On Deep-water Two-dimensional Waves produced by any given Initiating Disturbance. By Lord Kelvin.

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§ 1. Consider frictionless water in a straight canal, infinitely long and infinitely deep, with vertical sides. Let it be disturbed from rest by any change of pressure on the surface, uniform in every line perpendicular to the plane sides, and left to itself under constant air pressure. It is required to find the displacement and velocity of every particle of the water at any future time. Our initial condition will be fully specified by a given normal component velocity, and a normal component displacement, at every part of the surface.

§ 2. Taking O, any point at a distance h above the undisturbed water level, draw OX parallel to the length of the canal, and OZ vertically downwards. Let ξ, ζ be the displacement-components of any particle of the water whose undisturbed position is (x, z) . We suppose the disturbance infinitesimal; by which we mean that the change of distance between any two particles of water is infinitely small in comparison with their undisturbed distance; and the line joining them experiences changes of direction which are infinitely small in comparison with the radian. Water being assumed frictionless, its motion, started primarily from rest by pressure applied to the free surface, is essentially irrotational. Hence we have

$$\xi = \frac{d}{dx}\phi(x, z, t); \quad \zeta = \frac{d}{dz}\phi(x, z, t); \quad \dot{\xi} = \frac{d}{dx}\dot{\phi}; \quad \dot{\zeta} = \frac{d}{dz}\dot{\phi}. \quad (1);$$

where $\phi(x, z, t)$, or ϕ , as we may write it for brevity when convenient, is a function of the variables which may be called the displacement-potential; and $\dot{\phi}(x, z, t)$ is what is commonly called the velocity-potential. Thus a knowledge of the function ϕ , for all values of x, z, t , completely defines the displacement and the velocity of the fluid. And, by the fundamentals of hydrokinetics, a knowledge of ϕ for every point of the free

surface suffices to determine its value throughout the water ; in virtue of the equation

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dz^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The motion being infinitesimal, and the density being taken as unity, another application of the fundamental hydrokinetics shows that, as found by Cauchy and Poisson,

$$p - \Pi = g(z - h + \zeta) - \frac{d^2\phi}{dt^2} = g(z - h) + g \frac{d\phi}{dz} - \frac{d^2\phi}{dt^2} \quad . \quad (3);$$

where g denotes gravity ; Π the uniform atmospheric pressure on the free surface ; and p the pressure at the point $(x, z + \zeta)$ within the fluid.

§ 3. To apply (3) to the wave-surface, put in it, $z = h$; it gives

$$g \left(\frac{d\phi}{dz} \right)_{z=h} = \left(\frac{d^2\phi}{dt^2} \right)_{z=h} \quad . \quad . \quad . \quad . \quad . \quad (4);$$

and therefore if we could find a solution of this equation for all values of z , with (2) satisfied, we should have a solution of our present problem. Now we *can* find such a solution ; by a curiously altered application of Fourier's celebrated solution

$$\left[(t+c)^{-1} e^{\frac{-x^2}{4k(t+c)}}, \text{ for } \frac{dv}{dt} = k \frac{d^2v}{dx^2}, \right]$$

his equation for the linear conduction of heat. Change $t+c, x, k$, into $z+\iota, t, g^{-1}$ respectively :—we have (4), and we see that a solution of it is

$$\frac{1}{\sqrt{(z+\iota)}} e^{\frac{-gt^2}{4(z+\iota)}} \quad . \quad . \quad . \quad . \quad . \quad (5);$$

which also satisfies (2) because any function of $z+\iota x$ satisfies (2) if ι denotes $\sqrt{-1}$. Hence if {RS} denotes a realisation* by taking half sum of what is written after it with $\pm \iota$, we have, as a real solution of (4) for our problem

$$\phi(x, z, t) = \{RS\} \frac{1}{\sqrt{(z+\iota x)}} e^{\frac{-gt^2}{4(z+\iota x)}} \quad . \quad . \quad . \quad . \quad (6),$$

* A very easy way of effecting the realisations in (6) and (9) is by aid of De Moivre's theorem with, for one angle concerned in it, $\chi = \tan^{-1}x/z$; and another angle $= gt^2x/4(z^2+x^2)$.

$$= \frac{1}{\sqrt{2} \cdot \rho} \left[\sqrt{(\rho+z)} \cos \frac{gt^2 x}{4\rho^2} + \sqrt{(\rho-z)} \sin \frac{gt^2 x}{4\rho^2} \right] \epsilon^{\frac{-gt^2 z}{4\rho^2}} \left. \vphantom{\frac{1}{\sqrt{2} \cdot \rho}} \right\} \quad (7),$$

where $\rho^2 = z^2 + x^2$

$$= \sqrt{\frac{1}{\rho}} \sin \left(\frac{gt^2 x}{4\rho^2} + \theta \right) \epsilon^{\frac{-gt^2 z}{\rho^2}} \left. \vphantom{\frac{1}{\rho}} \right\} \quad (8).$$

where $\theta = \tan^{-1} \sqrt{\frac{\rho+z}{\rho-z}}$

The sign of $\sqrt{(\rho-z)}$ changes when x passes through zero.

Going back now to (5), and denoting by {RD} the difference of its values for $\pm \iota$ divided by 2ι , we have another solution of our problem essentially different from (6), as follows

$${}_2\phi(x, z, t) = \{RD\} \frac{1}{\sqrt{(z+\iota x)}} \epsilon^{\frac{-gt^2}{4(z+\iota x)}} \quad (9),$$

$$= \frac{1}{\sqrt{2} \cdot \rho} \left[\sqrt{(\rho+z)} \sin \frac{gt^2 x}{4\rho^2} - \sqrt{(\rho-z)} \cos \frac{gt^2 x}{4\rho^2} \right] \epsilon^{\frac{-gt^2 z}{4\rho^2}} \quad (10),$$

$$= \sqrt{\frac{1}{\rho}} \sin \left(\frac{gt^2 x}{4\rho^2} + \theta - \frac{\pi}{2} \right) \epsilon^{\frac{-gt^2 z}{4\rho^2}} \quad (11).$$

§ 4. The annexed diagram, fig. 1, represents for $t=0$ the solutions ${}_2\phi$ and ${}_1\phi$ as functions of x , with $z=1$ for convenience in the drawing. The formulas which we find by taking $t=0$ in (7) $\times \sqrt{2}$ and (10) $\times \sqrt{2}$ are

$${}_1\phi = \frac{\sqrt{[\sqrt{(x^2+z^2)}+z]}}{\sqrt{(x^2+z^2)}}; \quad {}_2\phi = \frac{\sqrt{[\sqrt{(x^2+z^2)}-z]}}{\sqrt{(x^2+z^2)}} \quad (12).$$

Before passing to the practical interpretation of our solutions, remark first that (12) contain full specifications of two distinct initiating disturbances; in each of which ϕ may be taken as a displacement-potential, or as a velocity-potential, or as a horizontal displacement-component or velocity, or as a vertical displacement-component or velocity. Thus we have really preparation for *six* different cases of motion, of which we shall choose one, $-\zeta = \sqrt{2} \times (7)$, for detailed examination.

§ 5. Taking $z=h=1$, for the water surface, let the two curves of figure 1 represent initial *displacements*, (12), of the water surface, left to itself with the water everywhere at rest. The displacements at any subsequent time t are expressed in real symbols by (7) (10) without the divisor $\sqrt{2}$, and by (8) (11) with a factor $\sqrt{2}$ introduced; either of which may be chosen according to convenience in calculation. One set has thus been calculated from (8), with

FIG. 1. $t=0$, $z=1$. Abscissas represent time; ordinates $\frac{\sqrt{(\rho+z)}}{\rho}$, $\frac{\sqrt{(\rho-z)}}{\rho}$.

$g = 4$, and $z = 1$, for six values of t ; $\cdot 5$, $1\cdot 5$, 2 , $2\cdot 5$, and 5 ; and for a sufficiently large number of values of x to represent the results by the curves shown in figs. 2 and 3. Except for the time $t = 5$, each curve shows sufficiently all the most interesting characteristics of the figure of the water at the corresponding time. The curve for $t = 5$ does not perceptibly leave the zero line at distances $x < 1\cdot 8$; but if we could see it, it would show us two and a half wavelets possessing very interesting characteristics; shown in the table of numbers, § 7 below, by which we see that several different curves with scales of ordinates magnified from one to one thousand, and to one million, and to ten thousand million, would be needed to exhibit them graphically.

§ 6. Looking to the curves for $t = 0$ and $t = \frac{1}{2}$; we see that at first the water rises at all distances from the middle of the disturbance greater than $x = 1\cdot 9$, and falls at less distances. And we see that the middle ($x = 0$) remains a crest (or positive maximum) till a very short time before $t = \frac{1}{2}$, when it begins to be a hollow. A crest then comes into existence beside it and begins to travel outwards. On the third curve, $t = 1$, we see this crest, travelled to a distance $x = 1\cdot 7$, from the middle where it came into being; and on the fourth, fifth, sixth, seventh curves (figs. 1, 2) we see it got to distances $2\cdot 9$, $4\cdot 8$, $6\cdot 5$, 22 , at the times $1\frac{1}{2}$, 2 , $2\frac{1}{2}$, 5 . This crest travelling rightwards on our diagrams has its anterior slope very gradual down to the undisturbed level at $x = \infty$. Its posterior slope is much steeper; and ends at the bottom of the hollow in the middle of the disturbance, at times from $t = \frac{1}{2}$ to $t = 1\frac{1}{2}$. At some time, which must be very soon after $t = 1\frac{1}{2}$, this hollow begins to travel rightwards from the middle, followed by a fresh crest shed off from the middle. At $t = 2$, the hollow has got as far as $x = \cdot 9$; at $t = 2\frac{1}{2}$, and 5 , respectively, it has reached $x = 1\cdot 75$, and $x = 6\cdot 7$. Looking in imagination to the extension of our curves leftwards from the middle of the diagram, we find an exact counterpart of what we have been examining on the right. Thus we see an initial elevation, symmetrical on the two sides of a convex crest, of height $1\cdot 41$ above the undisturbed level, sinking in the middle and rising on the two flanks. The crest becomes less and less convex till it gets down to height $1\cdot 1$, when it becomes concave; and two equal and similar wave-crests

are shed off on the two sides, travelling away from it rightwards and leftwards with accelerated velocities, each remaining for ever convex. Thus we see the beginnings of two endless processions of waves travelling outwards in the two directions; originating as infinitesimal wavelets shed off on the two sides of the middle line. Each crest and each hollow travels with increasing velocity. Each wave-length, from crest to crest, or from hollow to hollow, becomes longer and longer as it advances outwards; all this according to law fully expressed in (8) of § 3 above.

§ 7. Here is now the table of numbers promised in § 5 above; it practically defines the forms and magnitudes of the two and a half wavelets, between $x=0$ and $x=2$, which the space-curve for $t=5$ (figs. 2 and 3) fails to show.

$$\rho^2 = x^2 + h^2; h = 1; g = 4; t = 5; -\zeta = \sqrt{\frac{2}{\rho}} \sin\left(\frac{xt^2}{\rho^2} + \theta\right) \epsilon^{\frac{-t^2}{\rho^2}}.$$

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.	Col. 7.
x .	$\sqrt{\frac{2}{\rho}}$.	$\sqrt{\rho+1} = \sin \theta$.	Initial Elevation of Water-surface at Distance x . $\sin \theta \cdot \sqrt{\frac{2}{\rho}} = \zeta_0$.	$\sin\left(\frac{xt^2}{\rho^2} + \theta\right)$.	$\epsilon^{\frac{-t^2}{\rho^2}}$.	Elevation of Water-surface at Time t , and Distance x . $-\zeta$.
0	1.4142	1.0000	1.4142	1.0000	10^{-10} 1357	$+10^{-10}$ 1963
.05	1.4140	.9997	1.4140	.3434	„ 1478	„ „ .0717
.064	...	...	...	0	...	„ „ 0
.10	1.410	.9987	1.409	-.7541	„ 1778	-10^{-10} 1891
.15	1.407	.9972	1.403	-.8997	„ 3066	„ „ .3882
.20	1.401	.9952	1.393	-.0032	„ 3632	„ „ .0016
.202	...	...	...	0	...	„ „ 0
.30	1.384	.9894	1.370	.8997	„ 1.094	$+10^{-10}$ 1362
.363	...	...	...	0	...	„ „ 0
.40	1.362	.9820	1.338	-.5451	„ 4.366	-10^{-10} 243
.60	1.309	.9638	1.262	-.2341	„ 103.9	„ „ 31.84
.632	...	...	...	0	...	„ „ 0
.80	1.249	.9437	1.179	.7593	10^{-5} .02396	$+10^{-5}$ .0227
1.00	1.190	.9239	1.099	.8962	„ .2958	„ „ .3152
1.25	1.118	.9015	1.007	.6831	„ 5.793	„ „ 4.424
1.50	1.053	.8817	.9287	.4923	„ 45.63	„ „ 23.67
1.517	...	...	...	0	...	„ „ 0
1.75	.9961	.8651	.8616	-.6832	„ 212.5	-10^{-5} 144.6
2.00	.9456	.8506	.8043	-.9997	„ 848.2	„ „ 801.9
2.50	.8612	.8243	.7142	-.1633	„ .03180	„ „ 447.3
2.54	...	...	...	0	...	„ „ 0
3.00	.7952	.8113	.6450	.8296	„ .08210	„ .0542
3.50	.7411	.7980	.5917	.9473	„ .1516	„ .1064

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FIG. 2.

FIG. 3.

$$h = 1 ; g = 4 ; t = 5 ; -\zeta = \sqrt{\frac{2}{\rho}} \sin \left(\frac{xt^2}{\rho^2} + \theta \right) \epsilon^{\frac{-t^2}{\rho^2}}$$

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.	Col. 7.
$x.$	$\sqrt{\frac{2}{\rho}}$	$\sqrt{\frac{\rho+1}{2\rho}} \sin \theta.$	Initial Elevation of Water-surface at Distance $x.$ $\sin \theta. \sqrt{\frac{2}{\rho}} = \zeta_0.$	$\sin \left(\frac{xt^2}{\rho^2} + \theta \right).$	$\epsilon^{\frac{-t^2}{\rho^2}}.$	Elevation of Water-surface at Time t , and Distance $x.$ $-\zeta.$
4.0	.6965	.7882	.5490	.4856	.2298	.07771
4.41	...	...	...	0	...	0
4.5	.6588	.7798	.5139	-.0944	.3083	-.01917
5.0	.6262	.7733	.4843	-.5584	.3823	-.1336
5.5	.5981	.7678	.4592	-.8457	.4493	-.2273
6.0	.5733	.7629	.4375	-.9781	.5122	-.2872
6.5	.5513	.7587	.4185	-.9956	.5641	-.3096
7.0	.5318	.7555	.4018	-.9374	.6066	-.3024
7.5	.5150	.7522	.3868	-.8333	.6462	-.2773
8.0	.4980	.7494	.3734	-.7053	.6808	-.2392
9.0	.4699	.7451	.3501	-.4289	.7372	-.1486
10	.4461	.7416	.3308	-.1679	.8346	-.04753
10.62	...	...	...	0	...	0
11	.4255	.7385	.3142	.05698	.8147	.01975
12	.4076	.7359	.2999	.2428	.8416	.08375
13	.3916	.7339	.2874	.3940	.8644	.1334
14	.3775	.7318	.2762	.5175	.8808	.1721
15	.3648	.7302	.2663	.6163	.8954	.2013
16	.3533	.7286	.2574	.6953	.9082	.2231
18	.3331	.7266	.2419	.8098	.9260	.2498
20	.3160	.7256	.2290	.8831	.9396	.2622
22	.3014	.7230	.2179	.9313	.9497	.2666
24	.2885	.7216	.2082	.9627	.9579	.2661
26	.2772	.7206	.1997	.9815	.9638	.2622
28	.2672	.7193	.1923	.9915	.9685	.2565
30	.2581	.7187	.1856	.9979	.9727	.2505
32	.2500	.7181	.1795	.9999	.9759	.2439
34	.2425	.7173	.1740	.9993	.9786	.2371
38	.2294	.7163	.1643	.9933	.9828	.2240
42	.2182	.7155	.1561	.9840	.9847	.2188
46	.2084	.7147	.1490	.9734	.9883	.2005
50	.2000	.7141	.1429	.9623	.9902	.1905
55	.1906	.7135	.1360	.9486	.9917	.1794
60	.1826	.7129	.1302	.9361	.9931	.1697
70	.1690	.7120	.1203	.9125	.9949	.1535
80	.1581	.7114	.1125	.8931	.9961	.1407
100	.1415	.7108	.1005	.8626	.9977	.1217
∞	0	.7071	0	.7071	1.0000	0

§ 8. Look at the values shown in the previous table for the three factors which constitute ζ ;—we see that the first factor (col. 2) decreases slowly from $x=0$ to $x=\infty$; the second factor (col. 5) alternates between $+1$ and -1 with increasing distances (semi-wave-lengths) from zero to zero as x increases. The third

factor (col. 6) increases gradually from $e^{-\rho^2/h^2}$ at $x=0$, to 1 at $x=\infty$. At $x=50h$, the third factor is .99, which is so nearly unity that the diminution of amplitude is, for all greater values of x , practically given by the first factor alone, which diminishes from .2 at $x=50h$, to 0 at $x=\infty$.

§ 9. The diagrams hitherto given, figs. 1, 2, 3, may be called space-curves, as on each of them abscissas represent distance from the centre of the disturbance. Fig. 4 is a time-curve (abscissas representing time) for $x=2h$. It represents a very gradual rise, from $t=0$ to $t=.6$, followed by a fall to a minimum at $t=2.8$, and a succession of alternations, with smaller and smaller maximum elevations and depressions, and shorter and shorter times from zero to zero, on to $t=\infty$. The same words with altered figures describe the changes of water level at any fixed position farther from the centre of disturbance than $x=2$. The following table shows, for the case $x=100h$, all the times of zero less than $71h$, and the elevations and depressions at the intermediate times when the second factor (col. 5 of § 7) has its maximum and minimum values (± 1). These elevations and depressions are very approximately the greatest in the intervals between the zeros, because the third factor (col. 6, § 7) varies but slowly, as shown in the first column of the present table.

$$h=1; x=100; \rho=100.005.h; \theta=\tan^{-1}\sqrt{\frac{101}{99}}=45^\circ 18'.$$

$\frac{-t^2}{e^{\rho^2}}$	Times of Zero and of Approximate Maximum Elevation and Depression.	Approximate Maximum Elevations and Depressions.	$\frac{-t^2}{e^{\rho^2}}$	Times of Zero and of Approximate Maximum Elevation and Depression.	Approximate Maximum Elevations and Depressions.
.9922	8.333	+ .1403	.7718	50.90	+ .1091
...	15.33	0	...	52.42	0
.9616	19.80	- .1360	.7478	53.90	- .1058
...	23.43	0	...	55.34	0
.9317	26.57	+ .1317	.7247	56.74	+ .1025
...	29.38	0	...	58.10	0
.9031	31.94	- .1277	.7023	59.45	- .0993
...	34.31	0	...	60.75	0
.8750	36.54	+ .1237	.6806	62.03	+ .0962
...	38.62	0	...	63.29	0
.8480	40.61	- .1199	.6595	64.51	- .0933
...	42.50	0	...	65.72	0
.8219	44.31	+ .1162	.6392	66.90	+ .0904
...	46.04	0	...	68.07	0
.7964	47.72	- .1157	.6195	69.21	- .0876
...	49.34	0	...	70.34	0

FIG. 4. $x = 2h$, $g = 4$. Abscissas represent time. $-\zeta = \sqrt{\frac{2}{\sqrt{5}}} \sin\left(\frac{2t^2}{5} + \sin^{-1} \frac{\sqrt{5}+1}{2\sqrt{5}}\right) \epsilon^{\frac{t^2}{5}}$.

§ 10. Our assumption $h=1$ for the free surface involves no restriction of our solution to a particular case of the general formula (7). Our assumption $g=4$ merely means that our unit of abscissas is half the space fallen through in our unit of time. The fundamental formulas of § 3 may be geometrically explained by, as in § 2, taking O , our origin of co-ordinates, at a height h above the water level, and defining ρ as the distance of any particle of the fluid from it. When, as in §§ 5-9, we are only concerned with particles in the free surface (that is to say when $z=h$), we see that if x is a large multiple of z , $\rho \doteq x$. See for example the heading of the table of § 9. And if we are concerned with particles below the surface, we still have $\rho \doteq x$, if x is a large multiple of z . Thus we have the following approximation for (7) of § 3:—

$${}_1\phi(x, z, t) \doteq \frac{1}{\sqrt{2}x} \left[\sqrt{(x+z)} \cos \frac{gt^2}{4x} + \sqrt{(x-z)} \sin \frac{gt^2}{4x} \right] e^{\frac{-gt^2z}{4x^2}} \quad (13).$$

Suppose now $d\phi/dt$ to represent ζ (instead of ϕ , as in §§ 5-9); we have

$$\zeta = \frac{d}{dt} {}_1\phi(x, z, t) \quad . \quad . \quad . \quad . \quad . \quad (14),$$

which is easily found from (13) without farther restrictive suppositions. But if we suppose that z is negligibly small in comparison with x ; and farther that

$$\frac{gt^2z}{4x^2} \doteq 0 \quad . \quad . \quad . \quad . \quad . \quad (15),$$

we find by (14)

$$\zeta \doteq \frac{gt}{2\sqrt{2}x^{3/2}} \left(\cos \frac{gt^2}{4x} - \sin \frac{gt^2}{4x} \right) \quad . \quad . \quad . \quad . \quad . \quad (16).$$

This, except the sign $-$ instead of $+$, is Cauchy's solution;* of which he says that when the time has advanced so much as to violate a condition equivalent to (15), "le mouvement change avec la méthode d'approximation." The remainder of his Note XVI. (about 100 pages) is chiefly devoted to very elaborate efforts to obtain definite results for the larger values of t . This object is thoroughly attained by the exponential factor in (8) of § 3 above, without the crippling restriction $z/x \doteq 0$ which vitiates (16) for small values of x .

* *Œuvres*, vol. i. note xvi. p. 193.

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