m$; the $m - 1^{\text{th}}$ equation will involve the p 's in the second degree, as has been shown by Jacobi.

31. We shall next investigate the value of Δp_i , Δ being the operator referred to in Article 3. We have, from Article 8,

$$\dot{p}_m + z\dot{p}_{m-1} + z^2\dot{p}_{m-2} + \dots + z^{m-1}\dot{p}_1 = -\sqrt{A_0 f(z)},$$

for all values of z which make $\phi(z)$ vanish. If we now write

$$z = \frac{x}{y}, \quad a_r = \frac{a_r}{b_r},$$

where a_r is a root of $f(z) = 0$,

and r can have any integer value from $r = 1$ to $r = 2m$, we find, introducing these values,

$$y^m \dot{p}_m + y^{m-1} x \dot{p}_{m-1} + y^{m-2} x^2 \dot{p}_{m-2} + \dots + y x^{m-1} \dot{p}_1 = -\sqrt{U(x, y)}.$$

If we now operate on this equation with Δ , we readily obtain

$$y^m \Delta \dot{p}_m + y^{m-1} x \Delta \dot{p}_{m-1} + y^{m-2} x^2 \Delta \dot{p}_{m-2} + \dots + y x^{m-1} \Delta \dot{p}_1 \\ - \{m y^{m-1} x \dot{p}_m + (m-1) y^{m-2} x^2 \dot{p}_{m-1} + \dots + x^m \dot{p}_1\} = 0.$$

We must now substitute for x^m its value derived from the equation

$$x^m + p_1 x^{m-1} y + p_2 x^{m-2} y^2 + \dots + p_m y^m = 0,$$

and then equate to zero the coefficients of $x^{m-1}y, x^{m-2}y^2, \dots$, as explained in the rule laid down in Article 1.

We then arrive at the values of $\Delta p_m, \Delta p_{m-1}, \dots, \Delta p_1$, which are given by the following formulæ:

$$\left. \begin{aligned} \Delta p_m + p_1 p_m &= 0 \\ \Delta p_{m-1} - m p_m + p_1 p_{m-1} &= 0 \\ \Delta p_{m-2} - (m-1) p_{m-1} + p_1 p_{m-2} &= 0 \\ \dots &\dots \dots \dots \dots \\ \Delta p_1 - 2 p_2 + p_1 p_1 &= 0 \end{aligned} \right\} \dots\dots\dots(1).$$

We have also obviously

$$\Delta p_i = (i+1) p_{i+1} - p_i p_i$$

for all integer values of i from $i = 1$ to $i = m-1$.

32. If we now express $t-t^0$ in terms of T , we find

$$\sqrt{A_0} (t-t^0) = \log \left\{ \frac{1+T}{1-T} \right\},$$

and consequently we learn that

$$\Sigma I_{m-1}(z) - \Sigma I_{m-1}(\zeta) = \frac{1}{\sqrt{A_0}} \log \left\{ \frac{1+T}{1-T} \right\} = t-t^0 \dots\dots\dots(1).$$

We shall now find the value of

$$\Sigma I_m(z) - \Sigma I_m(\zeta).$$

To do so we have but to operate on the equation above given with Δ , and we find, since by Article 5,

$$\Delta I_r(z) = -(m-r-2) I_{r+1}(z),$$

$$\Sigma I_m(z) - \Sigma I_m(\zeta) = \Delta(t-t^0) \dots\dots\dots(2),$$

and the problem is reduced to finding $\Delta(t-t^0)$, or, what is the same thing, ΔT , the value of which we now proceed to investigate.

$$\text{We have } T = \frac{\sqrt{A_0}(p_1 - q_1)}{p_1 + q_1} = \frac{p_1 - q_1}{\sqrt{A_0}(p_1 + q_1 - P_1)};$$

hence

$$\begin{aligned} \Delta T &= \sqrt{A_0} \{ (p_1 + q_1) \Delta(p_1 - q_1) - (p_1 - q_1) \Delta(p_1 + q_1) \} / (p_1 + q_1)^2 \\ &\quad + \frac{1}{2} (p_1 - q_1) \sqrt{A_0} P_1 / (p_1 + q_1), \end{aligned}$$

which, after a few reductions, we find can be expressed in the form

$$\Delta T = -\frac{T}{2} (1-T^2)(p_1 + q_1 - P_1) = -\frac{T}{2} (1-T^2)(\pi_1 + \kappa_1) \dots(3);$$

operating now with Δ on the equation

$$\sqrt{A_0} (t-t^0) = \log \left\{ \frac{1+T}{1-T} \right\},$$

we find
$$\sqrt{A_0} \Delta (t-t^0) + \sqrt{A_0} (t-t^0) \frac{P_1}{2} = \frac{2\Delta T}{1-T^2} \dots\dots\dots(4),$$

and consequently,

$$\begin{aligned} \Delta (t-t^0) &= \Sigma I_m (z) - \Sigma I_m (\zeta) = -\frac{P_1}{2} (t-t^0) - \frac{T(\pi_1 + \kappa_1)}{\sqrt{A_0}} \\ &= -\frac{P_1}{2} (t-t^0) - \frac{(\pi_1 + \kappa)}{A_0} \dots\dots(5) \end{aligned}$$

We can now, by a second operation of Δ on the above equation, determine

$$\Sigma I_{m+1} (z) - \Sigma I_{m+1} (\zeta),$$

since we know the value of $\Delta (t-t^0)$, whence it follows that we can obtain the value of

$$\Sigma I_r (z) - \Sigma I_r (\zeta),$$

r being any positive integer, by continued application of the operator Δ .

33. We now proceed to find the value of

$$\Sigma I_{-1} (z) - \Sigma I_{-1} (\zeta),$$

which we shall denote by $r-r^0$, remarking that when we interchange each x^i and its corresponding y , and each a and its corresponding b , $t-t^0$ passes into $-(r-r^0)$, and T into $-T$, $\mathfrak{A}$ corresponding to $r-r^0$ in the same way that T does to $t-t^0$; we have, z standing for any root of

$$\phi (z) = 0,$$

$$\frac{az}{dt} = \frac{\sqrt{A_0 f(z)}}{\phi'(z)} = \frac{\sqrt{U(x, y)}}{\Phi'(x, y)} \left\{ \frac{y_1 y_2 \dots y_n}{y^3} \right\},$$

where

$$\Phi' (x, y) \equiv \phi' (z) (y_1 y_2 \dots y_n) y^{n-2}.$$

If we now interchange z and y , a and b , $U(x, y)$ remains unaltered,

and $\Phi'(x, y)$ becomes $(-1)^{m-1} \Phi'(x, y)$; hence $\frac{dz}{dt}$ becomes

$$\begin{aligned} & (-1)^{m-1} \frac{\sqrt{U(x, y)}}{\Phi'(x, y)} \left\{ \frac{x_1 x_2 \dots x_m}{x^3} \right\} \\ &= (-1)^{m-1} \frac{\sqrt{U(x, y)}}{\Phi'(x, y)} \left\{ \frac{y_1 y_2 \dots y_m}{y^3} \right\} \left\{ \frac{x_1 x_2 \dots x_m}{y_1 y_2 \dots y_m} \right\} \frac{y^3}{x^3} \\ &= (-1)^{m-1} \frac{dz}{dt} (-1)^m \frac{p_m}{z^3} = - \left(\frac{dz}{dt} \right) \frac{p_m}{z^3} = \frac{d}{dt} \left(\frac{1}{z} \right) p_m, \end{aligned}$$

Consequently dt becomes $\frac{dt}{p_m}$, and $I_r(z)$ becomes

$$-I_{m-r-2}(z).$$

Hence we have the following scheme of transformation :

$$\left. \begin{array}{lll} x \text{ into } y, & dt \text{ becomes } \frac{dt}{p_m} \\ a \text{ ,, } b, & \dot{p}_m \text{ ,, } -\frac{\dot{p}_m}{p_m} \\ z \text{ ,, } \frac{1}{z}, & T \text{ ,, } -\mathfrak{I} \\ a \text{ ,, } \frac{1}{a}, & \lambda_m \text{ ,, } \frac{\lambda_m}{P_{2m}} \\ t-t^0 \text{ ,, } -(r-r^0), & \dot{p}_r = \frac{p_m \dot{p}_{m-r} - p_{m-r} \dot{p}_m}{p_m} \end{array} \right\} \dots\dots\dots(1),$$

where
$$\mathfrak{I} = \frac{e^{\sqrt{A_{2m}}(r-r^0)} - 1}{e^{\sqrt{A_{2m}}(r-r^0)} + 1}.$$

Now $\mathfrak{I}$ is known, for T , which is equal to

$$\frac{\sqrt{A_0}(p_m - q_m)}{\dot{p}_m + \dot{q}_m},$$

becomes, in consequence of the interchanges above spoken of,

$$- \frac{\sqrt{A_{2m}}(p_m - q_m)}{\dot{p}_m q_m + p_m \dot{q}_m},$$

from which we learn that

$$\mathfrak{I} = \frac{\sqrt{A_{2m}}(p_m - q_m)}{\dot{p}_m q_m + p_m \dot{q}_m} \dots\dots\dots(2).$$

Consequently $\tau - \tau^0$ is known from the formula

$$\sqrt{A_{2m}} (\tau - \tau^0) = \log \left\{ \frac{1 + \mathfrak{F}}{1 - \mathfrak{F}} \right\} \dots\dots\dots (3),$$

$$\text{and, finally, } \Sigma I_{-1}(z) - \Sigma I_{-1}(\xi) = -\frac{1}{\sqrt{A_{2m}}} \log \left\{ \frac{1 + \mathfrak{F}}{1 - \mathfrak{F}} \right\} \dots\dots\dots (4).$$

34. We shall now proceed to find the value of

$$\Sigma I_{-2}(z) - \Sigma I_{-2}(\xi),$$

and to do so we shall express $\mathfrak{F}$ in terms of T , p_m , and q_m . From the fundamental equation, we have

$$\left. \begin{aligned} p_m^2 &= A_0 (p_m^2 - 2\lambda_m p_m + P_{2m}) \\ q_m^2 &= A_0 (q_m^2 - 2\lambda_m q_m + P_{2m}) \end{aligned} \right\} \dots\dots\dots (1).$$

Now, if we multiply the first of these equations by q_m^2 , and the second by p_m^2 , and then add, we find

$$p_m^2 q_m^2 + p_m^2 q_m^2 = A_0 \{ 2p_m^2 q_m^2 - 2\lambda_m p_m q_m (p_m + q_m) + P_{2m} (p_m^2 + q_m^2) \} \dots\dots\dots (2),$$

and, if we square the equation

$$\dot{p}_m + \dot{q}_m = \sqrt{A_0} T^{-1} (p_m - q_m),$$

$$\text{we obtain } 2\dot{p}_m \dot{q}_m = A_0 T^{-2} (p_m - q_m)^2 - \dot{p}_m^2 - \dot{q}_m^2,$$

and substituting the values of $\dot{p}_m^2$ and $\dot{q}_m^2$ from (1), we have

$$2\dot{p}_m \dot{q}_m = A_0 \{ T^{-2} (p_m - q_m)^2 - (p_m^2 + q_m^2) + 2\lambda_m (p_m + q_m) - 2P_{2m} \} \dots\dots\dots (3).$$

If we now multiply (3) by $p_m q_m$, and add to (1), we obtain

$$(p_m \dot{q}_m + \dot{p}_m q_m)^2 = A_0 \{ (T^{-2} - 1) p_m q_m + P_{2m} \} (p_m - q_m)^2 \dots\dots (4),$$

from which we find easily

$$\mathfrak{F}^{-2} - 1 = (T^{-2} - 1) \frac{A_0 p_m q_m}{A_{2m}} \dots\dots\dots (5).$$

If we now operate on this equation (5) with δ , we obtain

$$-\frac{2\delta\mathfrak{F}}{\mathfrak{F}^3} = \frac{A_0 (T^{-2} - 1)}{A_{2m}^2} \{ A_{2m} (p_m q_{m-1} + p_{m-1} q_m) - 2A_{2m-1} p_m q_m \};$$

$$\text{or } 2\delta\mathfrak{F} = -\mathfrak{F} (1 - \mathfrak{F}^2) \left\{ \frac{p_{m-1}}{p_m} + \frac{q_{m-1}}{q_m} - \frac{P_{2m-1}}{P_{2m}} \right\} \dots\dots\dots (6).$$

Now, $(\tau - \tau^0) \sqrt{A_{2m}} = \log \left\{ \frac{1 + \mathfrak{X}}{1 - \mathfrak{X}} \right\}.$

Operating on this equation with δ , we find

$$\sqrt{A_{2m}} \delta (\tau - \tau^0) + (\tau - \tau^0) \frac{2m A_{2m-1}}{2 \sqrt{A_{2m}}} = \frac{2 \delta \mathfrak{X}}{1 - \mathfrak{X}^2},$$

and, substituting the above value of $\delta T'$, we obtain

$$\sqrt{A_{2m}} \delta (\tau - \tau^0) = \mathfrak{X} \left\{ \frac{P_{2m-1}}{P_{2m}} - \frac{p_{m-1}}{p_m} - \frac{q_{m-1}}{q_m} \right\} \frac{-(\tau - \tau^0)}{2} \sqrt{A_{2m}} \times \frac{P_{2m-1}}{P_{2m}} \dots \dots \dots (7),$$

and, finally,

$$\Sigma I_{-2}(z) - \Sigma I_{-2}(\zeta) = \frac{\mathfrak{X}}{\sqrt{A_{2m}}} \left\{ \frac{P_{2m-1}}{P_{2m}} - \frac{p_{m-1}}{p_m} - \frac{q_{m-1}}{q_m} \right\} \frac{-(\tau - \tau^0)}{2} \frac{P_{2m-1}}{P_{2m}} \dots \dots \dots (8).$$

Having now obtained $\tau - \tau^0$ and $\delta (\tau - \tau^0)$, we can find $\delta^2 (\tau - \tau^0)$, and so, by repeated operations of δ , arrive at the value of $\delta^r (\tau - \tau^0)$. But the reader will perceive that this work is unnecessary if we have already arrived at the value of $\Delta^r (t - t^0)$; for the values of $\delta^r (\tau - \tau^0)$ and $\Delta^r (t - t^0)$ are not independent, but can be derived, one from the other, by means of the interchange of x and y , a and b , as noticed in Article 33.

If, then, we have calculated the ascending series

$$\Sigma I_m(z) - \Sigma I_m(\zeta), \quad \Sigma I_{m+1}(z) - \Sigma I_{m+1}(\zeta), \quad \Sigma I_{m+r}(z) - \Sigma I_{m+r}(\zeta),$$

the descending series

$$\Sigma I_{-2}(z) - \Sigma I_{-2}(\zeta), \quad \dots \quad \Sigma I_{-r-2}(z) - \Sigma I_{-r-2}(\zeta)$$

can be found from the rules laid down for transformation in Article 33.

35. It remains to show how to find the value of

$$\Sigma L(z, n) - \Sigma L(\zeta, n),$$

where $L(z, n)$ is, as before defined,

$$\int^s \frac{dz}{(z-n) \sqrt{A_0 f(z)}}.$$

Now, it is clear that $I_{-1}(z)$ passes into $L(z, n)$ when z and each

of the α 's is diminished by the same quantity n ; when this is the case,

$$\begin{aligned} A_0 & \text{ becomes } A_0 = N_0, \text{ say,} \\ A_1 & \quad ,, \quad A_1 + nA_0 = N_1, \\ A_2 & \quad ,, \quad A_2 + 2nA_1 + n^2A_0 = N_2, \\ \dots & \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ A_{2m} & \quad ,, \quad A_{2m} + 2mnA_{2m-1} + \dots n^{2m}A_0 = N_{2m}, \end{aligned}$$

and, adopting the notation of Article 28, p_m becomes $(-1)^m u(n)$, and q_m , in like manner, becomes $(-1)^m v(n)$.

If, now, in the formula of Article 33, viz.,

$$\tau - \tau^0 = \Sigma I_{-1}(z) - \Sigma I_{-1}(\zeta) = \frac{1}{\sqrt{A_{2m}}} \log \left\{ \frac{1 + \mathfrak{I}}{1 - \mathfrak{I}} \right\},$$

we suppose the z 's, ζ 's, and α 's all diminished by n , we find

$$\Sigma L(z, n) - \Sigma L(\zeta, n) = \frac{1}{\sqrt{N_{2m}}} \log \left\{ \frac{1 + \mathfrak{I}_n}{1 - \mathfrak{I}_n} \right\} \dots \dots \dots (1),$$

where $\mathfrak{I}_n$ denotes the value of $\mathfrak{I}$ when the facients are diminished by n , and is easily seen to assume the form

$$\mathfrak{I}_n = \frac{(-1)^m \sqrt{N_{2m}} \{u(n) - v(n)\}}{u(n) \frac{dv(n)}{dt^0} + v(n) \frac{du(n)}{dt}}.$$

36. We shall now examine the value of

$$\Sigma L(z, \alpha) - \Sigma L(\zeta, \alpha),$$

a particular case of the formula discussed in the last article, in which $n = \alpha$, a root of

$$f(z) = 0.$$

It is evident that N_{2m} vanishes in this case, since $n = \alpha$, and therefore, in order to obtain the value of

$$\Sigma L(z, \alpha) - \Sigma L(\zeta, \alpha),$$

we must evaluate $\frac{1}{\sqrt{N_{2m}}} \log \left\{ \frac{1 + \sqrt{N_{2m}} D_n}{1 - \sqrt{N_{2m}} D_n} \right\}$,

when $\sqrt{N_{2m}} = 0$,

where we have written for the sake of brevity

$$D_n = \frac{(-1)^n \{u(n) - v(n)\}}{u(n) \frac{dv(n)}{dt^0} + v(n) \frac{du(n)}{dt}},$$

and we readily perceive that the value we seek is $2D_n$, when $n = a$; hence we have

$$\Sigma L(z, a) - \Sigma L(\zeta, a) = 2D_a \dots\dots\dots (1).$$

If one of the ζ 's happened to be equal to a , then $v(a)$ vanishes, and D_a assumes the very simple form

$$D_a = \frac{(-1)^a}{\frac{dv(a)}{dt^0}}.$$

37. If we wish to determine the value of

$$\Sigma \int^z \frac{dz}{(z-u)^r \sqrt{A_0 f(z)}} - \Sigma \int^\zeta \frac{d\zeta}{(\zeta-u)^r \sqrt{A_0 f(\zeta)}},$$

we have only to diminish the facients by u in the formula for

$$\Sigma I_{-r}(z) - \Sigma I_{-r}(\zeta),$$

which we have shown how to find in Article 34, or we can differentiate the formula

$$\Sigma L(z, u) - \Sigma L(\zeta, u) = \frac{1}{\sqrt{N_{2m}}} \log \left\{ \frac{1 + \mathfrak{F}_n}{1 - \mathfrak{F}_n} \right\}$$

$n-1$ times with respect to u .

We have now shown that, if we have

$$\Sigma I_i(z) - \Sigma I_i(\zeta) = 0$$

for all integer values of i from $i = 0$ to $i = m-2$, we have

$$\Sigma I_{m-1}(z) - \Sigma I_{m-1}(\zeta) = \frac{1}{\sqrt{A_0}} \log \left\{ \frac{1+T}{1-T} \right\},$$

and have laid before the reader the proof that

$$\Sigma I_r(z) - \Sigma I_r(\zeta),$$

r being any positive or negative integer, can be obtained by an *operative process alone* on the equation

$$\Sigma I_{m-1}(z) - \Sigma I_{m-1}(\xi) = \frac{1}{\sqrt{A_0}} \log \left\{ \frac{1+T'}{1-T'} \right\}.$$

So that we have, r being a *positive* integer,

$$\Sigma I_{m+r}(z) - \Sigma I_{m+r}(\xi) = \frac{\Delta^{r+1}}{(r+1)!} \left\{ \frac{1}{\sqrt{A_0}} \log \left(\frac{1+T'}{1-T'} \right) \right\},$$

$$\Sigma I_{-r-2}(z) - \Sigma I_{-r-2}(\xi) = \frac{\delta^{r+1}}{(r+1)!} \left\{ \frac{1}{\sqrt{A_{2m}}} \log \left(\frac{1+\mathfrak{T}}{1-\mathfrak{T}} \right) \right\}.$$

38. We now turn to the discussion of Abelian or hyper-elliptic functions, of which we propose to give a definition and an investigation of their periodicity.

We shall consider the general case in which

$$I_r(z) \equiv \int^z \frac{z^r dz}{\sqrt{A_0} f(z)},$$

and $f(z)$ is of the degree $2m$ in z , and we shall take $m-1$ quantities $z_1, z_2, \dots, z_{m-1}$, instead of m quantities previously considered in connexion with the above integral, and we shall suppose the $m-1$ quantities to be the roots of the equation

$$z^{m-1} + p_1 z^{m-2} + p_2 z^{m-3} + \dots + p_{m-1} = 0 \dots\dots\dots(1).$$

If we write

$$\left. \begin{aligned} \Sigma I_0(z) &= J_0(z), & \text{or } J_0 & \text{if we have no} \\ \Sigma I_1(z) &= J_1(z), & \text{occasion to put the} \\ \dots &\dots & \text{z system in evidence} \\ \Sigma I_{m-2}(z) &= J_{m-2}(z), \end{aligned} \right\} \dots\dots\dots(2).$$

Σ denoting summation with regard to the $m-1$ quantities $z_1, z_2, \dots, z_{m-1}$, it is clear that, if we consider the $m-1$ coefficients which occur in equation (1) as known quantities, the $m-1$ sums $J_0, J_1, \dots, J_{m-2}$, are known; conversely, if we are given $J_0, J_1, \dots, J_{m-2}$, the $m-1$ quantities $p_1, p_2, \dots, p_{m-2}$ could be determined.

These latter quantities we shall call *primary* Abelian functions, and any rational functions of them Abelian functions.

We shall write, then,

$$\left. \begin{aligned} p_1 &= f_1 \{J_0, J_1, \dots J_{m-2}\} \\ p_2 &= f_2 \{J_0, J_1, \dots J_{m-2}\} \\ \dots &\dots \dots \dots \dots \dots \dots \\ p_{m-1} &= f_{m-1} \{J_0, J_1, \dots J_{m-2}\} \end{aligned} \right\} \dots\dots\dots(3).$$

39. We now turn to the general case discussed in this paper, namely,

$$\sum I_i(z) - \sum I_i(\xi) = 0,$$

where we have m quantities $z_1, z_2, \dots z_m$, and also m simultaneous values of these quantities $\xi_1, \xi_2, \dots \xi_m$. Let us now take $z_m = \alpha_1$, a root of

$$f(z) = 0,$$

and also take $\xi_m = \alpha_2$, another root of

$$f(z) = 0,$$

and we shall have, adopting the notation introduced in this article,

$$J_i(z) + I_i(\alpha) = J_i(\xi) + I_i(\alpha_2),$$

$$\text{or} \quad J_i(z) + \{J_i(\alpha_1) - I_i(\alpha_2)\} = J_i(\xi) \dots\dots\dots(1),$$

when the above equation holds for all integer values of i from $i = 0$ to $i = m-2$.

Now, T for this system is found from the equation

$$T = \frac{\sqrt{A_0} (p_m - q_m)}{p_m + q_m},$$

$$\text{or} \quad p_m + q_m = \sqrt{A_0} T^{-1} p_m - \sqrt{A_0} T^{-1} q_m.$$

Now, from Article 28, we learn that

$$\left. \begin{aligned} T \left\{ \frac{du(\xi)}{d\xi} + (-1)^{m-1} \sqrt{A_0 f(\xi)} \right\} &= \sqrt{A_0} u(\xi) \\ T \left\{ \frac{dv(z)}{dz} + (-1)^{m-1} \sqrt{A_0 f(z)} \right\} &= -\sqrt{A_0} v(z) \end{aligned} \right\} \dots\dots\dots(2),$$

and, if in the first of these equations we put $\xi = \alpha_2$, and in the

second $z = a_1$, we obtain

$$\left. \begin{aligned} T \frac{du(a_2)}{dt} &= \sqrt{A_0} u(a_2) \\ T \frac{dv(a_1)}{dt^0} &= -\sqrt{A_0} v(a_1) \end{aligned} \right\} \dots\dots\dots(3).$$

The first of these equations gives us

$$\sqrt{A_0} T^{-1} = \frac{du(a_2)}{u(a_1)} = \frac{\sum \frac{dz}{z-a}}{\frac{\sum \sqrt{A_0 f(z)}}{(z-a_1)(z-a_2)\theta'(z)}},$$

where

$$\theta(z) \equiv (z-z_1)(z-z_2) \dots (z-z_{m-1}),$$

or, if we write

$$Q(z) \equiv \frac{\sqrt{A_0 f(z)}}{\theta'(z)},$$

we have

$$\sqrt{A_0} T^{-1} = \frac{\sum Q(z)}{(z-a_1)(z-a_2)} \dots\dots\dots(4),$$

and from the second equation we obtain in like manner

$$\sqrt{A_0} T^{-1} = -\frac{\sum Q(\zeta)}{(\zeta-a_1)(\zeta-a_2)} \dots\dots\dots(5).$$

40. We now investigate the values of $\dot{p}_m$ and $\dot{q}_m$; we have

$$p_m = (-1)^m a_1 z_1 z_2 \dots z_{m-1},$$

$$q_m = (-1)^m a_2 \zeta_1 \zeta_2 \dots \zeta_{m-1};$$

differentiating, then, the first of these equations with respect to t , and the second with respect to t^0 , we obtain, remembering that $\frac{dz}{dt}$ vanishes when $z = a$, a root of

$$f(z) = 0,$$

$$\dot{p}_m = (-1)^m a_1 \left\{ z_2 z_3 \dots z_{m-1} \frac{\sqrt{A_0 f(z_1)}}{(z_1 - a_1) \theta'(z_1)} + \dots \right\},$$

or

$$\dot{p}_m = (-1)^m a_1 \sum \frac{Q(z_1) z_2 z_3 \dots z_{m-1}}{(z_1 - a_1)} \dots\dots\dots(1),$$

$$\text{and, similarly, } \dot{q}_m = (-1)^m a_2 \sum \frac{Q(\zeta_1) \zeta_2 \zeta_3 \dots \zeta_{m-1}}{(\zeta_1 - a_2)} \dots\dots\dots(2).$$

Introducing now these values of p_m , q_m , $\dot{p}_m$, and $\dot{q}_m$ into the equation

$$\dot{p}_m + \dot{q}_m = \sqrt{A_0} T^{-1} p_m - \sqrt{A_0} T^{-1} q_m,$$

we obtain

$$\alpha_1 \Sigma \frac{Q(z_1) z_2 z_3 \dots z_{m-1}}{(z_1 - a_1)} + \alpha_2 \Sigma \frac{Q(\xi_1) \xi_2 \xi_3 \dots \xi_{m-1}}{\xi_1 - a_2} \\ = \alpha_1 z_1 z_2 \dots z_{m-1} \sqrt{A_0} T^{-1} - \alpha_2 \xi_1 \xi_2 \dots \xi_{m-1} \sqrt{A_0} T^{-1} \dots (3).$$

Now, from the values of $\sqrt{A_0} T^{-1}$ given in (4) and (5) of the last article, we can write

$$\alpha_1 z_1 z_2 \dots z_{m-1} \sqrt{A_0} T^{-1} = \alpha_1 z_1 z_2 \dots z_{m-1} \Sigma \frac{Q(z_1)}{(z_1 - a_1)(z_1 - a_2)}, \\ \alpha_2 \xi_1 \xi_2 \dots \xi_{m-1} \sqrt{A_0} T^{-1} = -\alpha_2 \xi_1 \xi_2 \dots \xi_{m-1} \Sigma \frac{Q(\xi_1)}{(\xi_1 - a_1)(\xi_1 - a_2)},$$

and, introducing these values into the equation (3), it assumes the form

$$\alpha_1 \Sigma \frac{Q(z_1) z_2 z_3 \dots z_{m-1}}{(z_1 - a_1)} + \alpha_2 \Sigma \frac{Q(\xi_1) \xi_2 \xi_3 \dots \xi_{m-1}}{(\xi_1 - a_2)} \\ = \alpha_1 \Sigma \frac{Q(z_1) z_1 z_2 z_3 \dots z_{m-1}}{(z_1 - a_1)(z_1 - a_2)} + \alpha_2 \Sigma \frac{Q(\xi_1) \xi_1 \xi_2 \xi_3 \dots \xi_{m-1}}{(\xi_1 - a_1)(\xi_1 - a_2)},$$

$$\text{or } \alpha_1 \Sigma \frac{Q(z_1) z_2 z_3 \dots z_m}{(z_1 - a_1)(z_1 - a_2)} (z_1 - a_2 - z_1) + \alpha_2 \Sigma \frac{Q(\xi_1) \xi_2 \xi_3 \dots \xi_{m-1}}{(\xi_1 - a_1)(\xi_1 - a_2)} (\xi_1 - a_1 - \xi_1) \\ = 0,$$

and dividing by $\alpha_1 \alpha_2$, we obtain, finally,

$$\Sigma \frac{Q(z) z_2 z_3 \dots z_{m-1}}{(z_1 - a_1)(z_1 - a_2)} + \Sigma \frac{Q(\xi_1) \xi_2 \xi_3 \dots \xi_{m-1}}{(\xi_1 - a_1)(\xi_1 - a_2)} = 0 \dots (4).$$

This remarkable result furnishes us by successive operations of δ with $m-2$ other equations of a similar nature, for it must be observed that

$$\delta Q(z) = 0,$$

since $Q(z)$ is a function of the difference of the facients which enter into it. We remark further that this equation, and all those which flow from it by operations of δ , thus constituting a system of algebraic equivalents of the transcendental system, remain unaltered when we interchange a_1 and a_2 , so that the systems

$$J_i(z) + \{I_i(a_1) - I_i(a_2)\} = J_i(\xi),$$

$$J_i(z) + \{I_i(a_2) - I_i(a_1)\} = J_i(\xi)$$

give identical algebraic equivalents, so that, as far as algebraic equivalents are concerned, we get the same algebraic equations connecting the $m-1$ z 's, and the $m-1$ ζ 's, whether we write

$$J_i(z) + I_i \left(\frac{a_1}{a_2} \right) = J_i(\zeta)$$

or
$$J_i(z) - I_i \left(\frac{a_1}{a_2} \right) = J_i(\zeta),$$

where we have written $I_i(a_1) - I_i(a_2)$

in the form
$$I_i \left(\frac{a_1}{a_2} \right).$$

41. We shall now write

$$J_i(z) + I_i \left(\frac{a_1}{a_2} \right) = J_i(\zeta) \dots\dots\dots(1),$$

and then take a new set of $m-1$ quantities $\eta_1, \eta_2, \dots \eta_{m-1}$ connected with the ζ 's by the system of equations contained in the typical equation

$$J_i(\zeta) + I_i \left(\frac{a_1}{a_2} \right) = J_i(\eta) \dots\dots\dots(2),$$

and seek the algebraic relations connecting the z 's and the η 's, or, what is the same thing, the relations connecting the p 's and the r 's, the z 's and η 's being given respectively by the following equations :

$$z^{m-1} + p_1 z^{m-2} + p_2 z^{m-3} + \dots + p_{m-1} = 0,$$

$$\eta^{m-1} + r_1 \eta^{m-2} + r_2 \eta^{m-3} + \dots + r_{m-1} = 0.$$

We seek, in fact, the algebraic relations connecting the p 's and the r 's which constitute the algebraic equivalents of the transcendental system

$$J_i(z) + 2I_i \left(\frac{a_1}{a_2} \right) = J_i(\eta) \dots\dots\dots(3),$$

which results from (1) and (2).

Now the required algebraic equivalents of (3) are plainly to be found by writing down the algebraic equivalents of (1) connecting the z 's and ζ 's, and the algebraic equivalents of (2) connecting the ζ 's and the η 's, and from these $2m-2$ equations eliminating the $m-1$ ζ 's, thus obtaining $m-1$ equations between the $m-1$ z 's and $m-1$ η 's.

But the algebraic equivalents of (2) are precisely the same as those of

$$J_i(\zeta) - I_i \left(\frac{a_1}{a_2} \right) = J_i(\eta) \dots\dots\dots (4),$$

and this equation would, in conjunction with (1), lead us to the transcendental system

$$J_i(z) = J_i(\eta) \dots\dots\dots (5);$$

it follows then that the algebraic equivalents of (3) and (5) must be identical.

Now, with regard to the system

$$J_i(z) = J_i(\eta),$$

we observe that it results from the general case of

$$\Sigma I_i(z) = \Sigma I_i(\eta),$$

in which one of the z 's has become equal to one of the η 's, and has thus disappeared from the system of equations, leaving us

$$J_i(z) = J_i(\eta).$$

Now, let $z_m = \eta_m$ be the common value which has disappeared from the system, and adding to both sides of the typical equation

$$J_i(z) = J_i(\eta),$$

to the left-hand side $I_i(z_m)$, and to the right-hand side $I_i(\eta_m)$, we have

$$\Sigma I_i(z) = \Sigma I_i(\eta) \dots\dots\dots (6),$$

in which

$$z_m = \eta_m.$$

From Article 28, we have T for system (6) given by the equation

$$T \left\{ \frac{dn(\eta)}{dt} + (-1)^{m-1} \sqrt{A_0} f(\eta) \right\} = \sqrt{A_0} n(\eta),$$

and if we take $\eta = \eta_m = z_m$, it is clear that $n(\eta_m)$ vanishes, and as the expression within the brackets does not become zero, it follows that T must vanish for the system (6), and that we must have

$$p_s = r_s,$$

s being an integer which may have any value from $s = 1$ to $s = m$. Now, since

$$p_m = -z_m p_{m-1},$$

z_m being equal to η_m , $r_m = -\zeta_m r_{m-1}$,

it readily follows that we must have

$$p_s = r_s$$

for all integer values of s from $s = 1$ to $s = m-1$; consequently the $m-1$ algebraic equivalents of system (5) are contained in the typical equation

$$p_s = r_s,$$

which must also be expressive of the algebraic equivalents of the system

$$J_i(z) + 2I\left(\frac{a_1}{a_2}\right) = J_i(\eta),$$

from which it follows that $p_1, p_2, \dots p_{m-1}$

are periodic functions of the quantities

$$J_0(z), J_1(z), \dots J_{m-1}(z);$$

the periods being $2I\left(\frac{a_1}{a_2}\right)$, a_1 and a_2 being any two roots of the equation

$$f(z) = 0.$$

We shall not here enter on a discussion touching the reality of the roots of

$$f(z) = 0,$$

and the question of the real and imaginary portions of $I_i\left(\frac{a_1}{a_2}\right)$, but shall rest content with the statement that

$$p_s = f_s\{J_0(z), J_1(z), \dots J_{m-1}(z)\}$$

is a periodic function of the J 's, and remains unaltered when $J_i(z)$ is increased or diminished by one or more of the various quantities $2I_i\left(\frac{a_1}{a_2}\right)$.

42. If we now consider the periods, it would at first sight appear that we had $2m-1$ independent periods of the type $I_i\left(\frac{a_1}{a_2}\right)$; but such is not the case, as we shall now show. We wish to prove that a relation of the following kind exists between the $2m-1$ integrals

$$I_i\left(\frac{a_1}{a}\right), I_i\left(\frac{a_2}{a}\right), \dots I_i\left(\frac{a_{2m}}{a}\right);$$

α being some one root which we take as the inferior limit of the integral

$$\int_{\alpha}^z \frac{z' dz}{\sqrt{A_0 f(z)}},$$

in each case, viz.,

$$\pm I_i \left(\frac{a_1}{\alpha} \right) \pm I_i \left(\frac{a_2}{\alpha} \right) \pm \dots \pm I_i \left(\frac{a_m}{\alpha} \right) = \pm I_i \left(\frac{a_{m+1}}{\alpha} \right) \pm \dots \pm I_i \left(\frac{a_{2m}}{\alpha} \right) \dots (1)$$

holding for integer values of i from $i = 0$ to $i = m-1$, some one of the integrals vanishing when the superior limit becomes α .

If we take as a set of z 's $\alpha_1, \alpha_2, \dots, \alpha_m$, say, then, if we can show that a set of ζ 's can have the values $\alpha_{m+1}, \alpha_{m+2}, \dots, \alpha_{2m}$, the relations implied in (1) are proved to exist.

Now, we have in general

$$T = \frac{\sqrt{A_0} (p_m - q_m)}{\dot{p}_m + \dot{q}_m} = \frac{\sqrt{A_0} (p_{m-1} - q_{m-1})}{\dot{p}_{m-1} + \dot{q}_{m-1}} = \dots = \frac{\sqrt{A_0} (p_1 - q_1)}{\dot{p}_1 + \dot{q}_1} \dots (2),$$

and, since $\dot{p}_m, \dot{p}_{m-1}, \dots, \dot{p}_1; \dot{q}_m, \dot{q}_{m-1}, \dots, \dot{q}_1$ all vanish when the z 's and ζ 's have the values assigned above, it follows that we must have a corresponding transcendental set of relations which are those contained in (1). There are consequently but $2m-2$ independent periods of the type $2I_i \left(\frac{a_1}{\alpha_2} \right)$.

43. There is an interesting algebraic integral of the system

$$J_i(z) + I_i \left(\frac{a_1}{\alpha_2} \right) = J_i(\zeta),$$

which, I think, is worthy of notice.

$$\text{We have } T \left\{ \frac{dn(\zeta)}{dt} + (-1)^{m-1} \sqrt{A_0 f(\zeta)} \right\} = \sqrt{A_0} n(\zeta).$$

Now let us take $\zeta_m = \alpha_2$, and the above equation becomes

$$T \frac{dn(\alpha_2)}{dt} = \sqrt{A_0} n(\alpha_2),$$

an equation which, being integrated with respect to t , gives us

$$n(\alpha_2) = c \{ e^{\sqrt{A_0}(t-t'')} + e^{-\sqrt{A_0}(t-t'')} - 2 \},$$

c being a constant depending on $\zeta_1, \zeta_2, \dots, \zeta_{m-1}$.

If we now take

$$u_1(z) = (z_1 - z)(z_2 - z) \dots (z_{m-1} - z) \dots \dots \dots (1),$$

z being any quantity, we have

$$u(a_2) = (z_m - a_2) u_1(a_2).$$

In order to determine c we must determine the value of $\frac{d^2 u(a_2)}{dt^2}$, when

$$t - t^0 = 0.$$

$$\text{Now } \frac{d^2 u(a_2)}{dt^2} = (z_m - a_2) \frac{d^2 u_1(a_2)}{dt^2} + 2 \frac{dz_m}{dt} \frac{du_1(a_2)}{dt} + u_1(a_2) \frac{d^2 z_m}{dt^2},$$

but, since the first two terms of the right-hand side of the equation vanish when

$$t - t^0 = 0,$$

for then

$$z_m - a_2 = 0,$$

and

$$\frac{dz_m}{dt} = 0,$$

when z is a root of $f(z)$, we are only concerned with the value of

$$u_1(a_2) \frac{d^2 z_m}{dt^2},$$

when

$$t - t^0 = 0.$$

Now,

$$\frac{dz_m}{dt} = \frac{\sqrt{A_0 f(z_m)}}{\phi'(z_m)},$$

consequently, differentiating with respect to t , we find

$$\frac{d^2 z_m}{dt^2} = \frac{1}{\phi'(z_m)} \frac{\sqrt{A_0 f'(z_m)}}{2\sqrt{f(z_m)}} \frac{dz_m}{dt} + \sqrt{A_0 f(z_m)} \frac{d}{dt} \left\{ \frac{1}{\phi'(z_m)} \right\},$$

$$\text{or } \frac{d^2 z_m}{dt^2} = \frac{A_0 f'(z_m)}{2\{\phi'(z_m)\}^2} + \sqrt{A_0 f(z_m)} \frac{d}{dt} \left\{ \frac{1}{\phi'(z_m)} \right\} \dots \dots \dots (2).$$

If we now let

$$t - t^0 = 0,$$

z_m becomes a_2 , and we obtain

$$\frac{d^2 z_m}{dt^2} (\text{when } z_m = a_2) = \frac{A_0 f'(a_2)}{2\{v_1(a_2)\}^2} \dots \dots \dots (3),$$

where $v_1(z) \equiv (\zeta_1 - z)(\zeta_2 - z) \dots (\zeta_{m-1} - z)$,

and consequently

$$\frac{A_0 f(a_2)}{2v_1(a_2)} = c \frac{d}{dt^2} \{e^{\sqrt{A_0}(t-t^0)} + e^{-\sqrt{A_0}(t-t^0)} - 2\} = 2cA_0;$$

hence
$$c = \frac{f'(a_2)}{4v_1(a_2)},$$

and, finally,
$$(a_1 - a_2) u_1(a_2) v_1(a_2) = \frac{f'(a_2)}{4} \chi^2 \dots \dots \dots (4),$$

where
$$\chi = e^{\sqrt{A_0} \frac{1}{2}(t-t^0)} - e^{-\sqrt{A_0} \frac{1}{2}(t-t^0)},$$

or
$$u_1(a_2) v_1(a_2) = - \frac{\chi^2}{4} (a_2 - a_3)(a_2 - a_4) \dots (a_2 - a_{2m}) \dots \dots (5),$$

and, by parity of reasoning, we have also

$$u_1(a_1) v_1(a_1) = - \frac{\chi^2}{4} (a_1 - a_3)(a_1 - a_4) \dots (a_1 - a_{2m}) \dots \dots (6).$$

If we now divide the first equation (5) by (6), we obtain an *unique* result—

$$\frac{u_1(a_2) v_1(a_2)}{u_1(a_1) v_1(a_1)} = \frac{(a_2 - a_3)(a_2 - a_4) \dots (a_2 - a_{2m})}{(a_1 - a_3)(a_1 - a_4) \dots (a_1 - a_{2m})} \dots \dots (7),$$

or
$$\frac{(a_2^{m-1} + p_1 a_2^{m-2} + p_2 a_2^{m-3} + \dots + p_{m-1})(a_2^{m-1} + q_1 a_2^{m-2} + q_2 a_2^{m-3} + \dots + q_{m-1})}{(a_1^{m-1} + p_1 a_1^{m-2} + p_2 a_1^{m-3} + \dots + p_{m-1})(a_1^{m-1} + q_1 a_1^{m-2} + q_2 a_1^{m-3} + \dots + q_{m-1})}$$

$$= \frac{(a_2 - a_3)(a_2 - a_4) \dots (a_2 - a_{2m})}{(a_1 - a_3)(a_1 - a_4) \dots (a_1 - a_{2m})},$$

from which it follows that the above relation between the *p*'s and *q*'s is linear in both sets of quantities.

We can determine no other equation or equations from the above, by operation of δ , since it consists of functions of the differences of the various facients which enter into it, and is easily seen to be an absolute invariant, remaining unaltered when we write for the facients their respective reciprocals.

The results of this article when applied to the cases $m = 2$ and $m = 3$ are many, and interesting, and enable us to obtain the laws of the periodicity of elliptic functions and hyper-elliptic functions for the case $m = 3$, with ease, without the application of the general theorem of Article 41.

44. For the case $m = 2$, we may write

$$I_0(z) + I_0\left(\frac{a_1}{a_2}\right) = I_0(\zeta) \dots\dots\dots(1),$$

$$I_0(\zeta) + I_0\left(\frac{a_1}{a_2}\right) = I_0(\eta) \dots\dots\dots(2),$$

$$\text{from which we infer } I_0(z) + 2I\left(\frac{a_1}{a_2}\right) = I_0(\eta) \dots\dots\dots(3).$$

Now, from (3) of the last article, we obtain

$$\frac{(a_1 - z)(a_2 - \zeta)}{(a_1 - z)(a_1 - \zeta)} = \frac{(a_2 - a_3)(a_2 - a_4)}{(a_1 - a_3)(a_1 - a_4)} \dots\dots\dots(4),$$

$$\frac{(a_2 - \zeta)(a_2 - \eta)}{(a_1 - \zeta)(a_1 - \eta)} = \frac{(a_2 - a_3)(a_2 - a_4)}{(a_1 - a_3)(a_1 - a_4)} \dots\dots\dots(5),$$

from which it follows that $z = \eta$ is the algebraic equivalent of (3).

We must regard, then, z as a periodic function of $I_0(z)$, and its periods are, say,

$$2I_0\left(\frac{a_1}{a_2}\right), \quad 2I\left(\frac{a_3}{a_2}\right), \quad 2I\left(\frac{a_4}{a_2}\right),$$

and these three are easily seen to be connected, since $z = a_3$ and $\zeta = a_4$ are values which satisfy (4), and which correspond to the transcendental equation

$$\pm I_0\left(\frac{a_1}{a_2}\right) \pm I\left(\frac{a_3}{a_2}\right) \pm I\left(\frac{a_4}{a_2}\right) = 0.$$

45. For the case $m = 3$, we have

$$\left. \begin{aligned} J_0(z) + I_0\left(\frac{a_1}{a_2}\right) &= J_0(\zeta) \\ J_1(z) + I_1\left(\frac{a_1}{a_2}\right) &= J_1(\zeta) \end{aligned} \right\} \dots\dots\dots(1),$$

$$\left. \begin{aligned} J_0(\zeta) + I_0\left(\frac{a_1}{a_2}\right) &= J_0(\eta) \\ J_1(\zeta) + I_1\left(\frac{a_1}{a_2}\right) &= J_1(\eta) \end{aligned} \right\} \dots\dots\dots(2).$$

Now, from (5) and (6) of Article 43, we have, for the system (1),

$$\left. \begin{aligned} u_1(a_2) v_1(a_2) &= -\frac{\chi^2}{4} (a_2 - a_3)(a_2 - a_4)(a_2 - a_5)(a_2 - a_6) \\ u_1(a_1) v_1(a_1) &= -\frac{\chi^2}{4} (a_1 - a_3)(a_1 - a_4)(a_1 - a_5)(a_1 - a_6) \end{aligned} \right\} \dots\dots(3).$$

Now, I say that $t - t^0$ is the *same* for both the systems (1) and (2), as appears from the value of T^{-1} given in Article 39, and consequently χ will have the same value for both these systems.

We shall have, then, two corresponding equations to those given above in (3), which we may write

$$\left. \begin{aligned} v_1(a_2) w_1(a_2) &= -\frac{\chi^2}{4} (a_2 - a_3)(a_2 - a_4)(a_2 - a_5)(a_2 - a_6) \\ v_1(a_1) w_1(a_1) &= -\frac{\chi^2}{4} (a_1 - a_3)(a_1 - a_4)(a_1 - a_5)(a_1 - a_6) \end{aligned} \right\} \dots\dots(4),$$

where

$$w_1(z) \equiv (z - \eta_1)(z - \eta_2),$$

from which we infer

$$v_1(a_1) = w_1(a_1),$$

$$v_1(a_2) = w_1(a_1),$$

or

$$p_1 = r_1, \quad p_2 = r_2,$$

from which it follows that p_1 and p_2 are periodic functions of $J_0(z)$ and $J_1(z)$, the corresponding periods being of the types

$$I_0\left(\frac{a_1}{a_2}\right), \quad I_1\left(\frac{a_1}{a_2}\right),$$

a_1 and a_2 being two roots of $f(z) = 0$.

In a manner similar to that adopted in the discussion of the periods in the last article, we learn that there are but four independent periods of each class, the five integrals

$$I_0\left(\frac{a_1}{a_2}\right), \quad I_0\left(\frac{a_3}{a_2}\right), \quad \dots \quad I_0\left(\frac{a_6}{a_2}\right),$$

as well as

$$I_1\left(\frac{a_1}{a_2}\right), \quad I_1\left(\frac{a_3}{a_2}\right), \quad \dots \quad I_1\left(\frac{a_6}{a_2}\right),$$

being connected in each case by a linear relation.

The method exposed in this paper then completely establishes the nature and periodicity of elliptic and Abelian functions without, as remarked before, entering on any discussion of the reality of the roots of $f(z)$, or the real and imaginary parts of an integral, such as $I_i\left(\frac{a_1}{a_2}\right)$, a_1 and a_2 being any two roots of $f(z)$.

The results given in this paper will, I think, when applied to that important class of curves whose coordinates can be expressed in terms of Abelian functions, be found interesting and will furnish us with many new properties not unworthy of notice.