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by the upward current, from the horizontal impetus of the wind and consequently calm, need not be so spacious as to require more than a very few minutes for it to be passed over by a storm travelling at the rate of thirty miles per hour. Supposing the confluent blasts to proceed only from two opposite quadrants, the extent of the storm, in front and in rear of its path, would of course be doubled, so as to require forty miles distance to effect a reduction of the velocity from one hundred miles per hour to five miles.

I am, Gentlemen,

Respectfully yours,

Philadelphia, November 23, 1844.

ROBERT HARE.

XV. On the Mechanism of Glacial Motion.

By W. HOPKINS, Esq., M.A., F.R.S., &c.

To the Editors of the *Philosophical Magazine and Journal*.

GENTLEMEN,

IN my preceding communication I gave an explanation of the manner in which the general motion of glaciers may be accounted for according to the *sliding theory*. I now proceed to consider more particularly the mechanism of this motion.

1. The most important facts in the motion of glaciers, hitherto determined, appear to be the following:—

(1.) The motion of each point on the surface of a glacier is approximately parallel to the longitudinal axis of the glacier, and the sides of the glacial valley, whenever the axis and sides are approximately parallel, as is very generally the case in those portions of glaciers which have been subjected to detailed observations, and which are descending along valleys of well-defined and determinate forms. If the valleys converge or diverge in descending, the motion of different points of the glacier will partake of the same character.

(2.) If we take any number of points in a transverse line perpendicular to the axis of the glacier, the velocity of those points will be greatest which are nearest to the centre of the glacier.

(3.) This increase of the velocity, in passing from the sides to the centre, takes place much more rapidly near the sides than in other parts of the glacier.

(4.) The motion (at least during summer) is greater near the lower than it is near the upper extremity of a glacier.

(5.) The motion is greater in warm than in cold weather.

(6.) Nothing has been positively ascertained by observation respecting the relative velocities of the upper and lower surfaces

of a glacier. The continued verticality of transverse fissures is, however, a conclusive proof that, to the depth to which these fissures extend, the velocity must be sensibly the same as at the surface.

2. The facts above stated are in entire accordance with the conclusions deducible from the sliding theory. According to that theory, the velocity of any portion of a glacier will depend (1) on the inclination of its bed, (2) the disintegration of its lower surface by the internal heat of the earth, (3) on subglacial currents, (4) the depth of the mass*, and (5) local and lateral obstacles. The first and second causes will generally have nearly the same effect both in the central and lateral portions; but the third cause will manifestly produce in general the greatest acceleration in the central parts, and the fourth cause will produce a similar effect, if the glacier be deeper in the centre than at its sides, while the greatest retardation will be produced on the lateral portions by the last of the above-mentioned causes. These causes sufficiently account for the greater velocity of the centre of the glacier.

Again, the second of the above causes will probably act with approximate uniformity throughout the whole length of the glacier, but the third cause will act with the greatest energy at the lower extremity, because the subglacial currents will be increased by innumerable tributaries as they descend. This cause, therefore, will tend to make the velocity greater, as we approach the lower end of the glacier, while the greater depth of the mass at the upper extremity will tend to give the greater velocity to that part of the glacier. In winter the effect of the currents must be very inconsiderable, and we should consequently expect that there will be a tendency in the portions of the glacier in the higher regions to move faster than those in the lower, in which case there must be a longitudinal compression, and consequent closing up of transverse fissures in a greater or less degree. During the summer, on the contrary, the subglacial currents will be most efficient, and we should expect that they will give the greater velocity to the lower extremity of the glacier, in which case the mass will be brought into a state of longitudinal tension.

3. It is manifest that the more rapid motion of the central part of the glacier must bring the whole mass into a state of internal constraint. My first object will be to deduce certain conclusions respecting the internal tensions and pressures thus superinduced, while the continuity of the mass, or of any con-

* In the experiments described in my previous communication, the velocity of descent increased with the weight, whence it is inferred that the deeper parts of a glacier must, *cæteris paribus*, tend to move the fastest.

siderable portion of it, remains unbroken. In the second place, I shall show how the continuity will be broken when the internal tensions become greater than the cohesive power of the ice, and shall investigate the forms and positions of the lines or curves of fracture; and thence I shall explain how the progressive motion of the glacier may take place by a constant repetition of such dislocations.

It has been already stated that no conclusive observations have been made respecting the difference between the velocities of the upper and lower surfaces of a glacier. In the following investigations I shall assume these velocities to be the same as they will be approximately, if the general motion be due to *sliding*. There would be no difficulty introduced into our investigations by the adoption of the contrary hypothesis—that the lower surface moves much slower than the upper one—but the resulting formulæ would be of much greater complexity. For the sake of simplicity I have therefore adopted the first hypothesis. Any one wishing to pursue the subject under the latter hypothesis will find no difficulty in doing so. This, however, will not be necessary for our immediate purpose.

4. *Directions of Maximum Tension and Pressure.*—Of the above laws the first and second are those with which we shall be first more immediately concerned. Let $p'q'$ be the line of motion

Fig. 1.

of one set of particles of the glacier, and $r's'$ that for another set, the distance between them being small; these lines will be parallel to each other and approximately so to the axis of the glacier. Let M and N be particles in $r's'$ and $p'q'$ respectively. Assuming $r's'$ to be nearer the side of the glacier than $p'q'$, N will move faster than M, by the second law. If they moved with the same velocity, the physical line MN would neither be lengthened nor shortened by the motion we are considering; and therefore it will be to the difference of motion of M and N that extension or compression will be due. Consequently we may suppose M to be at rest, and N to move

with the difference of velocity of M and N, when our object is to calculate the extensions and compressions due to the motion of the mass. For the greater distinctness, let us conceive the mass to be at first in a state in which it is neither extended nor compressed. In a given time, considering M at rest, let Nn be the space due to the difference of motion of M and N, or to the relative velocity of N. The physical line whose unextended length was MN will now be *extended* into Mn; and in like manner, if N' be to the left of L (ML being perpendicular to p'q' and r's'), MN' will be *compressed* into Mn', where N'n' = Nn. This shows distinctly the manner in which the motion in question must necessarily produce extension and compression in the mass in different directions.

If the space Nn due to the relative motion of N become sufficiently great, the physical line MN will be so extended as to break, and an open fissure will be the consequence. Our first object, however, will be to find those directions in which the extension and compression have their maximum values, before the extension has become so great as to cause the fracture of the mass. And here it is important to remark, that so long as we keep within this limit, we have no concern whatever with the cohesive force of the mass, or the law according to which it may vary in passing from one point to another. That consideration will only enter when we proceed to the ulterior part of our investigation, where the object will be to determine the directions in which the mass will be fractured, when the internal tensions shall become greater than those which its cohesive force is able to resist.

5. The *extension* of the physical line MN will be correctly represented by the ratio $\frac{mn}{MN}$, the point m being so taken that Mm shall = MN. This ratio will manifestly be different for different lines, such as MN; our first object is to find the position of MN for which the ratio will be a maximum, subject to the condition that Nn shall be the same for all positions of N. The problem, as thus presented to us, is not mechanical but geometrical, and a very simple one. To render it still more simple in the calculation, I shall here suppose Nn a small quantity of the first order, and shall omit small quantities of the second and higher orders*. Let ML = a, Nn = β, and LMN = θ; then shall we have, to the requisite degree of approximation,

$$mn = \beta \sin \theta, \quad MN = \frac{a}{\cos \theta};$$

* A more complete investigation, and independent of this restriction, will be given afterwards.

and therefore

$$\frac{m n}{M N} = \frac{\beta}{2 a} \sin 2 \theta.$$

Consequently the extension is a maximum when $\theta = 45^\circ$; also it is a minimum when $\theta = 135^\circ$, or the *compression* is a *maximum* for this latter value of θ .

6. To pass generally from the extension and compression to the corresponding values of the *tension* and *pressure*, it is necessary to know the laws which connect them; but here we are not concerned with this more general question, our object being merely to deduce the directions of maximum tension and pressure from those of maximum extension and compression. Now, whatever may be the general law connecting tension and extension, there can be no doubt that when the one is a maximum or minimum the other must be so likewise. Hence the directions above determined will be those of maximum *tension* and *pressure*, so far as the tension and pressure are superinduced by the motion of the glacier. Consequently if $M N$ and $M N'$ be those directions for the point M , each will be inclined at an angle of 45° to the direction of motion, and they will be at right angles to each other.

The preceding method has the advantage of pointing out very simply and distinctly the manner in which the difference of velocity in different parts of the glacier produces tension and pressure, and the results above given are arrived at with great facility. But I shall now proceed to a more complete investigation of the problem, still preserving the same limitation as before in the hypothesis of the equality of the velocities of the upper and lower surfaces of the glacier. Our results will thus be rendered independent of the depth of the glacier, which may, therefore, be treated as a simple *lamina*. We shall thus only be concerned with space of two dimensions.

7. *Internal Tensions and Pressures of a Glacier.*—When a plain solid lamina, having a certain degree of compressibility and extensibility, is brought into a position of constraint by forces acting in the plane of the lamina, the particles on one side of a geometrical line will exert certain forces on the contiguous particles on the opposite side of the line. If the lamina were formed of fluid particles the resultant action at each point of this *line of separation* would be normal to it; but when the lamina is solid this will not be generally the case, and therefore the force at any point of the line may be resolved into two forces, one being normal and the other tangential to the line of separation; all forces being supposed to act in the plane of the lamina. Suppose the line of separation to be a straight

line $A'A$ parallel to the axis of x , and let $p q$ be a portion of it so small that the actions on every point of $p q$ may be considered equal. Let $Y_1.p q$ denote the normal force exerted by

Fig. 2.

the particles immediately above $p q$ in the annexed figure, on those immediately below it, estimated in the direction $q B$; and let $f_1.p q$ represent the tangential action on $p q$. Again, let the line of separation coincide with $B'B$, parallel to the axis of y , and perpendicular to $A'A$; and let $X_1.q s$ denote the normal force exerted by the particles immediately on the right of $q s$ on the contiguous particles immediately on the left of it, and $f'_1.q s$ the tangential action. Join p and s , and let a perpendicular to $p s$ make an angle θ with $A'A$ or the axis of x . Then if $X.ps$ and $Y.ps$ be the resolved parts of the forces which the particles on one side of $p s$ exert on those on the opposite side, estimated in the direction $q A$ and $q B$ respectively, we shall have

$$X = X_1 \cos \theta + f_1 \sin \theta,$$

$$Y = Y_1 \sin \theta + f'_1 \cos \theta.$$

To prove these formulæ, we have only to observe that the forces acting on the sides $p q$ and $q s$ of the triangular element $p q s$ must be in equilibrium with the forces $-X$ and $-Y$ acting externally on the side $p s$, neglecting small quantities of the third order. Hence we have

$$-X.ps + X_1.q s + f_1.p q = 0,$$

$$-Y.ps + Y_1.p q + f'_1.q s = 0,$$

which, since $\frac{pq}{ps} = \sin \theta$, and $\frac{qs}{ps} = \cos \theta$, prove the above formulæ*.

We have also the relation

$$f'_l = f_r.$$

To prove this equation, complete the rectangular element pqr . A tangential force will act on the element along the side rs in a direction opposite to that of the tangential force (f) acting along pq , the intensity of which will not differ from f by any finite quantity; and similarly, a force (f') will act on the side pr in the direction opposite to that on qs . The moments of these forces with respect to the middle point of the rectangular element, will be

$$\frac{1}{2} f_l \cdot pq \cdot qs, \text{ and } \frac{1}{2} f'_l \cdot pq \cdot qs.$$

The direction of the resultant of the normal forces on qs will pass at a distance from the middle point of the element small compared with qs ; that distance will therefore not exceed a quantity of the second order; and consequently the moment of the force $X_1 \cdot qs$ about the middle point of the element will not exceed a quantity of the third order, and may be neglected in comparison with the moments of the tangential forces f and f'_l , which are of the second order. Hence the equilibrium of the element requires that we should have

$$\frac{1}{2} f'_l \cdot pq \cdot qs = \frac{1}{2} f_l \cdot pq \cdot qs,$$

or

$$f'_l = f_r.$$

With this condition we have

$$\begin{aligned} X &= X_1 \cos \theta + f_l \sin \theta, \\ Y &= Y_1 \sin \theta + f_l \cos \theta. \end{aligned}$$

If a line be drawn through q parallel to ps , the distance between the two lines will be a small quantity of the first order, and therefore the action on the line through q may be considered to have for its resolved parts the forces X and Y , from which they cannot differ by quantities exceeding infinitesimals of the first order.

Let the length of ps , or of an equal and parallel line through q , = λ ; the resolved parts of the forces upon it will be λX and λY . Let λR be the force on λ estimated in a direction making an angle ϕ with the axis of x , then shall we have

$$\lambda R = \lambda X \cdot \cos \phi + \lambda Y \cdot \sin \phi,$$

or

$$R = X \cos \phi + Y \sin \phi;$$

* See Poisson's memoir, "Sur le Mouvement des Corps Elastiques," in the *Mémoires de l'Institut*, vol. iii. p. 383.

R is therefore a function of the two independent variables θ and ϕ ; and I shall now proceed to find the values of θ and ϕ , which render R a maximum or a minimum. Differentiating with respect to ϕ , we have

$$0 = X \sin \phi - Y \cos \phi,$$

which shows that for any assigned value of θ , or position of the line of separation, the maximum value of R will be that of the resultant of X and Y, and the corresponding value of ϕ , that of the angle which the direction of that resultant makes with the axis of x . Differentiating with respect to θ , we have

$$0 = \frac{dX}{d\theta} \cos \phi + \frac{dY}{d\theta} \sin \phi.$$

Substituting for X and Y in these two equations, we obtain

$$(X_1 \cos \theta + f_1 \sin \theta) \sin \phi - (Y_1 \sin \theta + f_1 \cos \theta) \cos \phi = 0,$$

$$(X_1 \sin \theta - f_1 \cos \theta) \cos \phi - (Y_1 \cos \theta - f_1 \sin \theta) \sin \phi = 0.$$

Eliminating ϕ , we have

$$(X_1 \cos \theta + f_1 \sin \theta) (X_1 \sin \theta - f_1 \cos \theta) - (Y_1 \sin \theta + f_1 \cos \theta) (Y_1 \cos \theta - f_1 \sin \theta) = 0,$$

$$\therefore (X_1 f_1 + Y_1 f_1) (\sin^2 \theta - \cos^2 \theta) + (X_1^2 - Y_1^2) \sin \theta \cos \theta = 0;$$

$$\therefore \tan 2\theta = \frac{2f_1}{X_1 - Y_1}. \quad \dots \quad (1.)$$

Again, from the two preceding equations containing θ and ϕ , we have

$$(X_1 + f_1 \tan \theta) \tan \phi - (Y_1 \tan \theta + f_1) = 0;$$

$$(X_1 \tan \theta - f_1) - (Y_1 - f_1 \tan \theta) \tan \phi = 0,$$

or

$$X_1 \tan \phi - Y_1 \tan \theta + f_1 \tan \theta \tan \phi - f_1 = 0,$$

$$X_1 \tan \theta - Y_1 \tan \phi + f_1 \tan \theta \tan \phi - f_1 = 0.$$

θ and ϕ enter exactly in the same manner in these two equations, and must therefore be equal. Hence

$$\tan 2\phi = \frac{2f_1}{X_1 - Y_1}. \quad \dots \quad (2.)$$

Equation (1.) shows that there are two positions of the line of separation through any proposed point, at right angles to each other, for one of which the resultant action between the particles on opposite sides of the line at the proposed point is a maximum, and for the other a minimum; and since ϕ determines the direction of the resultant action, equation (2.) proves that direction to coincide with the normal to the line of separation, whenever that line is in a position for which the resultant action is a maximum or minimum. These conclusions may also be arrived at by somewhat different though equivalent reasoning, as follows.

8. First, to find the value of θ which gives R a maximum or minimum, we have

$$R^2 = X^2 + Y^2,$$

and therefore

$$0 = X \frac{dX}{d\theta} + Y \frac{dY}{d\theta},$$

which by substitution and reduction gives

$$(X_1 f + Y_1 f) (\sin^2 \theta - \cos^2 \theta) + (X_1^2 - Y_1^2) \sin \theta \cos \theta = 0,$$

$$\text{or} \quad \tan 2\theta = \frac{2f_1}{X_1 - Y_1}.$$

And, secondly, taking ϕ as the angle which the resultant of X and Y makes with the axis of x , we have

$$\tan \phi = \frac{Y}{X} = \frac{Y_1 \sin \theta + f_1 \cos \theta}{X_1 \cos \theta + f_1 \sin \theta};$$

and if we put $\phi = \theta$, we shall determine that position of the line of separation for which the direction of the resultant action at any proposed point of it coincides with the normal. We thus obtain

$$\sin \theta \{X_1 \cos \theta + f_1 \sin \theta\} = \cos \theta \{Y_1 \sin \theta + f_1 \cos \theta\},$$

$$\text{or} \quad (X_1 - Y_1) \sin \theta \cos \theta = f_1 (\cos^2 \theta - \sin^2 \theta);$$

$$\therefore \tan 2\theta = \frac{2f_1}{X_1 - Y_1}.$$

This equation shows that that position of the line of separation for which $\phi = \theta$, is that which corresponds to the maximum or minimum action between the contiguous particles on opposite sides of the line, as before proved.

9. If the forces acting parallel to the axes of x and y be pressures, X_1 and Y_1 will be negative.

If $X_1 - Y_1 = 0$, $\theta = 45^\circ$. This accords with the result previously obtained in article 5, where X_1 and Y_1 were both $= 0^*$.

10. If (R) and (r) be the maximum and minimum values of R at q , they will act respectively in directions perpendicular and parallel to ps , the position of that line being determined by equation (1.). (R) will therefore be the maximum tension,

* It should be observed that in the investigation of the article above referred to, the condition of the transverse force Y_1 being $= 0$ after the point N (fig. 1.) has moved to n , is only secured by supposing β so small a quantity of the first order that all small quantities of higher orders may be neglected. In that case the compression along ML becomes a small quantity of the second order, and therefore such as may be neglected, if β be very small, and not otherwise. If β were more considerable, we should obtain by the same mode of investigation a different result. In this case, however, the second mode of investigation is much the best.

and (r) the minimum tension or maximum pressure at that point. To find their values we have generally,

$$R = X \cos \phi + Y \sin \phi,$$

and, substituting for X and Y ,

$$R = (X_1 \cos \theta + f_1 \sin \theta) \cos \phi + (Y_1 \sin \theta + f_1 \cos \theta) \sin \phi.$$

Also, when R is a maximum or minimum,

$$\phi = \theta, \quad \text{and} \quad \tan 2\theta = \frac{2f_1}{X_1 - Y_1},$$

whence we obtain, after reduction,

$$(R) = \frac{1}{2} \left\{ X_1 + Y_1 + \sqrt{(X_1 - Y_1)^2 + 4f_1^2} \right\} \quad \text{and} \quad (r) = \frac{1}{2} \left\{ X_1 + Y_1 - \sqrt{(X_1 - Y_1)^2 + 4f_1^2} \right\} \quad (3.)$$

If X_1 and Y_1 be negative (*i. e.* pressures), the values of (R) as well as of (r) may become negative, in which case (R) will be the *minimum pressure* and (r) the *maximum pressure* to which the mass is subjected at q .

11. If f denote the tangential force acting along ps , we have, resolving X and Y along ps ,

$$\begin{aligned} f &= X \sin \theta - Y \cos \theta \\ &= (X_1 \cos \theta + f_1 \sin \theta) \sin \theta - (Y_1 \sin \theta + f_1 \cos \theta) \cos \theta \\ &= \frac{1}{2} (X_1 - Y_1) \sin 2\theta - f_1 \cos 2\theta. \end{aligned}$$

Therefore, when f is a maximum or minimum, we have

$$0 = \frac{1}{2} (X_1 - Y_1) \cos 2\theta + f_1 \sin 2\theta,$$

$$\text{or} \quad \cot 2\theta = - \frac{2f_1}{X_1 - Y_1} \quad (4.)$$

Comparing this equation with (1.), it appears that the directions of the line of separation, for which f is a maximum and minimum respectively, differ from those for which R is a maximum or minimum by 45° . To find (f), the maximum value of f , we have only to substitute from (4.) the values of $\sin 2\theta$ and $\cos 2\theta$ in the above expression for f . We thus obtain

$$(f) = \pm \frac{1}{2} \sqrt{(X_1 - Y_1)^2 + 4f_1^2} \quad (5.)$$

This expression shows that the maximum and minimum values of f are the same in magnitude, and only differing in sign. This ought to be the case, for I have shown that f is the same for any two directions at right angles to each other, and from (4.) we conclude that the maximum and minimum

directions are also at right angles to each other. Consequently the maximum and minimum values of f cannot differ in intensity, and only therefore in sign.

12. From these results, with respect to R and f , we have the following conclusions:—Let q be any point in the mass in a state of constraint; then if RqR be the direction of maximum tension at q , rqr perpendicular to RqR , will be that of minimum tension; and if the two lines f_qf be perpendicular to each other, and make angles of 45° with RqR and rqr , they will be the directions in which the intensity of f will be the greatest. It should also be recollected that $f = 0$ in the two directions RqR and rqr .

Fig. 3.

There is also a simple relation between the maximum value (f) of f , and the maximum and minimum values (R) and (r) of R . It is obtained immediately from equations (3.) and (5.), which give

$$(R) - (r) = 2(f).$$

13. If we take a rectangular element at q , of which the sides are respectively parallel to RR and rr , it will be acted on by no tangential forces, and therefore will be held in equilibrium by the maximum and minimum normal tensions alone. Consequently, whatever may be the forces acting on the mass, the state of tension or pressure at any point (q) will be the same as if the forces impressed on that point were those of two systems of forces, of which the intensities should be (R) and (r), and acting in directions perpendicular to each other. This proposition is important, as leaving no doubt as to the direction in which the greatest tendency is exerted by the normal forces to fracture the mass, a point to be considered in the sequel.

That there may be no doubt as to the equivalence of forces asserted in this proposition, let us assume the forces impressed on the mass at q to be (R) and (r), instead of X_1 , Y_1 and f_p , and thence deduce the expressions for the forces acting normally and tangentially on the line qs (fig. 2). Calling these forces X and f , and considering (R) and (r) as *tensions*, we have

$$\begin{aligned} X &= (R) \cos^2 \theta + (r) \sin^2 \theta \\ &= \frac{1}{2} \{ (R) + (r) \} + \frac{1}{2} \{ (R) - (r) \} \cos 2 \theta \\ &= \frac{1}{2} (X_1 + Y_1) + \frac{1}{2} \sqrt{(X_1 - Y_1)^2 + 4f_1^2} \cos 2 \theta. \end{aligned}$$

But by equation (1.)

$$\cos 2 \theta = \frac{X_1 - Y_1}{\sqrt{(X_1 - Y_1)^2 + 4f_1^2}}$$

therefore

$$X = X_1,$$

as it ought to be. Similarly,

$$\begin{aligned} f &= \{ (R) - (r) \} \sin \theta \cos \theta \\ &= \frac{1}{2} \{ (R) - (r) \} \sin 2 \theta \\ &= \frac{1}{2} \sqrt{(X_1 - Y_1)^2 + 4f_1^2} \cdot \sin 2 \theta \\ &= f_1, \end{aligned}$$

by substitution from equation (1.).

14. It should here be borne in mind that, in the preceding investigations, the internal tensions are supposed to be insufficient to break the continuity of the mass, and that, consequently, our results are independent of the law according to which the cohesion may vary from one point to another, the consideration of which only becomes necessary when we have to determine the directions of fracture. It is also important to remark that the preceding results are independent of the degree of relative displacements of different elements of the mass, the continuity remaining unbroken; it is not necessary that the displacements should be small. Again, it should be observed, that, to render these results absolutely accurate, we must suppose the line $p s$ (fig. 2) of infinitesimal magnitude, but still containing a great number of molecules forming so many points of action on each side of this *line of separation*. This hypothesis would not be strictly applicable in the extreme limit, to a mass of which the perfect continuity should be interrupted by innumerable *pores* of sensible magnitude, in every part. If, however, $p s$ be of finite length, so as to contain a great many points of contact, there will be no sensible error in our results as applied to a mass constituted as just mentioned, provided the resultant of the actions at the points of contact pass through a point of $p s$ not deviating sensibly from its middle point. Thus if $p s$ should be conceived to be several inches, or even several feet in length, our results

would be sensibly correct as applied to a mass of the enormous extent of a glacier, or of any considerable continuous portion of it.

15. Some of the preceding results may be still further elucidated. If $X_1 = 0$ and $Y_1 = 0$, the only force acting on an element of the mass will be f_p and it has been shown (arts. 6. and 9.) that θ will then $= 45^\circ$. Let $p q r s$ be an element originally square. As already explained, it will be acted on by two couples of forces f_p as represented in the figure. It is seen at once, that the greatest tension produced by these forces must be in a direction parallel to the diagonal $r q$, or supposing the deviation from rectangularity very small (for the reason assigned in the note, art. 9), in a direction making an angle of 45° with $p q$, as before proved. Also, it is easily seen that there will be compression in the direction perpendicular to $r q$, and likewise that the compression must be greater in that direction than any other. The maximum tension (R) and the maximum pressure $-(r)$ may be easily expressed; for the whole tangential force on $p q = f_i \cdot p q$; its resolved part parallel to $r q = f_i \cdot p q \cdot \cos 45^\circ$. Similarly, the resolved part of the force on $q s = f_i \cdot q s \cdot \cos 45^\circ$. Therefore

Fig. 4.

$$(R) p q \cdot \sin 45^\circ = f_i \cdot p q \cdot \cos 45^\circ,$$

$$\text{or} \quad (R) = f_i.$$

Similarly, we obtain $-(r) = f_i$.

These same values would be obtained from equations (3.) (art. 10.), putting $X_1 = 0$ and $Y_1 = 0$.

16. We thus see also how the two forces (R) and $-(r)$ produce the same effect on the element $p q r s$, in distorting it from its original rectangular form, as the tangential forces f_i . If, however, we should take a rectangular element whose sides were perpendicular respectively to the forces (R) and $-(r)$, it is manifest that no distortion would be produced in it by (R) and $-(r)$. In fact the tangential forces on such an element would be 0. And in all cases, if we take a rectangular element of the mass in its original state of no constraint, such that the sides of that element shall be perpendicular to the di-

18. Let us now take the annexed diagram (fig. 6) to represent a glacier, or rather a horizontal section of it. Any trans-

Fig. 6.

verse element PQ will, by the more rapid motion of the centre, be brought into the position $P'Q'$, and the rectangular element $p'q's'r'$ will assume the form $p_1q_1s_1r_1$, the continuity of the mass still remaining unbroken. This distortion is produced by the tangential force (f_t) impressed on the element in the directions p_1q_1 along that side, and s_1r_1 along the opposite side, together with equal tangential forces called into action along the other sides in the directions p_1r_1 and s_1q_1 respectively. The longitudinal tension will be represented by X_1 in the previous formulæ, and the transversal tension by Y_1 , these quantities being negative whenever we have pressures instead of tensions. Hence, if at any point q we take pqr in the direction of motion, and take $tqr = \theta$ as determined by equation (1.), qt will be the direction of maximum tension, and a line perpendicular to qt , that of maximum pressure at the point q . They will also be the lines of no tangential force.

19. *Formation of Surfaces of Discontinuity in the Mass of the Glacier.*—In the previous investigations the continuity of the mass has been supposed to remain unbroken; let us now suppose the internal actions to become greater than the cohesion, and the continuity, consequently, to be broken. This breach of continuity may be produced in two ways; by the *normal force* (R), which measures what is properly called the *tension*, or by the *tangential force* (f). Open fissures will result in the first case, and merely surfaces of discontinuity in the latter. The directions of fracture may depend on the cohesion of the mass as well as on the internal forces to which it is subjected, but it is important to observe that they will depend on the internal forces alone, however the cohesion may

vary in passing from one point to another, provided it be the same at each point for every direction through that point. This is equivalent to there being no *surface of less cohesion* through any proposed point; for, in such case, it is manifest that the direction in which a fracture would begin or be continued through any point, could not be influenced by a cohesion which should be the same for all directions. If, however, surfaces of less cohesion exist in the mass (*i. e.* if its cohesion vary according to any discontinuous law), they may evidently affect the directions of fracture. This case I shall reserve for subsequent consideration, restricting myself in the first instance to cases in which the variation of cohesion is continuous.

20. Let us first consider the formation of open fissures.

It has been proved (art. 13.) that whatever forces may act at any point of the mass in its state of constraint, they must be equivalent to two systems of tensions whose intensities are (R) and (r), and whose directions are perpendicular to each other. If, therefore, an open fissure commence at any proposed point, it must clearly be in a direction perpendicular to that of the greater of these tensions, or the maximum tension at that point. The directions in which the fissure will be subsequently propagated through other points may be, in some degree, affected by other circumstances, into the explanation of which it would be useless to enter here*; but those directions will be determined approximately by the same rule as at the first point. Hence, then, the normal to a fissure at any proposed point may be considered to coincide approximately with the direction of maximum tension at that point just previous to the fracture.

Let us first suppose the glacial valley to become narrower in descending, as is very commonly the case; there will doubtless be a *transversal compression*, or Y_1 in equation (1.) must be negative. Also, it is probable that X_1 will more frequently be a *tension*, or, if it be a pressure, it will probably be much less than Y_1 . We shall thus have

$$\tan 2\theta = \frac{2f_1}{Y_1 \pm X_1},$$

the denominator being positive. Y_1 and X_1 may be taken as nearly constant for the same transversal section of the glacier, while f_1 will vary from its greatest value near the sides, where the tendency to *twist* each element of the mass is greatest, to zero at the centre, where there is no such tendency at all. Consequently θ will be the greatest at the sides of the glacier, and will diminish for points nearer to the centre, or the line

* See my memoir on Physical Geology, in the Transactions of the Cambridge Philosophical Society, vol. vi. part 1.

of fracture will be a *curve line*. Also its convexity will be turned towards the upper extremity of the glacier; for, by referring to fig. 5, it will be seen that the point in which the normal to the curve of fracture meets the axis of the glacier, lies towards the *lower* extremity of the glacier. The greatest value of θ , in the case before us, must be less than 45° , whatever may be the value of f_1 , and may be much less than that quantity, if $Y_1 \pm X_1$ be considerable. The general form of the curve of fracture is represented by MN (fig. 6), supposing the fissure continued across the glacier.

It is here deserving of remark, that the force (R) may be sufficient to produce transverse fissures, though there may be no longitudinal tension such as that denoted by X_1 , provided f_1 be great enough; for, if we put $X_1 = 0$ in the expression for (R) (art. 10.), we have

$$(R) = \frac{1}{2} \{ \sqrt{Y_1^2 + 4f_1^2} - Y_1 \}.$$

Consequently there may still be fissures due to (R) near the flanks of the glacier, but there can be no transverse fissures in the central portion, where f_1 must necessarily be very small. The existence of transverse fissures in this latter portion is as distinctly indicative of longitudinal tension as the absence of longitudinal fissures is indicative of transverse compression.

21. If the width of the valley should increase instead of decreasing, at any part, we shall obtain a different result, assuming what I conceive to be almost universally true,—that when the width of valley increases, its fall diminishes. In such case the portion of the glacier in the more expanded part of the valley will be pressed upon by the mass immediately behind it, tending to descend with greater rapidity. The longitudinal action will then become a pressure instead of a tension, or X_1 will become negative, and may be great. We shall then have

$$\tan 2\theta = - \frac{2f_1}{X_1 \pm Y_1}.$$

Hence, assuming Y_1 to be a tension, or, if a pressure, smaller than X_1 , $\tan 2\theta$ will be small and negative at points not remote from the centre of the glacier; and, therefore, θ will be either a very little less than 90° , or will be very small and negative. The former value will in this case manifestly correspond to the direction of greatest tension, and consequently, if a fissure be formed, it will tend to *parallelism* with the axis as it approaches the centre of the glacier, making a larger angle with the axis at points more remote from it, since f_1 will then be greater. Thus the fissure will become *longitudinal*

instead of transverse, as represented in the annexed diagram. This is the law recognised by Prof. Forbes in cases such as that

Fig. 7.

now considered. The form of the curves of fissure deduced in the first case agrees with that recognised by all observers.

22. If the mass be so constituted as to present internal planes or surfaces of *less cohesive power*, the only case we need consider is that in which these planes or surfaces are very near and parallel to each other. Such a system may possibly be formed by the alternate bands of blue and white ice described by Prof. Forbes, and appears to be the only one of which there is any indication. If the direction of any surface of such a system, through any proposed point, approximate sufficiently to perpendicularity to the line of maximum tension for that point, it is manifest that the fissure will be formed along the surface of less cohesion; but if, on the contrary, this surface approximate sufficiently to the direction of maximum tension, it is easily seen that it will have no influence whatever on the direction of fracture. Now, according to the observations of Prof. Forbes, the blue bands are very approximately perpendicular to the transverse fissures, and therefore nearly coincident with the lines of maximum tension; consequently those bands, allowing that they form surfaces of less cohesion, can exert no material influence on the formation of the transverse fissures.

23. From this investigation we may also see distinctly by what action the lateral parts of a glacier are made to expand with an expanding valley; for the directions of greatest pressure being perpendicular to those of greatest tension, they must be parallel to the fissures represented in fig. 7, and therefore this greatest pressure must necessarily force out the flanks of the glacier and cause the lateral expansion here spoken of.

24. *Internal state of tension or pressure of the Mass after the formation of transverse fissures.*—Since the fissures will be perpendicular to the direction of the maximum tension (R), and therefore parallel to that of (*r*), it is manifest that (R) will

be destroyed at all points not remote from each fissure, while the minimum tension or maximum pressure (r) will remain the same. Hence, if at any such points we would find the values of the internal forces, we must refer to the equations at the end of article 13, which become, if we write F for X , and $-(r)$ for (r) (since the transverse force is here a *pressure*),

$$F = (R) \cos^2 \theta - (r) \sin^2 \theta,$$

$$f = \frac{1}{2} \{(R) + (r)\} \sin 2 \theta,$$

where F denotes the normal and f the tangential force before the continuity of the mass is broken, on a line the normal to which makes an angle θ with the direction of maximum tension. In applying these equations after the formation of the fissure we must put $(R) = 0$, which gives

$$F = -(r) \sin^2 \theta,$$

$$f = \frac{1}{2} (r) \sin 2 \theta.$$

Consequently, if we take a portion of the mass between two fissures not remote from each other, it will be in a state of *pressure* (since F is negative) in every direction except that perpendicular to the fissures, in which there will be neither pressure nor tension. Such portion of the mass may therefore be said to be entirely relieved from its state of *tension*. The tangential force f will remain, but diminished in a ratio depending on that which (r) bears to (R) .

When the different portions of the mass are thus far relieved from internal constraint, the motion originally producing that constraint may be continued without causing immediately any further fracture; but by the continuance of the motion, the constraint must necessarily be again increased as at first, till the tension shall again become greater than the cohesion, when new fissures will be formed in the same manner as before, while the old ones, as they become transverse to the lines of greatest pressure by the more rapid motion of the centre of the glacier, will be gradually closed; and thus this opening of new fissures and closing of old ones may be conceived to go on for any length of time, independently of local causes constantly acting at particular points, and which, acting in conjunction with the general cause above considered, continually produce systems of fissures originating at those points, and deviating more or less from the general type which would be impressed on them if they were due solely to the more rapid motion of the centre of the glacier.

Hence we may understand how the centre of a glacier may move more rapidly than its sides, without actually producing

longitudinal planes of discontinuity, such that the portion of the mass immediately on one side of such plane shall *slide* past the contiguous portion on the opposite side without forming open fissures like those above considered. But though this is not a necessary consequence of the state of constraint of the mass, it is one of the ways, as already stated, in which the constraint *may* be destroyed. I proceed to consider the conditions under which this will take place.

25. If we take a geometrical line along the surface of a continuous portion of a glacier, the tendency to form an *open fissure* along that line will be measured by the intensity of the force which is normal to it; and in like manner the tendency to form a vertical plane or surface of discontinuity along the line will be measured by the tangential force f . If $f = 0$ along the proposed line, there can be no tendency in contiguous particles on opposite sides of this geometrical line to separate from each other by different velocities estimated along the line, since such a tendency must necessarily generate the tangential force (f) in question, and the greater the tendency the greater will be the force. Consequently there must be the greatest tendency to form these planes of discontinuity in those directions in which the force f is a maximum or minimum*. These directions are given by equation (4.) (art. 11), and are represented in diagram 3 of the same article. They are perpendicular to each other, and, in all cases, make angles of 45° with the directions of (R) and (r), those of maximum and minimum tensions. In these latter directions the values of f are always zero.

Hence, if there be no planes of less cohesion in the mass, and its cohesion should give way to the tangential force, before it yields to the normal tension, it must be along one of the lines for which f is a maximum or minimum, *i. e.* along a line inclined at an angle of 45° to the direction in which an open fissure would be formed, if formed at all, by the normal tension (R). If $X_1 = 0$ and $Y_1 = 0$, one of these directions will coincide with that of the axis of the glacier, to which the other will be perpendicular. Consequently there will be an equal tendency to fracture the mass in the way we are now considering, longitudinally and transversely. If the mass, instead of being constituted as just supposed, have a system of parallel planes of less cohesion, each fracture will take place along some one of such planes, provided they approximate sufficiently to those directions in which the fractures would take

* It must be recollected that the minimum only differs from the maximum value of f in *sign* and not in *magnitude*. Consequently there will be the same effort in both cases to overcome the cohesion of the mass.

place in a homogeneous mass; but if, on the contrary, the planes of less cohesion approximate to those directions in which $f=0$ (i. e. to those perpendicular to the open fissures), they will have no influence on the directions of the lines of fracture, for such fractures must either be formed exactly along the lines of less cohesion, or in the directions in which f is the greatest.

26. It may at first sight appear paradoxical that there should be the same tendency to fracture the mass by one particle sliding past another in a transversal as in a longitudinal direction; it is, however, unquestionably true. The difference between the two cases does not consist in any difference of conditions previous to fracture (or, therefore, in the tendency to produce fracture), but in the motion which may subsequently take place in the respective directions. The relative motion of particles on opposite sides of the transversal plane of fracture, will be limited to that which is just necessary to relieve the previous constraint of the mass; while the relative motion subsequent to a longitudinal fracture may be continued to any extent by the continued tendency of the mass on one side of such fracture to move faster than that on the opposite side.

Whether a continuous portion of ice of any considerable extent be ever fractured in the manner last considered, is uncertain, for no accurate observations have yet been made on the subject; but there can be no doubt that, in those portions of a glacier where it is much dislocated, the sliding of one portion past another must frequently take place. Such is doubtless the case along the dislocated flanks of glaciers, and thus the motion of the central part is much facilitated. If a glacier, continuous throughout its whole extent, could be formed, and then subjected to the existing conditions of glaciers, I have no doubt that the first fractures would be *transversal*, produced by the tension (R); and that after the mass had thus become considerably dislocated along its sides, the sliding of one finite portion past another would also take place; after which any internal constraint which might be superinduced by the peculiar motion of the glacier, would be relieved both by transverse fracture and by the sliding of one part past another.

27. If a column of glacial ice, of which the unextended length should $= l$, would just bear to be stretched to the length l' before it would break, the ratio $\frac{l'-l}{l}$ might be taken as a measure of the *extensibility* of glacial ice. If the extension should be produced within the comparatively short time of making an ordinary experiment of the kind, this ratio would, I think, unquestionably be very small; but if the extended

piece of ice should form a portion of an actual glacier, where the extension may be produced so slowly as to be continued for a number of years without producing fracture, it is very conceivable that the above ratio might become very much greater than in the former case. I have explained how the tension of the mass will be relieved in the direction of maximum tension by the formation of fissures. If the *elasticity* of the ice were *perfect*, a physical line in that direction would return to its original unstretched length, but will otherwise remain permanently elongated in a degree depending on its want of perfect elasticity. In a similar manner another line may become permanently compressed, and thus, by a repetition of the same processes, the linear dimensions of an element of the mass may become much altered in the course of a long period of time. In like manner a rectangular element may, in a long period, very slowly change its form so as to deviate very considerably from rectangularity without fracture. Thus it may happen that the flanks of a glacier may become dislocated with a much less degree of extension than the central portion, because the extension takes place much more rapidly near the sides than near the centre. As an example we may take the results of observations made by Prof. Forbes on the Mer de Glace. He ascertained the velocities at four stations in a transverse line, the first station being near the side of the glacier, and the second about 100 yards, and the third about 230 yards from the first; the fourth was near the centre of the glacier. The velocities were in proportion to the following numbers:—

A	B	C	D
1·000	1·332	1·356	1·367.

Consequently the *mean relative velocity* between A and B may be taken as =·332, and that between B and C

$$= (1·356 - 1·332) \frac{100}{130},$$

(since the distance B C = 130 and A B = 100)

$$= 0·2 \text{ nearly.}$$

The distance between C and D is not given, but assuming it to be nearly the same as between B and C, the mean relative velocity between C and D would be less than ·01, *i. e.* only about one-fortieth of that between A and B. Now, in the investigation of articles 4 and 5, $\frac{\beta}{a}$ is proportional to the relative velocity, and therefore the maximum extension, which $= \frac{\beta}{2a}$, is proportional to the relative velocity. Consequently,

in the case before us, the extension would increase forty times faster between the first and second stations than for a considerable space near the centre of the glacier; and if the extreme flanks were taken, the ratio would probably be still considerably greater. Thus, if the glacier at the present time were perfectly continuous, the flanks would probably be dislocated by transverse fissures in a few months, while the continuity of the central portion, independently of the action of local causes, might be preserved for many years, even if it should break with an extension no greater than that required to fracture the flanks. If also it be allowed that the extensibility becomes greater as the extension is produced more slowly, the glacier might move on for a still greater number of years without central dislocations due to the more rapid progression of its centre.

If this greater extensibility of glacial ice during a long period, or what may be called its *secular extensibility*, be allowed, we must also allow it to possess the property of *secular plasticity*, by which I mean that plasticity which shall require for its development the continuous application of force, possibly of great force, for a long period of time, in contradistinction to that plasticity by which a mass submits to an immediate and considerable change of form by the application of forces of comparatively small intensity. This latter kind of plasticity, when sufficiently great, becomes *semifluidity*; and if the observed phenomena of motion in a case like that of a glacier should justify the conclusion that the mass possesses this high degree of plasticity, we might be justified in asserting that the effectiveness of gravity to put the mass in motion was due to that property. But if, on the contrary, the observed circumstances of the motion only establish the existence of secular plasticity, and when, moreover, the motion is always accompanied by numerous and repeated fractures, we have no right thence to conclude that the motion is effectively due to the plasticity of the mass. I shall recur again to this point in the sequel.

28. With respect to the conclusions I have drawn from the preceding analysis of the problem of glacial motion, it is important to remark how entirely they are independent of the experimental determination of the values of the *constants* which afford us measures of such properties, for instance, as the elasticity, the extensibility, &c. of glacial ice; and also of all arbitrary constants and functions introduced into analytical results by the integration of the fundamental differential equations of the problem, and requiring for their determination a knowledge of the conditions of each particular problem. My

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conclusions have, in fact, been drawn from the conditions of equilibrium of an *element* of the mass, conditions, the analytical expression of which is only equivalent to the differential equations of the problem. The quantities involved in the preceding results are X_1 , Y_1 and f_1 . The directions in which the couples depending on f_1 act is determined by the observed fact of the central being greater than the lateral motion; the facts of the forces X_1 and Y_1 being pressures or tensions are inferred, and without any risk, as I conceive, of essential error, from the form and inclination of the glacial valley. And these are the points on which our conclusions depend; they are in a great degree independent of the actual values of the above forces. It would seem impossible to draw any accurate conclusions depending on less ambiguous circumstances. It may, however, be remarked that the whole investigation is tacitly based on the assumption of the continuity of the mass being preserved in the same sense as in the ordinary investigations of the motion of fluids. In the case before us the assumption has been, that, while a continuous mass having elasticity changes its form by the application of external forces, the particles which constitute an element of the mass, such as $p'q'r's'$ (fig. 6), in its original state of no constraint, continue to constitute a separate element, $p_1q_1r_1s_1$, during the whole change of form up to the state bordering on fracture. It is the common assumption on which all mathematical investigations of this nature are founded, and it is one which appears to me to possess the strongest *a priori* claims to our confidence, except under particular conditions with respect to the constitution of the mass. But I must reserve any further observations on this point for my next communication, in which, after this detailed exposition of my own views of the mechanics of the problem, I shall have some comments to make on those of Prof. Forbes.

I am, Gentlemen,

Your obedient Servant,

Cambridge, January 6, 1845.

W. HOPKINS.

Erratum in Mr. Hopkins's First Paper.

Page 15, line 2, for $\omega \sin \alpha$ read $\omega (\sin \alpha - \sin \beta) = \omega \sin \alpha$.

XVI. *On the Ventilation of the Coal-Mine Goaf.*

By MICHAEL FARADAY, Esq., D.C.L., F.R.S.

To Richard Taylor, Esq.

DEAR SIR,

YOU have honoured the Report by Mr. Lyell and myself with a place in the Philosophical Magazine, p. 16, and this induces me to send for insertion also, certain considerations.

Phil. Mag. S. 3. Vol. 26. No. 171. Feb. 1845.

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