XXXVI.—*On the Application of the Theory of Probabilities to the Question of the Combination of Testimonies or Judgments.* By GEORGE BOOLE, LL.D., Professor of Mathematics in Queen's College, Cork. Communicated by Bishop TERROT.

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1. The method for the solution of questions in the theory of probabilities applied in this paper, is that which was developed by the author in a treatise entitled, “An Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities.” The practical object of the paper is to deduce from that method certain conclusions relating to the combination of testimonies or judgments. Beside this, however, it will have a speculative reference to some more general questions connected with the theory of probabilities; and especially to the following question, viz.: To what extent the different modes in which the human mind proceeds, in the estimation of probability, may be considered as mutually confirming each other,—as manifestations of a central unity of thought amid the diversity of the forms in which that unity is developed.

The special problems relating to the combination of testimonies or judgments which are considered in this paper are the following: *1st*, That in which the testimonies to be combined are merely differing numerical measures of a physical magnitude, as the elevation of a star, furnished by different observations taken simultaneously; *2dly*, That in which the testimonies or judgments to be combined relate not to a numerical measure, but to some fact or hypothesis of which it is sought to determine the probability,—the probabilities furnished by the separate testimonies or judgments constituting our data.

2. I have, in the treatise to which reference has been made, described the method which will be practically applied in this paper as a general one. It will, I think, ultimately appear that there is a true and real sense in which the propriety of the description may be maintained. But at present I am anxious to qualify the appellation, and to speak of the method as general only with respect to problems which have been resolved into purely logical elements, or which are capable of such resolution. A more thorough analysis of the mental phenomena of expectation will, I think, tend to establish the position that all questions of probability, in the mathematical sense, admit of being resolved into primary elements of this nature, or, to speak more strictly, admit of being adequately represented by other problems whose elements are logical only. Postponing the consideration of this question, I will first endeavour to explain what is meant by the logical elements of a problem, and how the consideration of such elements affects the mode of its solution.

I regard the elements of a problem relating to probability as logical, when its data and its quæsitum are the probabilities of *events*. The reason for this appellation will shortly be seen. In *expression*, events may be distinguished as simple or compound. A simple event, *i.e.*, an event simple in expression, is one which is expressed by a single term or predication; a compound event, one which is formed by combining the expressions of simple events. “It rains,”—“it thunders,” would be simple events; “it rains and thunders,”—“it either rains or thunders,” &c., would be compound events. The constructions by which such combinations are expressed, although they belong to language, have their foundations in Logic. Thus the conjunctions *and*, *either*, *or*, &c., express merely certain operations of the faculty of Conception, the entire theory of which belongs to the science of Logic. The calculus of Logic, to which I shall have occasion to refer, is a development of that science in mathematical forms, in which letters represent things, or events, as subjects of Conception, and signs connecting those letters represent the operations of that faculty, the laws of the signs being the expressed laws of the operations signified. It is simply a mistake to regard that calculus as an attempt to reduce the ideas of Logic under the dominion of number. Such are the grounds upon which the class of problems to which I have referred are said to involve logical elements. The description is, however, not entirely appropriate, for the problems, as they are concerned with probabilities, in the mathematical acceptance of that term, involve numerical as well as logical elements; but it is by the latter that they are distinguished, and of them only is account taken in the nomenclature.

Thus, as an illustration of what has been said, that problem would be composed of logical elements, which, assigning for its numerical data the probabilities of the throwing an ace or six with each single die, should propose to determine the probability that the issue of a throw with two dice should be two aces, or that it should be an ace and a six, or that it should be either two aces or an ace and a six; and so on for any conceivable throw with any number of dice.

3. In the above example, the events whose numerical probabilities are given are simple events, of which the event whose probability is sought is a logical combination. But it might happen that the former events were themselves combinations of simple events. For instance, the data might be the probabilities that certain meteorological phenomena, as rain, thunder, hail, &c., would occur in certain definite combinations, and the object sought might be the probability that they would occur in certain other combinations; all these combinations being, such as it is within the province of language to express by means of conjunctions, and of the adverb *not*. Now this would still be a problem whose elements are logical.

4. But there are questions universally recognised as belonging to the theory of probabilities, whose elements cannot, in their direct significance, be regarded

as logical. The problem of the reduction of astronomical observations belongs to this class. Two observers, equally trustworthy, take an observation at the same place and time of the altitude of a star. One of them declares that it is $50^{\circ} 20'$, the other that it is $50^{\circ} 22'$. From these data, what shall we regard as the most probable altitude? We cannot, in this case, directly affirm that the numerical data are measures of probability at all. They are conflicting measures of a physical magnitude. And that which is sought is not the measure of a probability, but the most probable measure of the same magnitude. This is a problem evidently of a different kind from the one which we last considered. And accordingly it will be found that the principles of solution which have been actually applied to it are different from, perhaps we ought rather to say supplementary to, those which have sufficed for the solution of the others. In the problem of the dice, we have only to apply, and that directly, such principles as the following, viz., that when the probability of the occurrence of an event is p , that of its non-occurrence is $1-p$; that if the probabilities of two independent events are p and q , that of their concurrence is pq ; and so on. In the reduction of the conflicting elements of the observers' problem, another and quite distinct principle is usually employed, viz., the principle of the arithmetical mean, which affirms that if two different values are, on equal authority, assigned to a magnitude which is in itself single and definite, the mind is led to consider the arithmetical mean of those values as more likely to be its true measure than any other value. This is not the only principle which has been employed for the reduction in question. We shall refer to others. But it may justly be regarded as the most obvious of all which have been employed; and there is ground for considering it, as some eminent writers have expressly done, as primary and axiomatic in its nature.

5. The following is the typical form of problems whose elements are logical. If we represent the simple events involved in their expression by x, y, z , &c., then may all their data (we will suppose the number of data to be n) be expressed, in accordance with the principles of the calculus of Logic, under the general forms

$$\text{Prob. } \phi_1(x, y, z \dots) = p_1, \quad \text{Prob. } \phi_2(x, y, z \dots) = p_2, \dots \text{Prob. } \phi_n(x, y, z \dots) = p_n,$$

and the quæsitum, or object sought, will be the value of

$$\text{Prob. } \psi(x, y, z \dots),$$

where $\phi_1, \phi_2, \dots \phi_n$ and ψ denote different but given logical functions of x, y, z .

Although the method for the solution of questions in the Theory of Probabilities whose elements are logical has been developed at considerable length in a special chapter of the *Laws of Thought*, yet much that is essential for its proper and distinctive exhibition, has only been discovered since the publication of that work. For this reason it will be proper to offer some account here of the principles upon which the method rests.

6. I define the mathematical probability of an event as the ratio which the number of distinct cases or hypotheses favourable to that event bears to the whole number of distinct cases possible, supposing that to none of those cases the mind is entitled to give any preference over any other. Fundamentally, this definition agrees with that of LAPLACE. “La théorie des hazards consiste,” he remarks, “à réduire tous les évènements du même genre a un certain nombre de cas également possibles c’est à dire tels que nous soyons également indécis sur leur existence et à déterminer le nombre de cas favorables à l’évènement dont on cherche la probabilité. Le rapport de ce nombre a celui de tous les cas possibles est la mesure de cette probabilité.”—*Essai Philosophique sur les Probabilités*.

It is implied in this definition, that probability is relative to our actual state of information, and varies with that information. Of this principle LAPLACE gives the following illustration:—“Let there be three urns, A, B, C, of which we are only informed that one contains black and the other white balls; then, a ball being drawn from C, required the probability that the ball is black. As we are ignorant which of the urns contains black balls, so that we have no reason to suppose it to be the urn C rather than the urn A or the urn B, these three hypotheses will appear equally worthy of credit, but as the first of the three hypotheses alone is favourable to the drawing of a black ball from C, the probability of that event is $\frac{1}{3}$. Suppose now that, in addition to the previous data, it is known that the urn A contains only white balls, then our state of indecision has reference only to the urns B and C, and the probability that a ball drawn from C will be black is $\frac{1}{2}$. Lastly, if we are assured that both A and B contain white balls only, the probability that a black ball will issue from C rises into certitude.”—*Essai Philosophique sur les Probabilités*, p. 9.—(*Phil. Mag.*, p. 433.) Our estimate of the probability of an event varies not absolutely with the circumstances which actually affect its occurrence, but with our knowledge of those circumstances.

7. When the probabilities of simple events constitute our only data, we can, by virtue of the above definition, determine the probability of any logical combination of those events, and this either, 1st, absolutely; or, 2dly, conditionally. The reason why we can, in this case, more immediately apply the definition is, that not only is no connection expressed among the events whose probabilities are given, but none is implied, nor is any restraint imposed upon their possible combinations. This, as we shall see, is not the case when the data are the probabilities of compound events.

As an example, let us suppose that the probability of the conjunction of two events, x and y , is required, the data being simply that the probability of the event x is p , and that of the event y is q . Or, to express the problem in a form which we shall hereafter generally employ:

Given	Prob. $x=p$,	Prob. $y=q$,
Required	Prob. $x y$.	

Let a be the number of distinct cases favourable to the event x , out of m distinct cases equally possible, from the comparison of which the probability p has been assigned to the event x . In like manner let b be the number of distinct cases favourable to the event y , out of n distinct cases equally possible, from the comparison of which the probability q has been assigned to the event y . Then,

$$\frac{a}{m}=p \quad \text{and} \quad \frac{b}{n}=q.$$

Now the conjunction xy can only come to pass through the combination of some one of the a cases in which x happens, with some one of the b cases in which y happens, at the same time that we have an equal right to suppose that any one of the m cases in which x happens or fails may combine with any one of the n cases in which y happens or fails. To none of these combinations is the mind entitled to attach any preference over any other, Hence there exist ab distinct cases favourable to the conjunction of x and y out of a total of mn distinct and equally possible cases. Thus, by the definition, the probability of the conjunction of x and y will be represented by the product $\frac{ab}{mn}$ or pq .

Here the question may be asked,—Does, then, no difference exist between the case in which the events x and y are known to be independent, and that in which we are simply ignorant of the existence of any connection between them? I reply that there is none, so far as the numerical estimation of probability is concerned. There is, however, an important difference as respects the practical value of the numerical result. If the events x and y are known to be independent, and to have probabilities p and q , we know that, in the long run, the conjunction xy will tend to recur with a frequency which will be proportional to the magnitude of the fraction pq . We do not know that this will be the case if we are simply ignorant of any connection between x and y . This is the difference referred to, and it is an important one. But it does not affect the calculation of probability as flowing from the definition of its numerical measure.

8. As from the data Prob. $x=p$, Prob. $y=q$, we deduce Prob. $xy=pq$, so from the same data we should have, adopting the language of the calculus of Logic,

$$\text{Prob. } x(1-y)=p(1-q) \quad \text{Prob. } (1-x)(1-y)=(1-p)(1-q),$$

and so on. Here $x(1-y)$ denotes the compound event which consists in the occurrence of x conjointly with the non-occurrence of y ; $(1-x)(1-y)$, the compound event which consists in the joint non-occurrence of both x and y .

Extending this mode of investigation, we arrive at the theorem

$$\text{Prob. } \phi(x, y, z \dots) = \phi(p, q, r \dots) \quad . \quad . \quad . \quad (1)$$

where x, y, z , &c., denote any simple events whose probabilities (our only data) are $p, q, r \dots$, and $\phi(x, y, z \dots)$ denotes any event which can be expressed by

means of the simple events x, y, z , &c., in accordance with the notation of the calculus of Logic.

By the above theorem the probability of any compound event is determined absolutely, when the probabilities of its simple components are given.

9. And by the same mode of investigation, the probability of any combination may be determined conditionally, *i.e.*, the probability which the combination will have under a given condition consisting in the happening of some other combination. Thus, if our data are as before,

$$\text{Prob. } x=p, \quad \text{Prob. } y=q, \quad \text{Prob. } z=r, \quad \&c.$$

and if we require the probability that if the event $\phi(x, y, z \dots)$ present itself, the event $\psi(x, y, z \dots)$ will be present at the same time, we may demonstrate the following result, viz. :—

Prob. that if $\phi(x, y, z \dots)$ happen, $\psi(x, y, z \dots)$ will be present also

$$= \frac{\chi(p, q, r \dots)}{\phi(p, q, r \dots)} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

where the form of the function χ is determined by multiplying together, according to the principles of the calculus of Logic, the functions $\phi(x, y, z \dots)$ and $\psi(x, y, z \dots)$, and representing the result by $\chi(x, y, z \dots)$.—(*Laws of Thought*, p. 258, Prop. I.)

10. I postulate that when the data are not the probabilities of simple events, we must, in order to apply them to the calculation of probability, regard them, not as primary, but as derived from some anterior hypothesis, which presents the probabilities of simple events as its system of data, and exhibits our actual data as flowing out of that system, in accordance with those principles which have already been shown to be involved in the very definition of probability.

The ground of this postulate is, that to begin with the simple and proceed to the complex, seems to be, in all questions involving combinations such as we are here concerned with, a necessary procedure of the understanding. The calculation of probability depends upon combinations subject to a peculiar condition, viz., that they shall always present to us a series of cases or hypotheses, to none of which the mind is entitled to attach any preference over any other. We cannot, in endeavouring to ascend from the complex to the simple, secure the maintenance of this condition; but we can do so in descending from the simple to the complex. We have had an illustration of this truth in the reasoning by which we deduced the expression for the probability of the complex event xy from the probabilities of the simple events x and y , supposed to be given. And the methods which have been actually employed in the solution of problems whose immediate data were not the probabilities of simple events, have in fact rested upon the postulate above referred to. Thus in questions relating to juries, the immediate data are the probabilities, founded upon continued observation, that a decision will be unanimous, or that it will be pronounced by a given majority, &c.

But it is usual, in solving these problems, to regard such events as compound, and to derive them from a hypothesis which presents as its scheme or system of data, the probabilities of individual correctness of judgment in the members of the jury; the correctness of judgment in any such members being regarded as a simple event. And this mode of procedure is a very natural and very obvious one. For the degree of unanimity of a decision will so far depend upon the correctness of judgment in the members, that, if we knew what the probability of correctness in each member was, we could determine *à priori* the probability of any proposed measure of agreement in the body.

The only question which arises, indeed, is not concerning the necessity of the postulate, but concerning the mode in which it may be lawfully applied. How shall we lawfully construct the hypothesis by which the solution of a problem shall be made to depend upon the consideration of simple events. In answering this question, I will endeavour to show, *1st*, upon what the construction of the hypothesis does not depend; *2dly*, upon what it does depend.

11. The legitimate construction of the hypothesis in question cannot depend upon the accidents of language, or causes deeper than accident, which have led us to express particular things or events by simple terms, thus regarding them as simple events; and other events by combinations of these simple terms, thus presenting them as compound. The solution of a question in the theory of probabilities must depend upon the *information* conveyed in the data, not upon the peculiar elements and constructions of the language which is the vehicle of that information. Languages differ widely in these respects. Objects and events which in one language are expressed by simple terms, are in another expressed by combinations of simple terms. It is affirmed that a perfectly general method of solution must be independent of, and superior to, differences like these.

I will endeavour to illustrate this principle by an example. Let the problem to be resolved be the following. The probability of the concurrence of rain and snow is p , of the concurrence of snow and wind q and of the concurrence of wind and rain r ; required the probability of the concurrence of wind, rain, and snow.

Now suppose that we had to interpret the problem into a language in which there were no simple terms corresponding to the simple terms "wind," "rain," "snow," but in which there were simple terms for the three first of the concurrences above described.

We may, for simplicity, suppose that language to be a dialect of English, and the concurrence of rain and snow to be represented in it by the term "sleet," the concurrence of snow and wind by the term "drift," and the concurrence of wind and rain by the term "storm."

The event whose probability is sought, viz., the concurrence of rain, snow, and wind, would, in such a language, be represented by the combination either of two of the terms above defined (as of sleet with drift), or of all the terms together.

since the presence of any two of the phenomena "sleet," "drift," "storm," implies that of the third, and involves the conjunction of the phenomena of rain, snow, and wind.

The data of the problem we are considering might then, in the imagined dialect, assume the following form :

The probability of sleet is p , of drift q , and of storm r ; required the probability of the concurrence of the phenomena of sleet, drift, and storm.

But in this form the problem would not, in its data, express all the knowledge which the person using such language must possess of the connection of the events to which it related. He must know that it was impossible that any two of the events sleet, drift, and storm, should occur without the third, so that the problem, if so stated as to embody the same amount of actual knowledge as is conveyed in the previous statement of it, would assume the following form :

The probability of sleet is p , of drift q , of storm r , and these events are so connected, that no two of them can occur without the third occurring, What is the probability of their concurrence?

Now the principle affirmed declares that the solution of the problem must be the same whichever of these forms of statement we adopt.

As languages increase in affluence, the number of their simple terms becomes augmented, partly through the necessity of giving expression to new ideas, partly through the wish to give more convenient expression to definite and oft-recurring combinations of the old ones. With every term invented in subserviency to the latter purpose, a definition must be introduced. A dictionary, setting aside its philological portion (and even this not wholly), is a record of such definitions. As a consequence of such definitions of terms, spring up also propositions innumerable connecting these terms—propositions which in no degree add to the amount of our absolute knowledge, which are quite distinct from the discovered facts and laws of nature and of human history, but are merely logical deductions from the definitions. We might conceive of a language in which all possible combinations of ideas should be expressed by simple terms, with connecting definitions and propositions *ad infinitum*. The realization of such a conception is neither practicable nor desirable; but it is, nevertheless, the *limit* toward which all languages, which are not dead or decaying, do actually tend. The progressive action of this tendency does not affect the laws of expectation, neither, therefore, can it affect any consistent and scientific theory which is founded upon those laws.

We are not, therefore, permitted to assume that any events which, in the language of the problem, may be presented as simple events, must therefore be adopted as such into the hypothesis which is to form the basis of our method of solution. Nor, on the other hand, are we forbidden to employ transformations (sanctioned by the rules of Logic) which will have the effect of introducing an

entirely new scheme of simple events as the elements of the hypothesis in question.

12. To what conditions, then, must the hypothesis be subject? This question I now proceed to answer.

The hypothesis must be such that it may be consistently applied, without imposing upon the data any other conditions than those of possibility, *i.e.*, of accordance with a possible experience.

This principle is so obviously true, that it will only be needful to show how the conditions of possible experience are discovered. I shall subsequently show how their discovery limits and determines the hypothesis upon which the solution of questions in the theory of probabilities, whose elements are logical, depends.

The data of such problems are the probabilities of events. The object sought is also the probability of an event. The numerical values of these probabilities must be expressible by positive proper fractions. At any rate, they must not transcend the limits 0 and 1. This is one condition to which they are subject. Generally, however, there will exist other conditions dependent upon the mutual relations of the events whose probabilities are given.

Thus, if p were the probability that an event x will happen, q the probability that x and y will both happen, we have, as a necessary condition,

$$p \supseteq q.$$

Again, if p were the probability that x and y will both happen, q the probability that they will both fail, we must have the condition,

$$p + q \supseteq 1,$$

a condition which does not hold in the previous case.

I have, in the *Laws of Thought*, treated of these conditions, and of the principles by which they may be determined, in a special chapter (Cap. xix., *On Statistical Conditions*). A more simple, and at the same time perfectly general method, for their determination was afterwards discovered by me, and published in the *Philosophical Magazine*, Aug. 1854. As the method is of fundamental importance, I shall here illustrate it by an example, at the same time introducing a slight change in the mode of treatment, which leaves nothing to be desired in point of simplicity. The conditions to the discovery of which the method is applicable will be termed, in accordance with the language employed in the *Philosophical Magazine*,—the “Conditions of Possible Experience;” inasmuch, as whenever the numerical data of a problem are derived from actual experience these conditions will be satisfied, and whenever in data professing to be thus derived they are not satisfied, the presence of mistake or fraud may with certainty be affirmed.

Determination of the Conditions of Possible Experience.

13. To explain the method of effecting this object, by an example, I will first symbolically express the problem of Art. 11.

Let us then represent rain by x , snow by y , and wind by z . The problem in question then takes the following form:—

$$\text{Given} \quad \text{Prob. } xy=p, \quad \text{Prob. } yz=q, \quad \text{Prob. } xz=r \quad . \quad . \quad (1)$$

$$\text{Required} \quad \text{Prob. } xyz \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The value required we shall represent by u . It is our present object, not to solve this problem, but to ascertain the conditions which must connect p , q , and r , in order that the data may be possible, with the corresponding limitations of u . For if u were itself determined by experience, it would be subject to conditions of possibility similar to those which govern p , q , and r .

Now let us write, resolving the events in the problem into the possible alternations out of which they are formed,

$$\text{Prob. } xyz=u, \quad \text{Prob. } xy\bar{z}=\lambda, \quad \text{Prob. } x\bar{y}z=\mu, \quad \text{Prob. } y\bar{z}\bar{x}=\nu.$$

We have then

$$u+\lambda=p, \quad u+\nu=q, \quad u+\mu=r \quad . \quad . \quad . \quad (3)$$

The first of these equations only expresses that the probability of the concurrence of x and y is equal to the probability of the concurrence of x , y , and z , and the probability of the concurrence of x and y without z . To the equations (3) we must now add the inequations

$$\begin{aligned} u \geq 0, \quad \lambda \geq 0, \quad \mu \geq 0, \quad \nu \geq 0, \\ u+\lambda+\mu+\nu \leq 1 \end{aligned} \quad . \quad . \quad . \quad (4)$$

expressing the conditions to which u , λ , μ , ν , 1st, as probabilities, and, 2dly, as probabilities which do not altogether make up certainty, are subject.

First, we will eliminate λ , μ , and ν . Their values found from (3) are

$$\lambda=p-u \quad \mu=r-u \quad \nu=q-u.$$

Substituting these in (4) we have

$$\begin{aligned} u \geq 0 \quad p-u \geq 0 \quad q-u \geq 0 \quad r-u \geq 0 \\ p+q+r-2u \leq 1, \end{aligned}$$

Whence,

$$\left. \begin{aligned} u \leq p, \quad u \leq q, \quad u \leq r, \\ u \geq 0, \quad u \leq \frac{p+q+r-1}{2} \end{aligned} \right\} \quad . \quad . \quad . \quad (5)$$

Such are the conditions to which the quantity u is subject, conditions which the value of Prob. xyz must *a priori* satisfy.

To determine the conditions connecting p , q , and r , we must from (5) eliminate u . Now, if we have any two inequations of the form

$$u \leq a \quad u \leq b$$

the only condition connecting a and b which they establish is,

$$a \overline{\overline{=}} b.$$

Applying this principle to (5), we have

$$\begin{aligned} p \overline{\overline{=}} 0 & . \quad p \overline{\overline{=}} \frac{p+q+r-1}{2} . \\ q \overline{\overline{=}} 0 & . \quad q \overline{\overline{=}} \frac{p+q+r-1}{2} . \\ r \overline{\overline{=}} 0 & . \quad r \overline{\overline{=}} \frac{p+q+r-1}{2} . \end{aligned}$$

These may be reduced to the somewhat simpler form

$$\left. \begin{aligned} p \overline{\overline{=}} 0 & \quad q \overline{\overline{=}} 0 & r \overline{\overline{=}} 0 \\ p \overline{\overline{=}} q+r-1 \\ q \overline{\overline{=}} r+p-1 \\ r \overline{\overline{=}} p+q-1 \end{aligned} \right\} (6)$$

Such are the conditions of possible experience in the data.

Suppose, for instance, it was affirmed as a result of medical statistics, that in two-fifths of a number of cases of disease of a certain character, two symptoms, x and y , were observed ; in two-thirds of all the cases, the symptoms y and z were observed ; and in four-fifths of all the cases, the symptoms x and y were observed ; so that the number of cases observed being large, we might, on a future outbreak of the disease, consider the fractions two-fifths, two-thirds, and four-fifths, as the probabilities of recurrence of the particular combinations of the symptoms x , y , and z , observed. The above formulæ would show that the evidence was contradictory. For representing the respective fractions by p , q , and r , the condition

$$p \overline{\overline{=}} q+r-1$$

is not satisfied.

It is an evident consequence of the principle enunciated in Art. 11, that in determining the conditions of possible experience and of limitation, we may employ any *translated* form of the problem, just as well as the form in which it is originally expressed. Thus, if we take the translated form of the problem of that article, and represent sleet by s , drift by t , storm by u , we shall have as the data

$$\text{Prob. } s=p, \quad \text{Prob. } t=q, \quad \text{Prob. } u=r$$

with the conditions

$$s\bar{t}u=0, \quad u\bar{t}\bar{s}=0, \quad u\bar{s}\bar{t}=0 \quad . \quad . \quad . \quad . \quad (7)$$

the quæsitum being Prob. stu , which, as before, we shall represent by u .

Now if we write

$$\begin{aligned} \text{Prob. } stu=u, & \quad \text{Prob. } s\bar{t}u=0, & \text{Prob. } s\bar{u}t=0, & \text{Prob. } s\bar{u}\bar{t}=\lambda \\ \text{Prob. } t\bar{u}\bar{s}=0, & \text{Prob. } t\bar{u}\bar{s}=\mu, & \text{Prob. } u\bar{s}\bar{t}=\nu & . \quad . \quad . \quad (8) \end{aligned}$$

we have the following equations:—

$$\left. \begin{array}{l} u + \lambda = p \\ u + \mu = q \\ u + \nu = r \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

with the inequations

$$u \leq 0, \quad \lambda \leq 0, \quad \mu \leq 0, \quad \nu \leq 0, \quad u + \mu + \nu + \lambda \leq 1 \quad . \quad (10)$$

Determining from the equations λ, μ, ν , and substituting in the inequations, we get

$$u \leq 0 \quad p - u \leq 0, \quad q - u \leq 0, \quad r - u \leq 0, \quad p + q + r - 2u \leq 1 \quad . \quad (11)$$

a system which agrees with that obtained by the previous investigation (5) Art. 13.

14. The general rule for the determination of the conditions of possible experience and of limitation in a question of probability may be thus stated.

Resolve the events whose probabilities are either given or sought, into the mutually exclusive alternatives which they involve. If the calculus of Logic is employed, this is done by development.

Represent the probabilities of these alternations by λ, μ, ν , &c., and express the probabilities given and sought by the corresponding sums of these quantities. This will furnish a series of *equations*, which we will suppose to be n in number.

Determine from these equations any n of the quantities λ, μ, ν , in terms of the others.

Substitute the values thus obtained in the *inequations*

$$\lambda \leq 0 \quad \mu \leq 0 \quad \nu \leq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\lambda + \mu + \nu \quad . \quad . \quad \leq 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminate in succession such of the quantities $\lambda, \mu, \nu, \dots$ as are left in the above inequations after the substitution.

The elimination of any quantity as τ from the inequations, is effected by reducing each inequation to the form $\tau \leq a$, or to the form $\tau \leq b$, and observing that two such forms as the above give $a \leq b$.

If the “alternations” into which the events whose probabilities are given or sought are resolved, extend to all possible combinations of the simple events out of which they are formed, the inequations (2), must be replaced by the equation

$$\lambda + \mu + \nu \quad . \quad . \quad = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The rest of the process will be the same as before.

In the form of the above method developed in the *Philosophical Magazine* the quantities $\lambda, \mu, \nu, \dots$ represent the probabilities, not of those alternations alone, which are contained in the events whose probabilities are given or sought, but of all possible alternations which can be formed, by combining the simple events $x, y, z, \dots$. In this form, therefore, we have always an equation of the form (3), in the place of an inequation of the form (2). But though the result is the same, the form given to the method in this section is to be preferred, as it requires the elimina-

tions of a smaller number of symbols, except when the condition referred to in (3) is fulfilled, in which case, the methods are identical.

15. It remains to show how the conditions of possible experience as above determined, restrict us in the choice of the hypothesis, by the aid of which the final solution is to be obtained.

Taking for example the above problem of Art. 13, let us inquire whether it would be lawful to assume x , y , and z as the primary simple events of the problem.

If we make this assumption and then write

$$\text{Prob. } x = \alpha \quad \text{Prob. } y = \beta \quad \text{Prob. } z = \gamma$$

we find

$$\text{Prob. } xy = \alpha\beta \quad \text{Prob. } yz = \beta\gamma \quad \text{Prob. } zx = \gamma\alpha$$

whence comparing with (1) Art. 13—

$$\alpha\beta = p \quad \beta\gamma = q \quad \gamma\alpha = r$$

solving which equations we have

$$\alpha = \sqrt{\frac{qr}{p}} \quad \beta = \sqrt{\frac{rp}{q}} \quad \gamma = \sqrt{\frac{pq}{r}} \therefore \text{Prob. } xyz = \alpha\beta\gamma = \sqrt{pqr} \quad (4)$$

Now, α , β , γ , being by assumption probabilities, and therefore, lying numerically between the limits 0 and 1, we must have

$$qr \leq p \quad rp \leq q \quad pq \leq r \quad . \quad . \quad . \quad (5)$$

as the conditions to which p , q , and r (beside being fractional) must be subject. These conditions do not, however, agree with, and are not involved in, the conditions of possible experience, determined in (6) Art. 13. We may conclude, therefore, that the hypothesis upon which our solution is founded, involves elements, the introduction of which is unwarranted, and that the value of Prob. xyz determined is erroneous.

We may show, in fact, that the conditions (5) imply the conditions of possible experience, and something more. If $qr \leq p$ then, *à fortiori*, $qr \leq p + (1-q)(1-r)$ since $(1-q)(1-r)$ is essentially positive. Therefore,

$$\begin{aligned} qr &\leq p + 1 - q - r + qr \\ \text{whence } p &\geq q + r - 1 \end{aligned}$$

which is one of the conditions (6) Art 13. In the same way the other conditions in that article may be deduced from (5). The reverse reduction is, however, impossible.

16. The hypothesis upon which the method developed in the *Laws of Thought*, cap. xvii., for the solution of questions in the theory of probabilities whose elements are logical, is founded, seems to be the only one which satisfies the requirement referred to in Art. 12. It was not, however, upon such considerations as this,

that the method was founded. As presented in the *Laws of Thought*, it rests upon principles which, to my own mind, have something of an axiomatic character. Viewed in this light, its perfect accordance with the requirement above explained may be considered as a verification of it *à posteriori*. In itself, however, this accordance affords a sufficient ground of confidence in the legitimacy of the hypothesis. On the proof of this accordance I shall say something hereafter. At present I will only state the hypothesis, and show in what the accordance consists.

The hypothesis is the following:—Translating our problem by the aid of the calculus of logic into a language in which the events whose probabilities are given, appear as simple events subject to conditions founded on their definitions, Art. 11, we ascend above these simple events to another scheme of simple events, which are free, and which, when actually subjected to the conditions to which the before-mentioned simple events are necessarily subject, shall have the same probabilities, and shall in every respect take their place. The unknown probabilities of the free simple events, which form the elements of this hypothesis, must be so determined as to render the substitution possible, and to permit a formal construction of the problem, both in its data and by its quæsitum, out of those new elements.

The unknown probabilities being thus determined, the problem assumes a form in which its elementary data are the probabilities of simple events unrestricted by any condition. In this form the solution of the problem is possible by mere consequence of the fundamental definition of probability. The ground upon which this hypothesis was presented in the *Laws of Thought* was its intrinsic reasonableness. On this point I will only refer to my observations in the original work. The ground upon which, in the present essay, I wish to rest the hypothesis is, that it is the only one which does not impose upon the data other conditions than those of conformity with a possible experience. The conditions which must be fulfilled in order that $p', q', \&c.$, in the substituted and hypothetical data, may be measures of probability at all,—*i.e.*, may be positive proper fractions,—are precisely the conditions of possible experience in the original data. (See Appendix.)

17. The application of this hypothesis is so fully explained in the *Laws of Thought*, cap. xvii., that I shall here only describe the general method for the solution of questions in probabilities to which it leads, and show the connection which exists between the several parts of that method and the foregoing doctrine.

General Method.

Representing the problem to be solved under the form—

Given	Prob. $\phi_1(x, y, z, \dots) = p$	Prob. $\phi_2(x, y, z, \dots) = q, \&c.$
Required	Prob. $\psi(x, y, z, \dots)$	

and expressing the unknown value of Prob. $\psi(x, y, z \dots)$ by u , we form the logical equations:—

$$\begin{aligned} \phi_1(x, y, z \dots) &= s & \phi_2(x, y, z \dots) &= t, \text{ \&c.} \\ \psi(x, y, z \dots) &= w \end{aligned}$$

and hence, determining w as a developed logical function of $s, t \dots$ we have a result of the form

$$w = A + 0B + \frac{0}{0}C + \frac{1}{0}D \quad . \quad . \quad . \quad . \quad (1)$$

Here A, B, C, D are logical combinations of the simple events, s , t , &c., and the connection in which they stand to the event w and to each other is the following: A expresses those combinations of s , t , &c., which are entirely included in w ,—*i.e.*, which cannot happen without our being permitted to say that w happens. B represents combinations which may happen, but are not included under w ; so that when they happen, we may say that w does not happen. C represents those combinations, the happening of which leaves us in doubt whether w happens or not. D, those combinations, the happening of which is wholly interdicted.

Thus far we have only translated our problem into a language in which its data are the probabilities of simple events, viz. :—

$$\text{Prob. } s=p \quad \text{Prob. } t=q, \&c. \quad . \quad . \quad . \quad . \quad (2)$$

The condition, founded on definition, to which these simple events are subject is,

$$D=0$$

or, which amounts to the same thing,

$$A + B + C = 1$$

indicating that the combinations expressed by A, B, and C can alone happen. If we represent A+B+C by V, we have

$$w = A + \frac{0}{0} C \quad . \quad . \quad . \quad . \quad . \quad (3)$$

with the condition

$$V=1 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Of these equations, the latter expresses the conditions to which the simple events, s, t , &c., are subject; the former expresses w as a logical combination of those events.

We now, in accordance with the hypothesis, ascend to a new scheme of simple events, s, t , &c., unrestricted by any condition, and possessed of unknown probabilities, p, q , which are to be so determined that when s, t . . are subjected to the same condition (4) to which s, t . . are subject, they will have the same probabilities as s, t . . The system of equations to which we are thus led, and which contains the implicit solution of the problem, is the following (*Laws of Thought*, cap. xvii., p. 267) :—

$$\frac{V_s}{p} = \frac{V_t}{q} \therefore = \frac{A+C}{u} = V \quad (5)$$

V_s being formed by selecting those terms from V , which contain s as a factor; V

those, which contain t as a factor, &c. ; and then regarding s, t , &c., as algebraic quantities. From the system thus formed, we must determine u as a function of p, q . . and the arbitrary constant c , should it be present. This will be the solution of the problem.

The quantities s, t . . are the same as p', q' . . and represent the probabilities of the hypothetical simple events, represented by s', t' . . Accordingly, as probabilities, they must admit of being determined as positive proper fractions, and that the solution may not be ambiguous, they must admit of only one such determination. These conditions will be fulfilled whensoever the problem represents a possible experience, and it will be then only fulfilled. And in this way, or by directly investigating the conditions of possibility by the rule of Art. 14, a solution is made determinate.

The arbitrary constant c does not, as has been intimated, always present itself. When it does, it represents the unknown probability, that if the event C occur, w will occur. It indicates, therefore, the new experience which would be necessary in order to make the solution definite.

18. I will, for the sake of illustration, apply the method to the problem of Art. 11, and in so doing I will limit the solution by the conditions relative to s, t , &c.

The problem, as symbolically expressed in Art. 13, is as follows:—

$$\left. \begin{array}{llll} \text{Given} & \text{Prob. } xy=p & \text{Prob. } yz=q & \text{Prob. } zx=r \\ \text{Required} & \text{Prob. } xyz & & \end{array} \right\} \quad . \quad (1)$$

Translating the problem as directed in the first part of the rule, we write

$$\left. \begin{array}{lll} xy=s & yz=t & zx=v \\ & xyz=w & \end{array} \right\} \quad . \quad . \quad . \quad (2)$$

whence, by the calculus of logic,

$$\begin{aligned} w &= stv + 0 (s \bar{t} \bar{v} + t \bar{s} \bar{v} + v \bar{s} \bar{t} + s \bar{v} \bar{t}) \\ &\quad + \frac{1}{0} (s \bar{t} \bar{v} + s \bar{v} \bar{t} + t \bar{v} \bar{s}) \quad . \quad . \quad . \quad (3) \end{aligned}$$

Hence we find

$$V = stv + s \bar{t} \bar{v} + t \bar{s} \bar{v} + v \bar{s} \bar{t} + s \bar{v} \bar{t} \quad . \quad . \quad . \quad (4)$$

and are led to the algebraic system of equations

$$\begin{aligned} \frac{stv + s \bar{t} \bar{v}}{p} &= \frac{stv + t \bar{s} \bar{v}}{q} = \frac{stv + v \bar{s} \bar{t}}{r} \\ &= \frac{stv}{u} = stv + s \bar{t} \bar{v} + t \bar{s} \bar{v} + v \bar{s} \bar{t} + s \bar{v} \bar{t} \quad . \quad . \quad . \quad (5) \end{aligned}$$

These equations may be simplified by dividing every term by $\bar{s} \bar{t} \bar{v}$, and then assuming

$$\frac{s}{\bar{s}} = s' \quad \frac{t}{\bar{t}} = t' \quad \frac{v}{\bar{v}} = v' \quad . \quad . \quad . \quad (6)$$

They thus give

$$\begin{aligned} \frac{s' t' v' + s'}{p} &= \frac{s' t' v' + t'}{q} = \frac{s' t' v' + v'}{r} \\ &= \frac{s' t' v'}{u} = s' t' v' + s' + t' + v' + 1 \quad . \quad . \quad . \quad (7) \end{aligned}$$

The condition to which $s' t' v'$ are subject obviously is, that they shall be positive quantities, for this is equivalent to the condition that s, t, v shall be positive fractions.

From (7) we readily find

$$\frac{s'}{p-u} = \frac{t'}{q-u} = \frac{v'}{r-u} = \frac{s't'v'}{u} = s't'v' + s' + t' + v' + 1 \quad . \quad (8)$$

Whence

$$\left. \begin{aligned} s' &= \frac{p-u}{2u-p-q-r+1} \\ t' &= \frac{q-u}{2u-p-q-r+1} \\ v' &= \frac{r-u}{2u-p-q-r+1} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (9)$$

Substitute these values in the equation

$$\frac{s'}{p-u} = \frac{s't'v'}{u}$$

and reducing we get

$$(p-u)(q-u)(r-u) = u(2u-p-q-r+1)^2 \quad . \quad . \quad (10)$$

an equation for determining u .

And now let us inquire into the consequences which flow from the condition that $s' t' v'$ are positive quantities.

In the first place, the last member of (8), and therefore each other member of that system will be positive. This requires that the denominators, $p-u$, $q-u$, $r-u$, and u , should be positive, whence we have

$$\left. \begin{aligned} u &\geq 0 \\ u &\leq p, u \leq q, u \leq r \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (11)$$

Again, $p-u$, $q-u$, and $r-u$ being positive, the common denominators, $2u-p-q-r+1$ in (9) must be positive, whence

$$u \geq \frac{p+q+r-1}{2} \quad . \quad . \quad . \quad . \quad (12)$$

Such are the conditions relative to u . They agree in all respects with those assigned in the previous investigation, in (5), Art. 13; and, as in that article, the elimination of u leads to the conditions of possible experience,

$$\left. \begin{aligned} p &\geq 0 & q &\geq 0 & r &\geq 0 \\ p &\geq q+r-1 \\ q &\geq p+r-1 \\ r &\geq p+q-1 \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (13)$$

It may be well to notice, that these conditions involve the necessity of p, q , and r being fractional, though of course this does not exhaust their significance.

19. It remains to show that when the above conditions are satisfied, the system (7) will admit of but one solution in positive values of s', t', v' , and that (10) will furnish but one value of u satisfying the conditions (11) and (12).

Let us write 10 in the form

$$u(2u-p-q-r+1)^2 - (p-u)(q-u)(r-u) = 0 \quad . \quad (14)$$

or, for simplicity, in the form

$$U=0.$$

The lower limit of u is, by (11) and (12), either 0 or $\frac{p+q+r-1}{2}$, according as the latter quantity is positive or negative; the upper limit of u is the least of the quantities p, q, r ; suppose it p . First, let $\frac{p+q+r-1}{2}$ be positive, then, making u equal to this quantity, the value of U , as given in the first member of (14) becomes negative. Again, let $u=p$, then U becomes positive. Thus, as u varies from $\frac{p+q+r-1}{2}$ up to p , U changes from negative to positive. Now

$$\frac{dU}{du} = (2u-p-q-r+1)^2 + 4u(2u-p-q-r+1) + (p-u)(q-u) + (q-u)(r-u) + (r-u)(p-u) \quad (15)$$

which within the supposed limits is always positive. Hence U varies by continuous increase, and once only in its variation becomes equal to 0.

Secondly, let $\frac{p+q+r-1}{2}$ be negative, then u , varying from 0 up to p , U as before will vary by continuous increase from a negative to a positive value. See the first member of (14). Whence U , changing by continuous increase from a negative to a positive value, will still only once become equal to 0.

Wherefore, in either case, one root only of (10) will lie within the limits assigned to u in (11) and (12). And this one value substituted in (9) will give one set of values for s', t', v' .

20. The solutions which we have now obtained of the same problem on different hypotheses with respect to the selection of the simple events, set in clear light the principles upon which the due selection of such hypotheses depends. The hypothesis which seems most readily to present itself utterly fails, while the other, based quite as much upon an apparently remote speculation on language, as upon the study of the laws of expectation as usually conceived, finds a support and confirmation within the realm of pure mathematics which is of the most remarkable kind.

21. A practical simplification of the general method is suggested by that step of the preceding solution, which reduces (5) to the form (7). If we remove the traces (') from the letters in the latter system (and they do not at all affect the solution), we obtain what (5) would become if we replaced each of the symbols $\bar{s}, \bar{t}, \bar{v}$, by unity. Practically, therefore, we may modify the general rule in the following manner:—Having obtained V , replace each of the symbols $\bar{s}, \bar{t}$, &c., by unity, and proceed with the reduced value of V just as before, *i.e.*, let V_s represent that portion of V of which s is a factor, &c., then form the system of equations

$$\frac{V_s}{p} = \frac{V_t}{q} \dots = \frac{A + cC}{u} = V \quad . \quad . \quad . \quad . \quad (1)$$

and hence determine u as a function of $p, q \dots$ and c . The conditions of possible experience and of limitation will be found by supposing s, t , to admit of a single determination in positive values. Or as before, they may be found independently, and then applied to limit the solution.

22. We now proceed to the consideration of the problems referred to in Art. (1.) We shall first examine the problem which has for its object the determination of the most probable measure of a physical magnitude, two conflicting measures of which have been assigned by observation. The problem is not, as has been said, Art. 2, in its immediate presentation, one whose elements are logical, but it admits, as we shall see, of being so represented as to give it this character.

PROBLEM 1.

Two simultaneous observations of a physical magnitude, as the elevation of a star, assign to it the respective values p_1 and p_2 . The probability, when the first observation has been made, that it is correct, is c_1 , the corresponding probability for the second observation is c_2 . Required the most probable value of the physical magnitude hence resulting.

First Solution.

23. The numerical elements which are not, in their immediate presentation, probabilities, are p_1 and p_2 . But these become such if we contemplate the problem under another aspect. Let a quadrant be taken as the unit of magnitude, then p_1 and p_2 are proper fractions; p_1 actually expressing the probability afforded by the first observation, p_2 that afforded by the second observation, that a pointer, directed at random to that quadrant of elevation in which the star, regarded as a physical point, is situated, will point below the star. The problem thus regarded contains the following logical elements, which we shall express by appropriate symbols, viz.

The event which consists in the first observation, such as it is, being made $=x$.

The event which consists in the second observation, such as it is, being made $=y$.

The event which consists in the first observation being correct, $=m$.

The event which consists in the second observation being correct, $=v$.

The event which consists in a pointer, directed at random to the quadrant in which the star is situated, pointing below that star, $=z$.

We must now express symbolically the data, including therein whatever logical connections we can establish among the events, x, y, m, v , and z .

The probability that the first observation, when made, is correct, is c_1 . This is a conditional probability; or, to adopt a well-known form of expression, it is a probability *à posteriori*. Viewed from a point of time anterior to the observa-

tion, it is the probability that if the observation be made under its actual circumstances of care, personal fitness, instrumental accuracy, &c., it will be absolutely correct. Symbolically, it is the probability that if the event x take place, the event w will take place. The only mode of expressing this is by writing for the probability of x an arbitrary constant a_1 we have then

$$\text{Prob. } x = a_1 \quad \text{Prob. } xw = a_1 c_1 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The events x and w are not, however, independent. If we can affirm that a given observation is correct, we can affirm that that observation has been made. Symbolically, the occurrence of the event w implies the occurrence of the event x . Expressing this proposition in the language of the calculus of Logic we have the equation.

$$w \bar{x} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This forms a part of our data. It permits us to change also the form of one of the previous data, and instead of (1) to substitute

$$\text{Prob. } x = a_1 \quad \text{Prob. } w = a_1 c_1 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In like manner, representing the arbitrary probability of the event y by a_2 , we have

$$\text{Prob. } y = a_2 \quad \text{Prob. } yv = a_2 c_2 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

With the connecting condition

$$v \bar{y} = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which would permit us to substitute for (4) the system

$$\text{Prob. } y = a_2 \quad \text{Prob. } v = a_2 c_2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Again, when it is known that the first observation is a correct one, the probability that an indicator directed at random to the quadrant in which the star is situated will point below the star is p_1 . This, too, is a conditional probability. Symbolically, it is the probability that if the event w occur, the event z will occur. Hence, as the probability of the occurrence of w as $a_1 c_1$, we have

$$\text{Prob. } wz = a_1 c_1 p_1 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In like manner we find

$$\text{Prob. } vz = a_2 c_2 p_2 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Lastly, it is supposed that the values p and q are different. This involves the condition that the observations cannot both be correct. Whence we have the logical equation.

$$wv = 0 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This completes the analysis of the logical elements involved in the data of the problem. We now proceed to analyse those involved in its quæsitum or object proposed.

That object is to determine the probability of the event z , when the occurrence

of the events x and y is known. Symbolically expressed, it is the value of the fraction.

$$\frac{\text{Prob. } x y z}{\text{Prob. } x y}$$

or, as it may, by resolving the denominator, be written,

$$\frac{\text{Prob. } x y z}{\text{Prob. } x y z + \text{Prob. } x y \bar{z}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

To effect this object, we shall determine the value of Prob. $x y z$ and Prob. $x y \bar{z}$ separately.

Collecting the elements furnished by the preceding analysis, the first of the partial problems herein involved may be thus stated:—

$$\left. \begin{array}{ll} \text{Given} & \text{Prob. } x = a_1 \quad \text{Prob. } y = a_2 \\ & \text{Prob. } w = a_1 c_1 \quad \text{Prob. } v = a_2 c_2 \\ & \text{Prob. } wz = a_1 c_1 p_1 \quad \text{Prob. } vz = a_2 c_2 p_2 \end{array} \right\} \quad . \quad . \quad . \quad (11)$$

$$\text{with the conditions, } w\bar{x}=0, \quad v\bar{y}=0, \quad wv=0, \quad . \quad . \quad . \quad . \quad (12)$$

Required u , the value of Prob. xyz .

In selecting the above, I have chosen to employ (3) in place of (1), and (6) in place of (4). It makes no difference in the final result.

In accordance with the rule, let us write

$$wz=s, \quad vz=t, \quad xyz=\phi \quad . \quad . \quad . \quad (13)$$

we must then from (12) and (13) determine ϕ as a developed logical function of x, y, w, v, s , and t .

This problem admits of perfectly definite solution on the principles of the calculus of Logic. I shall here merely give the result, and point out a method by which it may be independently verified. We find

$$\begin{aligned} \phi = & x y w s \bar{v} \bar{t} + x y v t \bar{w} \bar{s} + 0 (x w s \bar{y} \bar{v} \bar{t} + y v t \bar{x} \bar{w} \bar{s} + x \bar{y} \bar{w} \bar{v} \bar{s} \bar{t} \\ & + y \bar{x} \bar{w} \bar{v} \bar{s} \bar{t} + \bar{x} \bar{y} \bar{w} \bar{v} \bar{s} \bar{t} + x y w \bar{v} \bar{s} \bar{t} + x w \bar{y} \bar{v} \bar{s} \bar{t} + x y v \bar{w} \bar{s} \bar{t} + y v \bar{x} \bar{w} \bar{s} \bar{t}) \\ & + \frac{0}{0} x y \bar{w} \bar{v} \bar{s} \bar{t} + \text{terms whose coefficient is } \frac{1}{0} \quad . \quad . \quad . \quad . \quad (14) \end{aligned}$$

We may verify this expansion by substituting for s and t their values wz and vz , paying attention to the conditions (12), and then comparing the result with the value of ϕ , viz., xyz .

Thus the term $x y w s \bar{v} \bar{t}$ becomes, on substitution

$$x y w \bar{v} \times w z \times (1 - v z) = x y z w \bar{v}$$

by the calculus of Logic. Now this represents a class entirely included in the class xyz , whence the coefficient of the term is unity.

The term $x w s \bar{y} \bar{v} \bar{t}$ reduces to $x w z \bar{y} \bar{v}$, and represents a class no part of which is included in xyz , whence the coefficient is 0.

The term $xy\bar{w}\bar{v}\bar{s}\bar{t}$, reduces to $xy\bar{w}\bar{v}$, and represents a class some part of which is included, and some part not included under the class xyz , whence the coefficient $\frac{0}{0}$; for an event included under the former class may or may not be included under the latter.

Lastly, any term whose coefficient in the expansion is $\frac{1}{0}$ would, on effecting the above-named substitutions, become 0, indicating the absolute non-existence of the class which it represents.

Resuming the value of ϕ , and adopting the simplification of Art. 21, we find for V the value

$$V = xyws + xyvt + xws + yvt + x + y + 1 + xyw + xw + xyv + yv + xy \\ = (x+1)(y+1) + yv(x+1)(t+1) + xw(y+1)(s+1) \quad (15)$$

And hence we have the following system of algebraic equations:

$$\frac{x(y+1) + xyv(t+1) + xw(y+1)(s+1)}{a_1} = \frac{y(x+1) + yv(x+1)(t+1) + xwy(s+1)}{a_2} \\ = \frac{xw(y+1)(s+1)}{a_1 c_1} = \frac{yv(x+1)(t+1)}{a_2 c_2} = \frac{xw(y+1)s}{a_1 c_1 p_1} = \frac{yv(x+1)t}{a_2 c_2 p_2} \\ = \frac{xyws + xyvt + cxy}{u} = (x+1)(y+1) + yv(x+1)(t+1) + xw(y+1)(s+1) \quad (16)$$

From these equations, if we assume

$$(x+1)(y+1) + yv(x+1)(t+1) + xw(y+1)(s+1) = \lambda,$$

λ being a subsidiary quantity introduced for convenience, we readily deduce

$$u = \frac{xyws + xyvt + cxy}{\lambda} \quad (17)$$

$$a_1 c_1 p_1 = \frac{xws(y+1)}{\lambda}$$

$$a_2(1-c_2) = \frac{y(x+1) + xwy(s+1)}{\lambda}$$

$$1 - a_2 c_2 = \frac{(x+1)(y+1) + xw(y+1)(s+1)}{\lambda}$$

$$\text{Hence} \quad \frac{a_1 c_1 p_1 \times a_2(1-c_2)}{1 - a_2 c_2} = \frac{xyws}{\lambda} \quad (18)$$

In like manner

$$\frac{a_2 c_2 p_2 \times a_1(1-c_1)}{1 - a_1 c_1} = \frac{xyvt}{\lambda} \quad (19)$$

Again we have

$$a_1(1-c_1) = \frac{x(y+1) + xyv(t+1)}{\lambda}$$

$$a_2(1-c_2) = \frac{y(x+1) + xwy(s+1)}{\lambda}$$

$$\begin{aligned}
1 - a_1 c_1 - a_2 c_2 &= \frac{(x+1)(y+1)}{\lambda} \\
1 - a_1 c_1 &= \frac{(x+1)(y+1) + yv(x+1)(t+1)}{\lambda} \\
1 - a_2 c_2 &= \frac{(x+1)(y+1) + xw(y+1)(s+1)}{\lambda}
\end{aligned}$$

Whence we find

$$\frac{a_1(1-c_1)a_2(1-c_2)(1-a_1c_1-a_2c_2)}{(1-a_1c_1)(1-a_2c_2)} = \frac{xy}{\lambda} \quad (20)$$

By means of (18), (19), and (20), we reduce (17) to the form

$$u = \frac{a_2(1-c_2)}{1-a_2c_2} a_1 c_1 p_1 + \frac{a_1(1-c_1)}{1-a_1c_1} a_2 c_2 p_2 + c \frac{a_1(1-c_1)a_2(1-c_2)(1-a_1c_1-a_2c_2)}{(1-a_1c_1)(1-a_2c_2)}$$

therefore effecting a slight reduction

$$\text{Prob. } xyz = \frac{a_1(1-c_1)a_2(1-c_2)}{(1-a_1c_1)(1-a_2c_2)} \left\{ \frac{1-a_1c_1}{1-c_1} c_1 p_1 + \frac{1-a_2c_2}{1-c_2} c_2 p_2 + c(1-a_1c_1-a_2c_2) \right\} \quad (21)$$

The arbitrary constant c , interpreted according to the rule, is the probability that if the event $xy\bar{w}\bar{v}\bar{s}\bar{t}$ take place, xyz will take place. Putting for $\bar{s}$ and $\bar{t}$ their values, and reducing as before, we find that c is the probability that if $xy\bar{w}\bar{v}$ take place, xyz will take place. In the end this amounts to the following statement.

c = probability that if both observations are incorrect, a pointer directed at random to the quadrant in which the star is situated will point below the star.

The value of Prob. xyz will be obtained from that of Prob. xyz by changing $p_1 p_2$ and c into $1-p_1$, $1-p_2$, and $1-c$. If we effect this change, and then substitute the expressions above found, in the formula,

$$\frac{\text{Prob. } xyz}{\text{Prob. } xyz + \text{Prob. } xyz}$$

We shall find

$$\begin{aligned}
\frac{\text{Prob. } xyz}{\text{Prob. } xy} &= \frac{\frac{1-a_1c_1}{1-c_1} c_1 p_1 + \frac{1-a_2c_2}{1-c_2} c_2 p_2 + c(1-a_1c_1-a_2c_2)}{\frac{1-a_1c_1}{1-c_1} c_1 + \frac{1-a_2c_2}{1-c_2} c_2 + 1-a_1c_1-a_2c_2} \\
&= \frac{\frac{1-a_1c_1}{1-c_1} c_1 p_1 + \frac{1-a_2c_2}{1-c_2} c_2 p_2 + c(1-a_1c_1-a_2c_2)}{1 + \frac{1-a_1c_1}{1-c_1} c_1 + \frac{1-a_2c_2}{1-c_2} c_2} \quad (22)
\end{aligned}$$

This expression involves an arbitrary constant c which we have no means of determining. This circumstance indicates that those principles of probability which relate to the combination of *events* do not *alone* suffice to enable us to combine into a definite result the conflicting measures of an astronomical observation.

The arbitrary character of the final solution might have been inferred from

the appearance of the symbol $\frac{0}{0}$ in (14). I have thought it better to complete the investigation, especially as it will serve as a model for the one which follows.

24. Before proceeding to the second solution of the problem, I will endeavour to explain the principle on which it will be founded. It is involved in the following definition.

Definition. The mean strength of any probabilities of an event which are founded upon different judgments or observations is to be measured by that supposed probability of the event *à priori* which those judgments or observations following thereupon would not tend to alter.

Thus, suppose we were considering the question of the suitability of a newly discovered island for the growth of a particular plant, and that the probability of its suitability, as dependent upon general impressions of the climate were r ; but that added special observations,—such as analysis of the soil, determination of allied species growing in the locality, &c., had some of them the effect of raising, others that of depressing, the general expectation before entertained. Now we might suppose that expectation to have had such a measure, that the added observations should, when united, leave the mind in the same state as before. I call that measure the *mean value* of the testimonies—the value about which, to adopt (for illustration, not for argument) a mechanical analogy, they balance each other. I conceive that in thus doing, I am only giving a scientific meaning to a term which has been hitherto used in a vague sense. I shall show that the formula of the arithmetical mean is a special determination applicable only to particular problems, of the more general mean of which I here speak, and that other determinations of it exist, applicable to other problems, but possessing, in common, certain definite characteristics.

To apply this principle to the problem under consideration, we must add to the data a new element, viz., the *à priori* value of Prob. z , i.e., the value which the mind is supposed to attach to it before the evidence furnished by the observations. We will suppose this value r . We must then seek, as before, the *a posteriori* value of Prob. z , i.e., its value after the observations, and, equating the two expressions, determine thence the value of r .

I shall, in referring to the above principle, speak of it as the “principle of the mean.”

Special solution of Problem I. founded upon the principle of the mean.

Assigning to z the *à priori* probability r , our data are the following:

$$\begin{array}{lll} \text{Prob. } x = a_1 & \text{Prob. } y = a_2 & \text{Prob. } z = r \\ \text{Prob. } wx = a_1 c_1 & \text{Prob. } vy = a_2 c_2 & \\ \text{Prob. } wxz = a_1 c_1 p_1 & \text{Prob. } vyz = a_2 c_2 p_2 & \end{array}$$

with the conditions $w\bar{x} = 0$ $v\bar{y} = 0$ $wv = 0$

and hence we are to seek, as before, the value of

$$\frac{\text{Prob. } xyz}{\text{Prob. } xyz + \text{Prob. } \overline{xyz}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Assuming then as before,

$$wz=s, \quad vz=t, \quad xyz=\phi$$

we find, by the calculus of Logic, the following expression for ϕ as a developed logical function of x, y, w, v, s, t , and z , viz. :

$$\begin{aligned} \phi = & x y w s z \bar{v} \bar{t} + x y v t z \bar{w} \bar{s} + x y z \bar{w} \bar{v} \bar{s} \bar{t} \\ & + 0\{x y v \bar{w} \bar{s} \bar{t} \bar{z} + y v z t \bar{x} \bar{w} \bar{s} + y v \bar{x} \bar{w} \bar{s} z \bar{t} + x y w s z \bar{v} \bar{t} + x w z s \bar{y} \bar{v} \bar{t} \\ & + x w \bar{y} z \bar{v} \bar{s} \bar{t} + x y z \bar{v} \bar{w} \bar{s} \bar{t} + x z \bar{y} \bar{v} \bar{w} \bar{s} \bar{t} + y z \bar{x} \bar{s} \bar{v} \bar{w} \bar{t} + x \bar{y} z \bar{v} \bar{w} \bar{s} \bar{t} \\ & y \bar{x} z \bar{v} \bar{w} \bar{s} \bar{t} + z \bar{x} \bar{y} \bar{v} \bar{w} \bar{s} \bar{t} + x \bar{y} z \bar{v} \bar{w} \bar{s} \bar{t}) \\ & + \text{terms with coefficient } \frac{1}{6} \end{aligned} \quad (2)$$

Hence, adopting the simplification of Art. 21, we have

$$\begin{aligned} \mathbf{V} &= xywsz + xyvtz + xyz + xyv + yvzt + yv + xyw + xwzs \\ &\quad + xw + xy + xz + yz + x + y + z + \mathbf{1} \\ &= xw(y+1)(zs+1) + yv(x+1)(zt+1) + (x+1)(y+1)(z+1) \end{aligned} \quad (3)$$

whence we form the algebraic system

$$\begin{aligned}
& \frac{xw(y+1)(zs+1) + xyv(zt+1) + x(y+1)(z+1)}{a_1} \\
&= \frac{xywy(zs+1) + yv(x+1)(zt+1) + (x+1)y(z+1)}{a_2} \\
&= \frac{xw(y+1)(zs+1)}{a_1 c_1} = \frac{yv(x+1)(zt+1)}{a_2 c_2} \\
&= \frac{xw(y+1)zs}{a_1 c_1 p_1} = \frac{yv(x+1)zt}{a_2 c_2 p_2} \\
&= \frac{xw(y+1)zs + yv(x+1)zt + (x+1)(y+1)z}{r} \\
&= \frac{xyzws + xyzvt + xyz}{u} \\
&= \frac{xw(y+1)(zs+1) + yv(x+1)(zt+1) + (x+1)(y+1)(z+1)}{1} = \lambda. \quad (4)
\end{aligned}$$

λ being a subsidiary quantity introduced for convenience.

From the above we find

$$\begin{aligned} \alpha_2 - \alpha_2 c_2 &= \frac{xywv(zs+1) + y(x+1)(z+1)}{\lambda} \\ \alpha_1 c_1 p_1 &= \frac{xwzs(y+1)}{\lambda} \\ 1 - \alpha_2 c_2 &= \frac{xw(y+1)(zs+1) + (x+1)(y+1)(z+1)}{\lambda} \end{aligned}$$

whence

$$\frac{\alpha_2(1-c_2)}{1-\alpha_2 c_2} \alpha_1 c_1 p_1 = \frac{xyzws}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In like manner we find

$$\frac{\alpha_1(1-c_1)}{1-\alpha_1 c_1} \alpha_2 c_2 p_2 = \frac{xyzvt}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Again, since we have from the above

$$\frac{\alpha_2 - \alpha_2 c_2}{1 - \alpha_2 c_2} = \frac{y}{y+1} \quad \text{and} \quad \frac{\alpha_1 - \alpha_1 c_1}{1 - \alpha_1 c_1} = \frac{x}{x+1}$$

we have

$$\frac{\alpha_1(1-c_1) \alpha_2(1-c_2)}{(1-\alpha_1 c_1)(1-\alpha_2 c_2)} = \frac{xy}{(x+1)(y+1)}$$

moreover,

$$r - \alpha_1 c_1 p_1 - \alpha_2 c_2 p_2 = \frac{(x+1)(y+1)z}{\lambda}$$

Multiplying the two last equations together we find

$$\frac{\alpha_1(1-c_1) \alpha_2(1-c_2)}{(1-\alpha_1 c_1)(1-\alpha_2 c_2)} (r - \alpha_1 c_1 p_1 - \alpha_2 c_2 p_2) = \frac{xyz}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now,

$$u = \frac{xyzws + xyzvt + xyz}{\lambda}$$

Substituting in this expression the values found for its several terms in (5), (6), and (7), we have

$$u = \frac{\alpha_2(1-c_2)}{1-\alpha_2 c_2} \alpha_1 c_1 p_1 + \frac{\alpha_1(1-c_1)}{1-\alpha_1 c_1} \alpha_2 c_2 p_2 + \frac{\alpha_1(1-c_1) \alpha_2(1-c_2)}{(1-\alpha_1 c_1)(1-\alpha_2 c_2)} (r - \alpha_1 c_1 p_1 - \alpha_2 c_2 p_2)$$

This is the value of Prob. xyz . That of Prob. xyz will be found by simply changing in the above expression p_1, p_2 , and r , into $1-p_1, 1-p_2$, and $1-r$ respectively. These expressions admit of some reductions, and give

$$\text{Prob. } xyz = \frac{\alpha_1(1-c_1) \alpha_2(1-c_2)}{(1-\alpha_1 c_1)(1-\alpha_2 c_2)} \left\{ \frac{1-\alpha_1}{1-c_1} c_1 p_1 + \frac{1-\alpha_2}{1-c_2} c_2 p_2 + r \right\} \quad . \quad . \quad . \quad (8)$$

$$\text{Prob. } xyz = \frac{\alpha_1(1-c_1) \alpha_2(1-c_2)}{(1-\alpha_1 c_1)(1-\alpha_2 c_2)} \left\{ \frac{1-\alpha_1}{1-c_1} c_1 (1-p_1) + \frac{1-\alpha_2}{1-c_2} c_2 (1-p_2) + 1-r \right\} \quad . \quad (9)$$

whence we find for the *a posteriori* value of Prob. z ,

$$\frac{\text{Prob. } xyz}{\text{Prob. } xy} = \frac{\frac{1-\alpha_1}{1-c_1} c_1 p_1 + \frac{1-\alpha_2}{1-c_2} c_2 p_2 + r}{\frac{1-\alpha_1}{1-c_1} c_1 + \frac{1-\alpha_2}{1-c_2} c_2 + 1}$$

Equating this to r we have

$$\frac{1-\alpha_1}{1-c_1} c_1 p_1 + \frac{1-\alpha_2}{1-c_2} c_2 p_2 + r = \left(\frac{1-\alpha_1}{1-c_1} c_1 + \frac{1-\alpha_2}{1-c_2} c_2 + 1 \right) r$$

Whence

$$r = \frac{\frac{1-\alpha_1}{1-c_1} c_1 p_1 + \frac{1-\alpha_2}{1-c_2} c_2 p_2}{\frac{1-\alpha_1}{1-c_1} c_1 + \frac{1-\alpha_2}{1-c_2} c_2} \quad . \quad . \quad . \quad . \quad (10)$$

26. Such is the final general expression for the probable altitude of the star. The following observations may throw light upon its real nature:—

1st, In the analysis by which this expression was obtained, p_1 and p_2 are the observed altitudes of the star, a quadrant of the celestial arc being taken as unity. Considered, however, as the expression, not of a *probability*, but of the most probable measure of a physical magnitude, the truth of the formula will of course be independent of the unit of magnitude.

2dly, The formula is independent of mechanical analogy. We may place it in the well-known form

$$r = W_1 p_1 + W_2 p_2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which, as the subject is usually treated W_1 and W_2 are called the *weights* of the observations. Here, however, these quantities are determined as functions of the initial data—these data being probabilities. We have

$$W_1 = \frac{\frac{1-a_1}{1-c_1} c_1}{\frac{1-a_1}{1-c_1} c_1 + \frac{1-a_2}{1-c_2} c_2} \quad W_2 = \frac{\frac{1-a_2}{1-c_2} c_2}{\frac{1-a_1}{1-c_1} c_1 + \frac{1-a_2}{1-c_2} c_2} \quad . \quad (2)$$

3dly, The initial probabilities, of which W_1 and W_2 are functions, are neither foreign nor imaginary elements. They may be difficult to determine, but theoretically their determination rests upon considerations which are entirely proper to the subject. When an observation has been made, the question whether it is correct or not is a question of probability. We can never predicate absolute correctness. We can seldom affirm absolutely that an observation is incorrect. Our knowledge of the circumstances of the observation, Art. 22, leads us to regard the probability in question as sometimes greater, sometimes less. To suppose it capable of a numerical value, as we have done, by the introduction of the constants c_1, c_2 , is then perfectly legitimate. It has been said that an estimate of the correctness of the observation rests upon the circumstances by which it was accompanied. These circumstances, taken in the aggregate, are themselves a subject of probability. This we express by the introduction of the constants a_1, a_2 . The probability after an observation is made that it is correct, and the probability before it is made that the state of things shall be such as to give to the result that particular probability of correctness, are quite different things.

4thly, In the same course of observations made by the same individual with consciously uniform regard to personal and instrumental accuracy the values of a_1 and a_2 would be sensibly equal. The formula (10) would thus reduce to the following, viz.:—

$$r = \frac{\frac{c_1}{1-c_1} p_1 + \frac{c_2}{1-c_2} p_2}{\frac{c_1}{1-c_1} + \frac{c_2}{1-c_2}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{Here } W_1 = \frac{\frac{c_1}{1-c_1}}{\frac{c_1}{1-c_1} + \frac{c_2}{1-c_2}} \quad W_2 = \frac{\frac{c_2}{1-c_2}}{\frac{c_1}{1-c_1} + \frac{c_2}{1-c_2}} \quad (4)$$

If $c_1=1$, we have

$$W_1=1 \quad W_2=0$$

and

$$r=p_1.$$

This accords with the condition that if either of the observations is believed to be correct, the value which it furnishes for the altitude of the star must be taken as the true one.

5thly, If $c_1=c_2$, i.e., if we have no right to give preference to one observation over the other, we have

$$r = \frac{p_1+p_2}{2} \quad (5)$$

the formula of the arithmetical mean.

6thly, From the form of W_1, W_2 in (4), it is evident that the weights, so to speak, of the observations vary in a higher ratio than that of the simple probabilities of correctness of the observations. The practical lesson to be drawn from this is, that we ought to attach a greater weight to good observations, and a smaller to bad ones, than, according to usual modes of consideration, we should be disposed to do.

The above are the most important observations suggested by the formula to which the last investigation has led. One or two remarks remain to be offered upon the analysis by which it was obtained.

Although the two forms of investigation which we have exhibited differ, there is nothing inconsistent in the results to which they lead. If we compare corresponding formulæ in the two, e.g., the values of Prob. xyz , or those of Prob. $xy\bar{z}$, we shall find that the one investigation assigns a definite but consistent value to what the other left arbitrary. Either comparison gives

$$c = \frac{r - a_1 c_1 p_1 - a_2 c_2 p_2}{1 - a_1 c_1 - a_2 c_2}$$

We may prove, either by the “conditions of possible experience,” or independently, that this value is necessarily a proper positive fraction, and this accords with the interpretation of c as a probability. Art. 23.

27. But a much more important consideration is the following. It is a plain consequence of the logical theory of probabilities, that the state of expectation which accompanies entire ignorance of an event is properly represented, not by the fraction $\frac{1}{2}$, but by the indefinite form $\frac{0}{0}$. And this agrees with a conclusion at which Bishop TERROT, on independent, but as I think just grounds, has arrived.* Now this shows, why, if the consideration of the *a priori* probability of z is, from

* Transactions of the Royal Society of Edinburgh, vol. xxi. p. 375.

the insufficiency of the remaining data, necessary in order to give to the *à posteriori* probability of $\approx$ a definite value, the solution obtained when that *à priori* value is neglected should involve the symbol $\frac{0}{0}$. The presence of this symbol in a solution always indicates insufficiency in the data. And herein, as it seems to me, consists the reason why the mind, impatient of incertitude even while dealing with the very science of uncertain knowledge, is led to seek escape from its doubts, by calling in the aid, in some form or another, of that adventitious principle which I have denominated the principle of the mean. I say in some form or another; for I can conceive of another form of the same principle connected more directly with the idea of a *limit* than with that of a mean. Thus as testimonies which are insufficient of themselves to produce a definite expectation may definitely modify a definite expectation previously formed, we have suggested to us the idea of that limiting state to which perpetual and independent repetition of the same series of testimonies would cause the mind, whatever its starting point of expectation might be, to tend. And as this limiting state would be one which a further repetition would not alter, we should thus arrive in effect at the same solution as is indicated by the principle of the mean, in its direct expression.

28. I have extended the preceding analysis to the case in which three observations are to be combined, a case which, in connection with the previous one, is sufficient to determine the general law. The result is what the preceding analysis suggests, and may be expressed in the following theorem:—

If n conflicting observations assign to the altitude of a star the respective values $p_1 p_2 \dots p_n$; if, moreover, $a_1 a_2 \dots a_n$ are the antecedent probabilities that the observations will be such as they prove to be with respect to those circumstances which determine their relative accuracy, and $c_1 c_2 \dots c_n$ their respective probabilities of correctness to a mind acquainted with these circumstances, *i.e.*, to the mind of the observer after the observations have been made, then the most probable altitude of the star will be

$$\frac{\frac{1-a_1}{1-c_1}c_1p_1 + \frac{1-a_2}{1-c_2}c_2p_2 \dots + \frac{1-a_n}{1-c_n}c_np_n}{\frac{1-a_1}{1-c_1}c_1 + \frac{1-a_2}{1-c_2}c_2 \dots + \frac{1-a_n}{1-c_n}c_n}$$

This expression admits of the same deductions as the one before obtained, for the case in which the observations are two in number, and in particular it leads, when the circumstances of the observations are judged to be in all respects the same, to the principle of the arithmetical mean expressed by the formula

$$\frac{p_1 + p_2 \dots + p_n}{n}$$

29. I have remarked that the principle of the arithmetical mean has some

claim to be regarded as axiomatic. In the preceding sections it presents itself as a special result of a very complex analysis founded upon the logical theory of probabilities. Now I wish to observe, that there is nothing in these circumstances which we have a right to regard as denoting inconsistency. Of the theory of probabilities it is eminently true that modes of investigation, which to our present conceptions must appear fundamentally different, habitually lead us to the same result. A profounder acquaintance with the laws of the human mind, and a deeper insight into the relations of things, might perhaps show us that principles which appear to us to have nothing in common may yet have a necessary connection with each other,—may possibly spring up from a common origin. I will endeavour to make my meaning clear by two illustrations, which will present this question in somewhat different lights.

30. An idea which seems naturally to suggest itself in connection with the theory of probabilities is that of mechanical analogy. Evidence of this we see in the language, already referred to, which attributes *weight* to observation. The complete and scientific development of the idea will be found in a memoir by Professor DONKIN,* who, establishing a kind of metaphysical statics on proofs of the same nature as those which are employed in deducing *à priori* the laws of ordinary statics, has arrived, by legitimate deduction, at the remotest consequences of GAUSS's theory of the combination of observations. The mind, in the developed analogy, is compared to a lever acted upon by different weights, or to a mechanical system subject to given forces, and seeking, under this action, a position of equilibrium. Now it is at least a very remarkable circumstance, that an analogy of this kind should not only admit of exact scientific expression, but should, through a long train of analytical consequences, present the same laws and results, and suggest the same methods, as the principle of the arithmetical mean already referred to. All the abstract terms by which mental states and emotions are expressed, derive, if philology be of any value, their origin from outward and material things. And hence, though it might be impossible to ascend historically to the first employment of those expressions which describe the mind under the action of *forces*, and speak of the *balancing* of opinions, we cannot doubt that a perceived analogy was their source. But it could hardly have been anticipated that this analogy should remain complete and unimpaired through so lengthened a range of scientific deductions.

To what I have said above I will only add, that it is as instruments of expression and communication, rather than of thought, that material symbols, and the analogies which they furnish, seem to me to possess importance. Even the analogy which we have been considering cannot of itself occupy the place of a first principle, but seems to be a particular manifestation of that deeper

* Sur la Theorie de la Combinaison des Observations. LIOUVILLE's Journal de Mathematiques, tom. xv., 1850.

truth of which LEIBNITZ had a glimpse when he spoke of the principle of fitness and congruity—" *principe de la convenance*,"*—the ground of rational mechanics. Of course, I do not contemplate this or any subjective principle whatever, as affording us the slightest ground for affirming that the constitution of nature must, *à priori*, possess such and such a character. But it does seem to be a fact that the material system has been constituted in a certain degree of accordance with our rational faculties. The study of this accordance, *à posteriori*, is a perfectly legitimate object; and I think it the more interesting, when it brings before our view the scientific form of any of those analogies which commended themselves to the minds of the fathers of our race, which are embodied in our common speech, and without which we could apparently never hold converse with our fellows, except upon material objects.

31. The second illustration which I have to offer is the following. Many of the most important applications of the theory of probabilities, the method of least squares, for example, rest upon what has been termed the law of facility of error. This consists in the position, that in seeking to determine by observation a physical magnitude, as the elevation of a star, the probability that any measure will deviate by a quantity x from the true value, will vary directly as the function $e^{-k^2 x^2}$, where k is a constant quantity. The probability that our measure will fall between the limits x and $x + dx$ being expressed by the function

$$\frac{k}{\sqrt{\pi}} e^{-k^2 x^2} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

GAUSS has shown that this is the only "law of facility" consistent with the assumption that, in a series of observations of the same magnitude, the arithmetical mean of the several measures obtained is the most probable value. It may even be shown, that whatever the actual "law of facility," under given circumstances, may be, and it is plain that it must vary with circumstances, such as the constitution of the instrument and the character of the observer, &c., the probability that the arithmetical mean of a very large number of values determined by observation will deviate from some fixed value by a quantity x , will vary directly as $e^{-k^2 x^2}$, k being a constant dependent upon the nature of the observations.† Such, at least, is the limiting form of the function to which the law of deviation approaches as the number of observations is increased. Now it is remarkable that considerations of a totally different kind, and founded mainly upon our conceptions of space, lead to a similar result. The probability of linear deviation (measured in a given direction) of a ball from a mark at which it is aimed, seems to obey the same law; the principle upon which that law is deter-

* ERDMANN's Edit., p. 716.

† For some very interesting illustrations of this doctrine, see the letters of M. BRAVAIS, published in the notes to QUETELET's Letters on the Theory of Probabilities.

mined being, not that of the arithmetical mean, but rather a principle of geometrical consistency, intimately connected with our ideas of the composition of motion.

The principle was first stated in a popular and somewhat inexact form, by Sir JOHN HERSCHEL, I believe, in the *Edinburgh Review*.* It was afterwards made the subject of an adverse criticism in the *Philosophical Magazine*, by Mr LESLIE ELLIS.† There is no living mathematician for whose intellectual character I entertain a more sincere respect than I do for that of Mr ELLIS; and even while stating the grounds upon which I differ from him, with respect to the value of Sir JOHN HERSCHEL's principle, I avail myself of his labours, in giving to that principle a more scientific form and expression, and in developing its consequences. The language adopted in the following statement, will be, as far as possible, that of the author of the principle,—the analysis will be that of Mr ELLIS.

Suppose a ball dropped from a given height, with the intention that it shall fall on a given mark. Now, taking the mark as the origin of two rectangular axes, let it be assumed, that the actual deviation observed is a compound event, of which the two components are the corresponding deviations measured along the rectangular axes. Grant, also, that the latter deviations are independent events. Further, let us represent by $f(x^2)$, $f(y^2)$, the probabilities of the respective component deviations measured along the axes x and y ,—we give to them this form, because, positive and negative deviations being equally probable, the function expressing probability must be an *even* one, *i.e.*, must not change sign with the error. Hence the probability of the actual deviations observed will be $f(x^2)f(y^2)$. Let it be observed that this is not the probability of a deviation to the extent $\sqrt{x^2+y^2}$ from the mark, but of a deviation to that extent *in a particular line of direction*. Now, let the principle be assumed, that this expression is independent of the position of the axes, *i.e.*, that we may regard component deviations along any two rectangular axes as independent events, by the composition of which the actual deviation is produced. We have then x' and y' representing two new component deviations,

$$f(x^2)f(y^2)=f(x'^2)f(y'^2) \quad . \quad . \quad . \quad . \quad (2)$$

If $y'=\sqrt{x^2+y^2}$ then $x'=0$ and we have

$$f(x^2)f(y^2)=f(0)f(x^2+y^2) \quad . \quad . \quad . \quad . \quad (3)$$

An equation of which the complete solution is,

$$f(x^2)=A\epsilon^{hx^2}$$

A and h being constants. The condition that the probability of the error must

* Vol. xcii. p. 17. Art. QUETELET on Probabilities.

† Vol. xxxvii. p. 321, "Letter addressed to J. D. FORBES, Esq., Professor of Natural Philosophy in the University of Edinburgh, on an alleged proof of the Method of Least Squares."

diminish as the amount of the error increases, requires that h should be negative. We may therefore write, $-k^2$, for h . Whence

$$f(x^2) = A e^{-k^2 x^2} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To apply this result to the case in which the ball is supposed at some point on the plane which, projected on the axis of x , will fall between x and $x + \delta x$, we must give to A the form $C \delta x$.

Thus we get the expression,

$$C e^{-k^2 x^2} \delta x$$

Lastly, the *certainty* that the ball must fall at some point for which the value of x lies between $-\infty$ and ∞ gives us the equation

$$\int_{-\infty}^{\infty} C e^{-k^2 x^2} \delta x = 1$$

whence $C \frac{\sqrt{\pi}}{k} = 1$ and $C = \frac{k}{\sqrt{\pi}}$. Thus, the probability of a deviation from the axis y to a distance lying between x and $x + \delta x$ will be given by the formula

$$\frac{k}{\sqrt{\pi}} e^{-k^2 x^2} \delta x \quad . \quad . \quad . \quad . \quad (5)$$

an expression which agrees with (1).

In like manner, the probability that the ball will deviate to a distance greater than y and less than $y + \delta y$ from the axis x will be

$$\frac{k}{\sqrt{\pi}} e^{-k^2 y^2}$$

whence the probability that it will actually fall upon the elementary area $\delta x \delta y$ will be

$$\frac{k^2}{\pi} e^{-k^2 (x^2 + y^2)} \delta x \delta y$$

Now, this result admits of a remarkable confirmation. For it is manifest that the probability that the ball will fall *somewhere* between the distances x and $x + \delta x$ from the axis y , ought to be equal to the above expression integrated with respect to y between the limits $-\infty$ and ∞ . But that probability has been already determined to be $\frac{k}{\sqrt{\pi}} e^{-k^2 x^2} \delta x$; we ought therefore to have

$$\int_{-\infty}^{\infty} \left(\frac{k^2}{\pi} e^{-k^2 (x^2 + y^2)} \delta x \right) \delta y = \frac{k}{\sqrt{\pi}} e^{-k^2 x^2} \delta x \quad . \quad . \quad (6)$$

an equation which is actually true.

Mr ELLIS considers this as showing, that the principle from which the demonstration sets out, viz., that the actual deviation of the ball from the mark may be regarded as a compound event, of which the two independent components are the deviations from the axes, involves either a mistake or a *petitio principii*. But consistency of results can never be a proof of mistake in the principles from which they are deduced; and *alone*, it offers no adequate ground for the suspicion of a

petitio principii. It is to be observed, that it is only the probability of deviation from a fixed axis which follows, according to the above investigation, the law expressed by GAUSS'S function. The probability of deviation in *any* direction to a distance between r and $r + \delta r$ from the mark, is expressed by a different function. This would be fatal to any hypothesis which should represent GAUSS'S function as determining, *à priori*, the actual law of deviation. There are indeed few cases in which it can be determined what the law is, and writers on probability have been far too anxious to interpret nature in accordance with their formulæ. No one has shown this more clearly than Mr ELLIS. The precise value of Sir JOHN HERSCHEL'S principle, as corrected by him, I conceive to be this,—that it establishes an identity between the law of facility of error expressed by GAUSS'S function and the law which in a special problem, involving the consideration of space and motion, seems to accord with our most elementary conceptions of these things; and this identity I apprehend to be, not an accidental thing, but a very distinct expression of that harmonious relation which binds together the different spheres of thought and existence.

33. We proceed next to the consideration of the second general problem,—that in which it is proposed to determine the combined force of two testimonies or judgments in support of a fact, the strength of each separate testimony being given.

The problem has a material as well as a formal aspect. Thus oral testimonies differ from the judgments which are furnished by the immediate personal observation of facts. And although no definite general laws have, so far as I am aware, been assigned concerning the mode in which the material character of the evidence affects expectation, it is not to be doubted that an influence does proceed from this source. As respects testimony alone, there are cases in which we feel that it is cumulative,—there are cases in which we feel that it is not so; and this difference we also feel depends upon the nature of the testimony itself. But in the majority of cases, we should probably feel that the elements upon which this difference of character depends are blended together, some decided preponderance being due to the one or to the other. Testimony will be chiefly or entirely cumulative which is given quite independently by different persons, and is at the same time based upon different grounds. In proportion as these conditions fail of being satisfied, the testimony partakes less and less of the cumulative character. Still this possession of cumulative character may be regarded as the standard by which the distinctive qualities of testimonies, as affecting belief or expectation, may be estimated. In judgments founded upon the personal observation of facts, though this character may be observed, the standard seems to be different. When different modes of considering a subject—different courses of experiment or inquiry—lead to different probabilities of a fact, some making it more probable, some less, we generally feel that a kind of mean ought to be taken among them. Perhaps the most succinct general statement would be,

that it belongs to testimony, in its normal character, to be cumulative,—to judgment, to require the application, in some form or other, of the principle of means or averages; but that all departures from these normal states involve the blending of the two elements together, in proportions determined by the degree of the deflection.

Now, although it does not belong to the theory of probabilities, in its formal and scientific character, to pronounce upon the material character of a problem, and to say whether its data are in their own nature cumulative or not, yet the results to which the theory leads are, in a very remarkable degree, accordant with the distinctions which have just been pointed out. I shall show that the solution of the problem of the combination of testimonies, when the data are presented in a purely formal character, and without any adventitious principle, involves arbitrary constants, and is therefore indefinite,—being capable, however, under certain circumstances, of assuming a definite form. I shall show that such a form is assumed when the circumstances are such as to give to the testimonies the highest degree of cumulative character. I shall then solve the problem a second time, introducing that adventitious principle which I have already exemplified in the problem of the reduction of astronomical observations, and which appears to me to contain the true theory of means or averages. The form of the solution thus obtained, which is also perfectly definite, will apply to the case, in which it is our object, not to combine testimonies, in the ordinary sense of the term, but to determine the mean of expectations founded upon the issues of conflicting judgments. To one point of importance I must again, before entering upon the analytical investigation, ask the attention of the reader. It is, that in the present subject, the question of the right application of a formula is quite distinct from that of the validity of the processes by which that formula is derived from its data. The latter is a question of formal science, the former involves considerations which belong rather to the philosophy of the human mind.

I will first express the problem which we have to consider in a general form, equally applicable to the combination of testimonies or of judgments. I shall consider the fact of a testimony having been borne, or an observation made, as a circumstance or event affecting our expectation of the event to which it has reference.

PROBLEM II.

34. *Required the probability of an event z , when two circumstances x and y are known to be present,—the probability of the event z , when we only know of the existence of the circumstance x being p ,—and its probability when we only know of the existence of y being q .*

Here we are concerned with three events, x , y , and z . For convenience and uniformity I shall, in the solution of the problem, speak of x and y as *events*, as

well as of z . A circumstance is an event—a state of things which comes to pass, or has come forth—*evenit*.

The data leave wholly arbitrary the probabilities of the event x and y . Thus p and q are *conditional* probabilities; p is the probability that if the event x occur, the event z will occur; q is the probability that if the event y occur, the event z will occur. Hence

$$p = \frac{\text{Prob. } xy}{\text{Prob. } x}, \quad q = \frac{\text{Prob. } yz}{\text{Prob. } y} \quad . \quad . \quad . \quad . \quad (1)$$

Our object is to determine the probability that if the events x and y both occur, the event z will occur. We have therefore to seek the value of the fraction

$$\frac{\text{Prob. } xyz}{\text{Prob. } xy}$$

or, as for our present purpose it is more convenient to say, of

$$\frac{\text{Prob. } xyz}{\text{Prob. } xyz + \text{Prob. } \overline{xyz}} \quad . \quad . \quad . \quad . \quad (2)$$

In seeking the value of Prob. xyz , which we shall represent by u , the formal statement of our *data* and *quæsitum* will therefore be

$$\text{Given } \left\{ \begin{array}{ll} \text{Prob. } x=c, & \text{Prob. } y=c' \\ \text{Prob. } xz=cp, & \text{Prob. } yz=c'q \end{array} \right\} \quad . \quad . \quad . \quad . \quad (3)$$

Required Prob. xyz .

c and c' being arbitrary constants expressing the unknown probabilities of the events x and y .

A misconception may here arise respecting the meaning of Prob. x , Prob. y , which it is worth while to anticipate. In the case of testimony, Prob. x would not mean the probability that a testimony would be borne, but the probability that the particular kind of testimony actually recorded considered with reference to its object, credibility, &c., would be borne. Testimonies differ, not merely as to their degree of credibility, but as to their *unexpectedness*—as to the *surprise* which they occasion. And it is, I think, matter of personal experience that this unexpectedness is in itself an element affecting the strength of that expectation which combined testimonies produce. So, too, if x and y are facts of observation, *e.g.*, observed symptoms of a disease z , the probability of that disease, when both symptoms present themselves, is not determined by the strength of the separate presumptions merely, but is consciously increased by our knowledge of the rarity of the symptoms themselves. And thus the elements Prob. x Prob. y , which have been introduced by a formal necessity of the statement of the problem are seen to belong to the very matter of its solution.

Making

$$xz=s, \quad yz=t, \quad xyz=\phi,$$

we find, by the calculus of logic,

$$\phi = x y s t + 0 (x s \bar{y} \bar{t} + y t \bar{x} \bar{s} + x y \bar{s} \bar{t} + x \bar{y} \bar{s} \bar{t} + y \bar{x} \bar{s} \bar{t} + \bar{x} \bar{y} \bar{s} \bar{t}) \\ + \text{ terms whose coefficients are } \frac{1}{0} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

a result which may be verified by the method applied to (14) in Art. 23.

Hence we find, adopting the simplification of Art. 21,

$$V = x y s t + x s + y t + x y + x + y + 1,$$

and since we have

$$\left. \begin{array}{l} \text{Prob. } x = c, \quad \text{Prob. } y = c', \quad \text{Prob. } s = cp, \quad \text{Prob. } t = c'q \\ \text{Prob. } xyz = u, \end{array} \right\} \quad . \quad . \quad . \quad (5)$$

we find, as an algebraic system of equations,

$$\left. \begin{array}{l} \frac{xyst + xs + xy + x}{c} = \frac{xyst + yt + xy + y}{c'} \\ = \frac{xyst + xs}{cp} = \frac{xyst + yt}{c'q} \\ = \frac{xyst}{u} = xyst + xs + yt + xy + x + y + 1 \end{array} \right\} \quad . \quad . \quad . \quad (6)$$

This system is easily reduced to the form

$$\left. \begin{array}{l} \frac{xs}{cp - u} = \frac{yt}{c'q - u} = \frac{xy + x + y + 1}{1 + u - cp - c'q} \\ = \frac{x + 1}{1 + u - cp - c'} = \frac{y + 1}{1 + u - c - c'q} = \frac{xyt}{u} \end{array} \right\} \quad . \quad . \quad . \quad (7)$$

And if we equate the respective products of the first three and of the last three members of the above, we find

$$(cp - u)(c'q - u)(1 + u - cp - c'q) = (1 + u - c' - cp)(1 + u - c - c'q)u \quad . \quad . \quad (8)$$

a quadratic equation by which the value of u must be determined.

If, in like manner, we assume

$$\text{Prob. } xy\bar{z} = t$$

we shall find

$$(\bar{c}\bar{1} - \bar{p} - t)(\bar{c}'\bar{1} - \bar{q} - t)(1 + t - \bar{c}\bar{1} - \bar{p} - \bar{c}'\bar{1} - \bar{q}) = (1 + t - c' - \bar{c}\bar{1} - \bar{p})(1 + t - c - \bar{c}'\bar{1} - \bar{q})t \quad (9)$$

From these equations the values of u and t being determined, we have finally

$$\frac{\text{Prob. } xyz}{\text{Prob. } xy} = \frac{u}{u + t} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Before we can apply this solution, we must determine the conditions of possible experience, and the conditions limiting the values of u and t . For this purpose writing

$$\begin{array}{llll} \text{Prob. } xyz = u, & \text{Prob. } xy\bar{z} = t, & \text{Prob. } xzy = \mu, & \text{Prob. } xy\bar{z} = \nu, \\ \text{Prob. } yz\bar{x} = \rho, & \text{Prob. } y\bar{z}x = \sigma, & & \end{array}$$

we have, from the data, the equations,

$$\begin{aligned} u + t + \mu + \nu &= c \\ u + t + \rho + \sigma &= c' \\ u + \mu &= cp \\ u + \rho &= c'q, \end{aligned}$$

to which must be added the inequations

$$\begin{aligned} u \geq 0 \quad t \geq 0 \quad \mu \geq 0 \quad \nu \geq 0 \quad \rho \geq 0 \quad \sigma \geq 0 \\ u + t + \mu + \nu + \rho + \sigma \leq 1 \end{aligned}$$

Proceeding as in Art. 14, we find ultimately as the conditions of possible experience

$$cp \leq 1 - c'(1 - q) \quad c'q \leq 1 - c(1 - p) \quad . \quad . \quad . \quad . \quad (11)$$

together with the usual condition that c , c' , p and q must be positive proper fractions, or at any rate must not transcend the values 0 and 1. We find, too, as the conditions limiting u and t ,

$$\left. \begin{aligned} u &\leq cp & u &\leq c'q \\ u &\geq c + c'q - 1 & u &\geq c' + cp - 1 & u &\geq 0 \\ t &\leq c(1 - p) & t &\leq c'(1 - q) \\ t &\geq c + c'(1 - q) - 1 & t &\geq c' + c - p - 1 & t &\geq 0 \end{aligned} \right\} \quad . \quad . \quad . \quad (12)$$

The solution of the problem assumes, therefore, the following form and character:—

1st, It involves two constants c and c' , which are arbitrary, except in that they are subject to the conditions (11).

2ndly, The values of u and t , determined from (8) and (9), in subjection to the conditions (12), are to be substituted in the formula (10).

3dly, In the absence of any means of determining c and c' , the value obtained will be indeterminate, except for particular values of p and q . Some general conclusions may nevertheless be deduced from its expression indicating the manner in which expectation is influenced by circumstances insufficient of themselves to give to it a definite amount of strength. This will appear from the following analysis.

Analysis of the Solution.

35. The solution is contained in the numbered results, from (8) to (12) inclusive, of the preceding Article. Of these, (11) expresses the conditions of possible experience, (12) the conditions limiting u and t . From (8) and (9), these quantities are to be determined in accordance with (12), and the resulting values substituted in (10).

By a proper reduction of (8) and (9), the solution may also be put in the following form:—

$$a'u^2 + (cc'm - aa')u - acc'pq = 0 \quad . \quad . \quad . \quad . \quad (1)$$

$$at^2 - (cc'm + aa')t - a'cc'(1 - p)(1 - q) = 0 \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{where} \quad a = cp + c'q - 1 \quad a' = c(1 - p) + c'(1 - q) - 1 \quad m = p + q - 1.$$

The values of u and t hence found, in accordance with the limitations expressed by (12), are to be substituted in the equation

$$\frac{\text{Prob. } xyz}{\text{Prob. } xy} = \frac{u}{u+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The following special deductions may now be noted.

1st, If either of the quantities p and q is equal to 1, the probability sought is equal to 1, whatever the values of c and c' may be.

Thus let $p=1$. Then (2) gives $t=0$; the only value which satisfies the conditions (12), in connexion with (11). The equation (1) is not satisfied by $u=0$, whence

$$\frac{u}{u+t} = \frac{u}{u} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This result is obviously correct. If, for example, of two symptoms which are present, and which furnish ground of inference respecting a particular disease, one be of such a nature as to make the existence of the disease a matter of certainty, the fact of that existence is established, however adverse to such a conclusion the presumption furnished by the other symptom, supposing it our only ground of inference, would be.

So, too, the verdict of an authority deemed infallible is consistently held to annul and make void all opposing testimony or argument, however powerful such testimony or argument, considered in itself, may be.

2ndly, If either of the quantities p and q is equal to 0, the probability sought reduces to 0, as it evidently ought to do.

3dly, If $p = \frac{1}{2}$ and $q = \frac{1}{2}$, the equations for determining u and t become identical. Hence $u=t$, and

$$\text{Probability sought} = \frac{u}{u+u} = \frac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This result is quite independent of the values of c and c' . And it is obviously a correct result. If the causes in operation, or the testimonies borne, are, separately, such as to leave the mind in a state of equipoise as respects the event whose probability is sought, united they will but produce the same effect, whatever the *à priori* probability may be that such causes will come into operation, or that such testimonies will be borne.

4thly, If $c=1$, and at the same time c' is not equal to 0, we find, for the equations determining u and t ,

$$\begin{aligned} (p-u)(c'q-u)(u-p-c'q+1) &= u(u-c'-p+1)(u-c'q) \\ (1-p-t)(c'1-q-t)(t-c'1-q+p) &= t(t-c'+p)(t-c'1-q) \end{aligned}$$

These give

$$u=c'q \qquad t=c'(1-q)$$

as the only values which satisfy (12). Hence

$$\text{Probability sought} = \frac{c'q}{c'q + c'1 - q} = q \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This result is evidently correct. The probability that such an event will take place when two other events, x and y , are present, is the same as the probability that it will take place when the event y is present, if it is known that the other event x is never absent.

5thly, If $c=c'$ and $q=1-p$, we find in like manner $u=t$, whence

$$\text{Probability sought} = \frac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This result is evidently correct. If the events or testimonies x and y are equally likely to happen, and if the first yields the same presumption *in favour* of that event whose probability is sought as the other yields *against* it, the chances are equally balanced, and the probability required is $\frac{1}{2}$.

6thly, But if $q=1-p$, while c and c' are *not* equal, then the value of the probability sought is no longer $\frac{1}{2}$. It may be shown, by a proper discussion of the formulæ, that the presumption afforded by the event x , whether favourable or unfavourable, is stronger than the *opposite* presumption afforded by the event y , whenever c is less than c' , and *vice versa*. And hence it follows, that if there be two events which, by themselves, afford equal presumptions, the one for and the other against some third event, of whose probability nothing more is known, then, if the said two events present themselves *in combination*, that one will yield the stronger presumption, which is itself, of the more rare occurrence. This, too, is agreeable to reason. For in those statistical observations by which probability is determined, we can only take account of co-existences and successions. We do not attempt to pronounce whether the presence of the event z in conjunction with the event x is due to the efficient action of the event x , or whether it is a product of some other cause or causes. The more frequent the occurrence of x , the less entitled are we to assert that those things which accompany or follow it derive their being from it, or are dependent upon it. If, for instance, x were a standing event, or a state of things *always* present, the probability that any event z would occur when x and y were jointly present, would be the same as the simple probability of that event z when y was present, and it would be wholly uninfluenced by the presence of x . This is the limiting case of the general principle.

7thly, The case in which $c=c'$ and $p=q$, is a very interesting one. A careful analysis leads to the following results.

If there be two events z and y , which are in themselves equally probable, the probability of each being c , and if when the event x is known to be present, while

it is not known whether y is present or not, the probability of z is p , the same probability being assigned to z , when it is known that y is present, but not known whether x is present or not; then, considering p as a presumption for or against z , according as p is greater or less than $\frac{1}{2}$.

1. That presumption is strengthened if the events x and y are known to be jointly present, *i.e.*, the probability of z is greater than p , if p is greater than $\frac{1}{5}$, but less than p in the contrary case.

2. The strengthening of the presumption is greatest when c is least. In other words, the less likely the events x and y are to happen, the more does their actual concurrence strengthen the presumption, favourable or unfavourable, which either of them alone must afford.

8thly, If we suppose c and c' both to approximate to 0, the values of u and t also approximate to 0, and the ratio $\frac{u}{u+t}$ assumes at the limit the form $\frac{0}{0}$. It may, however, be shown that its actual value at the limit is

$$\frac{pq}{pq + (1-p)(1-q)} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (8)$$

This is most readily obtained from (1) and (2), by rejecting the terms $\alpha' u^2$ and αt^2 , which we may do when u and t are infinitesimal. We thus find that u and t tend to assume the values $cc'pq$ and $cc'(1-p)(1-q)$, whence

$$\frac{u}{u+t} = \frac{pq}{pq + (1-p)(1-q)}$$

It is interesting here to inquire whether the appearance of the limiting value $\frac{pq}{pq+1-p} \frac{1}{1-q}$ is due merely to the *smallness* of c and c' . In studying this question, it occurred to me that it is generally not the mere improbability of events, or the mere unexpectedness of testimonies considered in themselves, but the improbability of the *concurrence* of such events or testimonies which gives to their *union* the highest degree of force. I therefore anticipated, that, if I should introduce among the primary data of the problem, the probability of the concurrence of the events x and y , assigning to it a value m , it would appear that, whenever m approached to 0, the presumptions with reference to the event z , founded upon x and y , would receive strength, whatever the values of c and c' might be. And this expectation was verified. On taking for the data

$$\text{Prob. } x=c, \quad \text{Prob. } y=c', \quad \text{Prob. } xy=m, \quad \text{Prob. } xz=cp, \quad \text{Prob. } yz=c'q$$

and representing the sought value of $\frac{\text{Prob. } xyz}{\text{Prob. } xy}$ by w , I found, for the determination of w , the equation

$$(cp-mw) (\overline{c'q-mw}) (1-w) = w(\overline{c1-p-m1-w}) (\overline{c'1-q-m1-w}) \quad . \quad . \quad (9)$$

which is substituted for the true one is the following. The probability that A speaks truth is p , that B speaks truth is q ; what is the probability that, if they both make assertions, and these assertions are both true or both false, they are both true? Whether A and B make the same assertion or not is assumed to be a matter of indifference. But this assumption is, in point of fact, as erroneous as it is unwarranted. The problem which we have solved in the preceding sections, interpreted in relation to testimony, is the following. Two witnesses, A and B, assert a fact. The probability of that fact, if we only knew of A's statement, would be p , if we only knew of B's, would be q ; what is its probability when we know of both? The formal expression of this problem will be seen in Art. 34. The most complete formal expression of the problem which has been substituted for it, taking into account all its elements, is as follows. Let x and y represent the testimonies of A and B, w and z the facts to which these testimonies respectively relate. Observe that no hypothesis is here made as to the connection, by sameness or difference, of w and z . And the simple absence of any such hypothesis is properly signified by expressing the events by different symbols, unaccompanied by any logical equation connecting these symbols.

If we wish to indicate that the events w and z are identical, we must write as a connecting logical equation,

$$w = z$$

though it must be simpler to express the identity by the employment of a single symbol as before. Any other definite relation may be expressed in a similar way.

The Problem now stands thus:—

$$\text{Given} \quad \left. \begin{array}{ll} \text{Prob. } x = c, & \text{Prob. } xw = cp, \\ \text{Prob. } y = c', & \text{Prob. } yz = c'q, \end{array} \right\} \quad \cdot \quad \cdot \quad \cdot \quad (13)$$

$$\text{Required} \quad \frac{\text{Prob. } xywz}{\text{Prob. } xywz + \text{Prob. } xy\bar{w}\bar{z}} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (14)$$

First, we will seek the value of Prob. $xywz$.

$$\text{Let} \quad xw = s, \quad yz = t, \quad xywz = v$$

From these logical equations we must now determine v as a developed logical function of x , y , s , and t . The result is

$$\begin{aligned} v = & x y s t + 0(x y s \bar{t} + x y t \bar{s} + x y \bar{s} \bar{t} + x s \bar{y} \bar{t} + x \bar{y} \bar{s} \bar{t} \\ & + y t \bar{x} \bar{s} + y \bar{x} \bar{s} \bar{t} + \bar{x} \bar{y} \bar{s} \bar{t}) \\ & + \text{terms whose coefficients are } \frac{1}{0}. \end{aligned}$$

Let u be the value of Prob. v . Then, by the simplification of Art. 21. we have

$$\begin{aligned}
\frac{xyst + xys + xyt + xy + xs + x}{c} &= \frac{xyst + xys + xyt + xy + yt + y}{c'} \\
&= \frac{xyst + xys + xs}{cp} = \frac{xyst + xyt + yt}{c'q} \\
&= \frac{xyst}{u} = xyst + xys + xyt + xy + xs + x + yt + y + 1
\end{aligned}$$

Equating the product of the third and fourth to that of the fifth and sixth members of the above system, we have

$$u = cc'pq = \text{Prob. } xywz$$

whence

$$cc' (1-p) (1-q) = \text{Prob. } xy\bar{w}\bar{z}$$

And hence

$$\frac{\text{Prob. } xywz}{\text{Prob. } xywz + \text{Prob. } xy\bar{w}\bar{z}} = \frac{pq}{pq + (1-p)(1-q)} \quad . \quad . \quad (15)$$

Here it will be noted, that although the arbitrary constants c and c' were necessarily introduced into the expression of the data of problem, they have no place in its solution. The result, it will also be seen, agrees with (8); and it thus shows that that formula would express the true solution of the problem originally proposed, if it were permitted to neglect the circumstance that it is to the *same* fact that the testimonies have reference, and so to regard their agreement as merely an agreement in being true or in being false, but not in being true or in being false about the same thing.

Special Solution of Problem II. founded upon the principle of the limit.

36. In the present investigation we employ the principle stated in Art. 24, our object being to determine the mean between p and q , when they represent probabilities founded upon different judgments, just as in Art. 25 we have determined the mean between p and q , when they represent different observed values of a physical magnitude.

To the previous data, viz.,

$$\text{Prob. } x=c, \quad \text{Prob. } y=c', \quad \text{Prob. } xz=cp, \quad \text{Prob. } yz=c'q \quad . \quad . \quad (1)$$

we now add, as the supposed *à priori* value of Prob. z ,

$$\text{Prob. } z=r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From these collective data we determine the fraction

$$\frac{\text{Prob. } xyz}{\text{Prob. } xy} \quad \text{or} \quad \frac{\text{Prob. } xyz}{\text{Prob. } xyz + \text{Prob. } xy\bar{z}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

representing the *à posteriori* value of Prob. z , and, equating the *à priori* and *à posteriori* values, determine x . The principle upon which the investigation proceeds, is, that we attribute to the mean strength of the probabilities p and q such a value, that if the mind had previously to the evidence been in the state of expectation which that value is supposed to measure, the evidence would not have

tended to alter that state. By the evidence I mean, of course, that which forms the basis of the judgments.

Making, as before,

$$xz=s \qquad yz=t \qquad xyz=v$$

and determining v as a developed logical function of x, y, z, s , and t , we find

$$\begin{aligned} v = & x y z s t + 0 (x y z \bar{s} \bar{t} + x z s \bar{y} \bar{t} + y z t \bar{x} \bar{s} + x \bar{y} z \bar{s} \bar{t} + y \bar{x} z \bar{s} \bar{t} \\ & + z \bar{x} \bar{y} \bar{s} \bar{t} + \bar{x} \bar{y} \bar{z} \bar{s} \bar{t}) \\ & + \text{terms whose coefficients are } \frac{1}{0}. \end{aligned}$$

Hence, availing ourselves of the simplification of Art. 21, we have

$$\begin{aligned} \frac{xyzst + xy + xzs + x}{c} &= \frac{xystz + xy + ytz + y}{c'} \\ &= \frac{xystz + xsz}{cp} = \frac{xystz + ytz}{c'q} \\ &= \frac{xystz + xsz + ytz + z}{r} = \frac{xystz}{u} \\ &= xystz + xy + xsz + ytz + x + y + z + 1 \end{aligned}$$

If we equate the product of the third and fourth to that of the fifth and sixth members of the above system, we find

$$\text{Prob. } xyz = \frac{cc'pq}{r}$$

whence by symmetry,

$$\text{Prob. } xyz = \frac{cc'(1-p)(1-q)}{1-r} \quad . \quad . \quad . \quad (4)$$

Substituting these values in (3), we have

$$\frac{\text{Prob. } xyz}{\text{Prob. } xy} = \frac{pq(1-r)}{pq(1-r) + (1-p)(1-q)r} \quad . \quad . \quad . \quad (5)$$

Before proceeding further, it will be well to note that in this formula p and q represent, not the general probabilities which the testimonies or evidences upon which our judgments are founded would give to the event z , but the probabilities which they would separately produce in a mind imbued with a previous expectation of the event z , the strength of which is measured by r . And there are some curious confirmations of the truth of the theorem, two of which I shall notice.

If we represent the *a posteriori* value of Prob. z by R , and accordingly make

$$\frac{pq(1-r)}{pq(1-r) + (1-p)(1-q)r} = R \quad . \quad . \quad . \quad (6)$$

we find, on solving the equation relatively to r ,

$$\frac{pq(1-R)}{pq(1-R) + (1-p)(1-q)R} = r \quad . \quad . \quad . \quad (7)$$

from which it appears, that if r is the *à priori* expectation of an event z , and if evidences are presented which severally would change r to p and q , and unitedly would change it to R ; then, reciprocally, if R measured the *à priori* expectation of the event, and evidences were received which would severally change it to p and q , unitedly they would reduce it to r . Now this is evidently what ought to be the case, since testimonies simply countervailing those by which r was changed to R , would simply undo what was done, and again reduce R to r .

We see that p and q being the same, R is greater when r is less, and less when r is greater; and this, though it is contrary to what we might at first expect, is agreeable to reason. For the effect of evidence is to be measured, not by the state of expectation which exists after it has been offered, but by the degree in which the previous state of expectation has been changed by it. Suppose p and q much greater than r , which we will conceive to be a small quantity, then the separate evidences greatly increase the probability of an event which was before very improbable; and unitedly they do this in a much higher degree than if the separate evidences had merely been such as to raise to the measures p and q , an expectation which was before not much below these measures.

Now, introducing the principle of the mean already explained, Art. 24, let us in (6) make $R=r$, we have

$$\frac{pq(1-r)}{pq(1-r) + (1-p)(1-q)r} = r$$

and solving this equation relatively to r , we find

$$r = \frac{\sqrt{pq}}{\sqrt{pq} + \sqrt{(1-p)(1-q)}} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

the formula required.

37. Upon this result, the following observations may be made:—

In the first place it may be shown, from the formula itself, that it always expresses a value intermediate between the values p and q . Thus we have

$$\begin{aligned} r-p &= \frac{\sqrt{pq}}{\sqrt{pq} + \sqrt{(1-p)(1-q)}} - p \\ &= \frac{\sqrt{q(1-p)} - \sqrt{q(1-q)}}{\sqrt{pq} + \sqrt{(1-p)(1-q)}} \times \sqrt{p(1-p)} \quad . \quad (9) \end{aligned}$$

on reduction. In like manner we have

$$\begin{aligned} r-q &= \frac{\sqrt{p(1-q)} - \sqrt{p(1-p)}}{\sqrt{pq} + \sqrt{(1-p)(1-q)}} \times \sqrt{q(1-q)} \\ &= - \frac{\sqrt{q(1-p)} - \sqrt{p(1-q)}}{\sqrt{pq} + \sqrt{(1-p)(1-q)}} \times \sqrt{q(1-q)} \quad . \quad (10) \end{aligned}$$

As p and q are positive fractions, the values of $r-p$ and $r-q$, given in (9) and (10), are clearly of opposite signs. whence r must lie between p and q .

In the second place, it may be shown, that r approaches more nearly to that one of the two values p and q , which most nearly approaches either of the limits 0 and 1. To show this, let us suppose q greater than p , and let us first inquire, under what circumstances r approaches more nearly to q than to p .

We must then assume

$$q - r < r - p$$

Substituting the values of these members from (9) and (10) we have

$$\frac{\sqrt{q(1-p)} - \sqrt{p(1-q)}}{\sqrt{pq} + \sqrt{(1-p)(1-q)}} \times \sqrt{q(1-q)} < \frac{\sqrt{q(1-p)} - \sqrt{p(1-q)}}{\sqrt{pq} + \sqrt{(1-p)(1-q)}} \times \sqrt{p(1-p)}$$

Now, q being by hypothesis greater than p , it is evident that $\sqrt{q(1-p)} - \sqrt{p(1-q)}$ will be positive. Rejecting, then, the common positive factor on both sides of the inequation we have

$$\sqrt{q(1-q)} < \sqrt{p(1-p)}$$

$$q - q^2 < p - p^2$$

$$q - p < q^2 - p^2$$

and dividing both sides by the positive factor $q - p$

$$1 < p + q$$

$$\therefore 1 - q < p$$

a condition which shows that q must be nearer to 1 than p is to 0.

On the other hand, as would appear from the very same analysis, changing only the signs $<$ into $>$, the condition that r may approach more nearly to p than to q , is that p may be nearer to 0 than q is to 1.

Now, 1 and 0, as limiting the measures of probability of the event z , indicate, the one that it certainly will, the other that it certainly will not occur. And the approach of any measure of probability to these limits indicates the approach of the probability to certainty. We see, then, that when p and q are measures of the probability of an event founded on different judgments, the *mean* between these measures, as determined by (8), will not be the usual arithmetical mean, but will always fall nearer to that one of the two values p and q which expresses a probability the most nearly approaching to certainty.

Now, this seems to be in accordance with reason. Evidence of any kind which enables us to pronounce a judgment with certainty, entirely preponderates over that which only enables us to affirm a probable judgment, Art. 35. And the more of the character of certainty that is possessed, the greater is the weight which is due to the evidence to which it belongs.

38. By an analysis similar to that which is applied in the previous sections. I have determined the general value of r , when the number of judgments is n , and

the values which they respectively give to the probability of the event z are p_1 $p_2 \dots p_n$. The result is

$$r = \frac{(p_1 p_2 \dots p_n)^{\frac{1}{n}}}{(p_1 p_2 \dots p_n)^{\frac{1}{n}} + ((1-p_1)(1-p_2) \dots (1-p_n))^{\frac{1}{n}}} \quad (1)$$

This is the general formula of the mean in reference to judgments, and much as it differs from the formula of the mean, in reference to the observations of a physical magnitude, some remarkable points of analogy exist. I will notice but one. The arithmetical mean is not altered if to the quantities among which it is taken we add another equal to the previous mean. Thus we have

$$\frac{p_1 + p_2 \dots + p_{n+1}}{n+1} = \frac{p_1 + p_2 \dots + p_n}{n}$$

provided that $p_{n+1} = \frac{p_1 + p_2 \dots + p_n}{n}$. Or representing $\frac{p_1 + p_2 \dots + p_n}{n}$ by P_n we have

$$P_{n+1} = P_n$$

provided that

$$p_{n+1} = P_n \quad (2)$$

The same relation may readily be shown to hold also, if P_n represent the mean of judgment, as expressed in (1).

39. The following is a brief summary of the conclusions established in this paper.

1st, The solution of the problem of astronomical observations by the logical theory of probabilities is, in its general form, indefinite.

2ndly, It becomes definite, if we introduce the general principle of means. The result is in accordance with the usual formulæ, but expresses the so-called *weights* of the observations as determinate functions of certain probabilities relating to the correctness of the observations, and the character of the observers.

3dly, When, as respects the two last elements, the observations are considered equal, the formula is reduced to the expression of the arithmetical mean.

4thly, The complete solution of the problem of the combination of two probabilities of an event founded upon different testimonies or judgments is indefinite, but admits, in various cases, of being reduced to a definite form.

5thly, This indefiniteness is due to the circumstance indicated by the formula, that the strength of the probabilities in combination is due, not to the strength of the separate probabilities alone, but also to the degree of unexpectedness of the testimonies or judgments themselves.

6thly, Combined presumptions, whether for or against an event, are generally strengthened by the unexpectedness of the combination.

7thly, When probabilities as $p_1, p_2, \dots, p_n$ are in a high degree cumulative, owing

to the exceeding improbability *à priori* of their combination, the expression for their united force tends to assume the form

$$\frac{p_1 p_2 \dots p_n}{p_1 p_2 \dots p_n + (1-p_1)(1-p_2) \dots (1-p_n)}$$

commonly assumed to express the *general* solution.

8thly, When the probabilities are so far from being cumulative that we feel that we ought to take a mean between them, the above formula is replaced by the following, viz.:—

$$\frac{\left\{ p_1 p_2 \dots p_n \right\}^{\frac{1}{n}}}{\left\{ p_1 p_2 \dots p_n \right\}^{\frac{1}{n}} + \left\{ (1-p_1)(1-p_2) \dots (1-p_n) \right\}^{\frac{1}{n}}}$$

9thly, This formula takes, in reference to ordinary judgments, the place of the arithmetical mean, with relation to the problem of astronomical observations, both being expressions of a more general principle.

40. It will probably appear to some of the readers of this paper, that I have dwelt more upon questions of philosophy and of language, than it is usual to do in mathematical treatises, and that I have also, in various parts, assumed the office of a critic, rather than that of an expositor of original views. Respecting the first of these points, I will only express a hope, that I have nowhere in this paper entered into discussions that are not strictly relevant to the subject. Upon the second, I have to observe, that the theory of probabilities is one in which as it seems to me, the critical office is especially needed. I do not think that it is likely to gain much advance from mere analysis. As respects the original portions of this paper, it is my strongest wish, that they should be regarded chiefly as materials for future judgment. Thus it is possible that the theory which I have developed with reference to problems of which the elements are logical, may be found to involve inconsistencies as a scientific theory, though I do not think this likely to be the case. But whether that theory shall finally be accepted or not, it is, I conceive, of some present importance, to establish the necessary dependence of *any* theory, professing to deal with the same class of problems, upon what I have termed the conditions of possible experience,—to show how those conditions may be determined, and how they are to be applied. As respects the so-called principle of the mean, applied in certain portions of this paper, it is open to inquiry whether it in all cases leads to results possessing the characteristic property, noted in Art. 38, and the decision of this question would materially affect our estimate of its value. Lastly, it is, I think, highly probable, that conditions which we do not yet know of may be discovered, affecting, not the *possibility* of the data of a problem as discussed in this paper, but their *adequacy*, and the principles which, in statistical research especially, ought to guide us in their selection. I am so conscious how limited, imperfect, and in some cases fluc-

The general proof consists in showing, that if the proposition is true for a particular value of n , it is true for the next greater. Whence, being true for the case of $n=1$, it is true universally. I will exemplify the method by showing how the truth of the proposition, when $n=2$ is dependent upon its truth when $n=1$.

Let $n=2$, then we have to consider the system (2), which may be reduced to the form

$$\frac{axy + bx}{axy + bx + cy + d} = p \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{axy + cy}{axy + bx + cy + d} = q \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Let us represent by Y the variable value of the first member of (4), when x and y are supposed to vary in subjection to the single condition (3). We have then

$$Y = \frac{axy + cy}{axy + bx + cy + d} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now differentiating (3) and (5) relatively to x and y , we find, after slight reductions,

$$(ay + b)(cy + d)dx + (ad - bc)xdy = 0 \quad . \quad . \quad . \quad . \quad (6)$$

$$dY = \frac{(ad - bc)y}{V^2} dx + \frac{(ax + c)(bx + d)}{V^2} dy \quad . \quad . \quad . \quad . \quad (7)$$

where, as before, $V = axy + bx + cy + d$. Substituting in (7) the value of dx found from (6), we have

$$dY = \frac{(ax + c)(bx + d)(ay + b)(cy + d) - (ad - bc)^2 xy}{(ay + b)(cy + d)V^2} dy$$

The numerator of this expression may be reduced to the form

$$V(abcxy + abdx + acdy + bcd)$$

whence

$$\frac{dY}{dy} = \frac{abcxy + abdx + acdy + bcd}{(ay + b)(cy + d)V} \quad . \quad . \quad . \quad . \quad (8)$$

This represents the differential coefficient of Y taken with respect to y as independent variable, x being regarded as a function of y determined by (3). The expression is always positive, if x and y are positive.

Now let y vary from 0 to ∞ through the whole range of positive magnitude. Writing (3) in the form

$$\frac{Ax}{Ax + B} = p \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $A = ay + b$, $B = cy + d$, the quantity x must, by reference to the case of $n=1$, have a positive value, since A and B are positive and p fractional. Whence, as y varies from 0 to ∞ , the value of $\frac{dY}{dy}$ is always positive.

Now when $y=0$, $Y=0$, and when $y=\infty$, $Y=1$, as is evident from (5). Therefore, as y increases from 0 to ∞ , Y continuously increases from 0 to 1. In this variation

it must once, and only once, become equal to q . Wherefore the system (3) (4) admits of one, and only one, solution in positive values of x and y .

The reasoning might also be presented in the following form. The condition of Y having a maximum or minimum value is expressed by the equation

$$abcxy + abdx + acdy + bcd = 0 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

It is obvious that this, as all the terms in the first member are positive, can never be satisfied by positive values of x and y . Hence Y has no maximum or minimum, consistently with (3) being satisfied, and thus it never resumes a former value, and is only once, in the course of its variation, equal to q .

In the case of $n=3$, we have

$$V = axyz + byz + czx + dxy + ex + fy + gz + h$$

and the system to be considered is

$$\frac{axyz + czx + dxy + ex}{V} = p \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{axyz + byz + dxy + fy}{V} = q \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\frac{axyz + byz + czx + gz}{V} = r \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Let the first number of the last equation, considered as a variable function of x, y, z be represented by Z , and suppose x, y , and z to vary in subjection to the conditions (11) (12). Just as before, it may be shown that Z increases continuously with z . The condition of Z having a maximum or minimum value, will be expressed by the following equation:

$$\begin{aligned} & (D + H + E + F) (ABC + ACG + ABG + BCG) \\ & + (A + B + C + G) (DHE + DHF + DEF + HEF) \\ & + (AC + BG) (DF + DH + EF + EH) \\ & + (AG + BC) (DF + EH + DE + FH) \\ & + (AB + CG) (DE + DH + FE + FH) \\ & + 4 AGFE + 4 BCDH = 0 \quad . \quad . \quad . \quad . \quad . \quad (14) \end{aligned}$$

Wherein

$$\begin{array}{llll} A = axyz & B = byz & C = czx & D = dxy \\ E = ex & F = fy & G = gz & H = h \end{array}$$

And as this equation has positive values only in its first member, it cannot be satisfied by positive values of x, y, z ; whence, by the same reasoning as before, the system (11), (12), (13) cannot have more than one solution in positive values of x, y, z .

To show that it will have one such solution, let z vary from 0 to α , then Z continuously increases from 0 to 1, and once becomes equal to r . At every stage of its variation we may give to (11) and (12) the form

$$\frac{Axy + Bx}{V} = p$$

$$\frac{Axy + Cy}{V} = q$$

which corresponds with the form of the general system (3) (4) in the case of $n=2$. Whence, for each positive value of z , one positive set of values of x and y will be found. The system (11), (12), (13) admits, therefore, of one solution in positive values of x , y , z , and of only one.

To prove the proposition generally, it ought to be shown that the function exemplified in the first members of (10) and (14), for the cases of $n=2$ and $n=3$ possesses universally the same property of consisting only of positive terms. I have proved that it does for the case of $n=4$, and the analysis was such as to leave no doubt whatever of its general truth.

I will now offer a few remarks on the application of the above proposition.

The system of equations for determining s and t . . , Art. 21, is of the form

$$\frac{V_s}{p} = \frac{V_t}{q} \dots = V \quad (15)$$

V being a function of the same general character as the one discussed in the foregoing proposition, but with this difference, that its coefficients, if we regard it as a complete function, are all equal either to 1 or to 0.

Thus in Art. 18, we have

$$V = stv + s + t + v + 1$$

Here the terms st , tv , and vs , must be considered as present, but with the coefficient 0.

This limitation does not affect the essentially positive character of the determining function exemplified in (10) and (14). Whence the system (15) cannot have more than one solution in positive values of s , t , &c. *This shows that the solution of the system of equations furnished by the general method can never be ambiguous.*

The vanishing of some of the coefficients of V does, however, affect the reasoning by which it has been shown, that for the general form of V discussed in the last proposition, one solution of the algebraic system in positive values will exist. Thus Y in (5) does not vanish with y , if both b and d vanish. And generally this vanishing of coefficients in V entails conditions among the quantities p , q , r . . , in addition to that of their being fractional, in order that the derived algebraic system may admit of a solution in positive values.

Thus if we take, as in (7) Art. 18,

$$V = stv + s + t + v + 1$$

with the derived algebraic system

$$\frac{stv + s}{V} = p \quad \frac{stv + t}{V} = q \quad \frac{stv + v}{V} = r$$

it is evident that if s , t , and v are positive quantities, and if we write

$$\frac{stv}{V} = u, \quad \frac{s}{V} = \lambda, \quad \frac{t}{V} = \mu, \quad \frac{v}{V} = \nu$$

u , λ , μ , and ν must be positive fractions, whence, in addition to the equations

$$u + \lambda = p$$

$$u + \mu = q$$

$$u + \nu = r$$

we shall have the inequations

$$\begin{aligned} u &\geq 0 & \lambda &\geq 0 & \mu &\geq 0 & \nu &\geq 0 \\ u + \lambda + \mu + \nu &\leq 1 \end{aligned}$$

This system is identical with the one obtained in 10, Art. 13, for the determination of the conditions of possible experience in the particular question of Probabilities, in which the above function V presents itself. And a very little attention will show, that if in any case we express as above the relations which must obviously be fulfilled in order that s , t , &c., may be positive quantities, we shall form a system of equations and inequations precisely agreeing with those which we should have to form in order to obtain the conditions of possible experience, if we sought those conditions, not from the data in their original expression, but from the translated data, as employed in Art. 13.

Hence, in order that s , t . . in the system of Art. 21, may be positive, or in the prior system, positive fractions, the problem of which these systems of equations involve the solution must represent a possible experience.

Conversely if that problem represent a possible experience, the quantities s , t . . will admit of being determined in the system of Art. 21, in positive values, or in the prior system, in positive fractional values.

I have not succeeded in obtaining a perfectly rigorous proof of the latter, or converse proposition in its general form, but I have not met with any individual cases in which it was not true. I will here only exemplify it in Problem II., Art. 34.

Here the value of V is

$$V = xyst + xs + yt + xy + x + y + 1$$

and the algebraic system employed in the determination of Prob. xyz is

$$\begin{aligned} \frac{xyst + xs + yt + xy + x}{c} &= \frac{xyst + yt + xy + y}{c'} \\ &= \frac{xyst + xs}{cp} = \frac{xyst + yt}{c'q} = \frac{xyst}{u} \\ &= xyst + xs + yt + xy + x + y + 1 \quad . \quad . \quad . \quad (16) \end{aligned}$$

For the determination of u we hence find the following equation—

$$u(u - \overline{c' + cp - 1})(u - \overline{c + c'q - 1}) - (cp - u)(c'q - u)(u - \overline{cp + c'q - 1}) = 0 \quad (17)$$

the conditions of limitation being

$$\begin{array}{ccccccc} u > 0 & u > c' + cp - 1 & u > c + c'q - 1 & & & & \\ & u < cp & u < c'q & . & . & . & . \end{array} \quad (18)$$

Now, since u is greater then $c + c'q - 1$ it is *a fortiori* greater then $cp + c'q - 1$. Thus within the limits assigned to u , all the factors of each term of (17) will be positive.

If then we give to u the value which belongs to the highest of its inferior limits, the first member of (17) will be reduced to its second term, and will be negative. If we give to u the value which belongs to the lowest of its superior limits, the first member of (17) will be reduced to its first term, and will be positive. Moreover, that member is a quadratic function of u . Hence there is one root, and only one, within the limits specified.

We must now express x , y , s , and t , in terms of u . Their values determined from the system (16) are as follows, viz. :—

$$\begin{aligned} x &= \frac{c(1-p)}{u-(c'+c'q-1)} & y &= \frac{c'(1-q)}{u-(c'+cp-1)} \\ s &= \frac{u-(c+c'q-1)}{c(1-p)} \times \frac{u}{c'q-u} \\ t &= \frac{u-(c'+cp-1)}{c'(1-q)} \times \frac{u}{cp-u} \end{aligned}$$

All these expressions become positive when u is determined in accordance with the conditions (18).

It would seem from the above, as well as from reasonings analogous to those of Proposition I., that when the algebraic system belonging to a problem in the theory of probabilities is placed in the form

$$\frac{V_x}{V} = p, \frac{V_y}{V} = q \dots$$

the limits of variation of the first member of any equation subject to the condition, that the variables shall all be positive, and shall vary in subjection to all the other equations of the system, will not in general be 0 and 1, as in the case contemplated in Prop. I., but will correspond with the limits of value of the second member of the same equation as determined by the conditions of possible experience.

This conclusion I have in various cases independently verified. The analytical theory still, however, demands a more thorough investigation.

APPENDIX B.

A note to Archbishop WHATELY's Logic, Book III., sec. 14, contains a rule for computing the joint force of two probabilities in favour of a conclusion which, as actually applied, is at variance with the preceding results. For this reason,

and also because the validity of the rule in question has been made the subject of recent controversy, I design to offer a few remarks upon the subject here. The rule is contained in the following extract. “As, in the case of two probable premises, the conclusion is not established except on the supposition of their being *both* true, so, in the case of two (and the like holds good with any number) distinct and independent indications of the truth of some proposition, unless *both* of them *fail*, the proposition must be true: we therefore multiply together the fractions indicating the probability of *failure* of each,—the chances *against* it;—and the result being the total chances against the establishment of the conclusion by these arguments, this fraction being deducted from unity, the remainder gives the probability *for* it. *E. g.*, A certain book is conjectured to be by such and such an author, partly, *1st*, from its resemblance in style to his known works, partly (*2dly*), from its being attributed to him by some one likely to be pretty well informed: let the probability of the Conclusion, as deduced from one of these arguments by itself, be supposed $\frac{2}{5}$, and, in the other case $\frac{3}{7}$; then the *opposite* probabilities will be, respectively, $\frac{3}{5}$ and $\frac{4}{7}$; which multiplied together give $\frac{12}{35}$, as the probability *against* the Conclusion; *i. e.*, the chance that the work may *not* be his, notwithstanding those reasons for believing that it is: and consequently the probability in *favour* of that Conclusion will be $\frac{23}{35}$, or nearly $\frac{2}{3}$.”

A confusion may here be noted between the probability that a conclusion is proved, and the probability in favour of a conclusion furnished by evidence which does not prove it. In the proof and statement of his rule, Archbishop WHATELY adopts the former view of the nature of the probabilities concerned in the data. In the exemplification of it, he adopts the latter. He thus applies the rule to a case for which it was not intended, and to which it is in fact inapplicable.

The rule is given, and the conditions of its just application are assigned in Professor DE MORGAN'S *Formal Logic*, p. 201. Its origin may be thus explained. Let there be two independent causes, A and B, either of which, when present, necessarily produces an effect E. Let a be the probability that A is present, b the probability that B is present; then $1-a$ is the probability that A is absent, $1-b$ the probability that B is absent, $(1-a)(1-b)$ the probability that they are both absent; finally, $1-(1-a)(1-b)$ the probability that they are not both absent. This, then, is the probability that one at least of the causes is present; and therefore it is the probability that the event E, so far as it is dependent upon these causes, will occur. In its special application to arguments viewed as causes of belief or expectation, it would lead to the following theorem. If there are two independent arguments in favour of a conclusion which the premises of either, if granted, are sufficient to establish, the doubt only existing as to the truth of the premises, and if the probability that the premises of the first argument are true is a , the probability that the premises of the second argument are true b , then the probability that the conclusion is established is $1-(1-a)(1-b)$. Interpreted, this formula gives

Archbishop WHATELY's rule, but the conditions of its valid application are evidently not fulfilled in the example which he has given. To satisfy these conditions, this problem ought to be changed into the following: "There exists a certain quality of style, the possession of which would *prove* the work to be by the author supposed. The probability that the work possesses that quality is $\frac{2}{5}$. There is a person so well informed that his attributing the work to the supposed author would be *conclusive*. The probability that he does attribute it to the author in question is $\frac{3}{7}$. Required the probability, on these grounds, that the supposed author is the real one." But this is evidently not the sense in which the problem was meant to be understood. Thus to take one point, it is not the quality of the style that is a matter of probability, but the mode in which a known and observed quality affects the question of authorship.

Taking the problem in its intended meaning, each of the fractions $\frac{2}{5}$, $\frac{3}{7}$, measuring, not the probability of the truth of certain premises, but the probability drawn from these premises, as conditions, in favour of a certain supposition (I use this word in preference to conclusion), we are no longer permitted to apply the formula above determined. And we are not permitted to do so, because the probabilities with which we are concerned are conditional, and their possession of this character greatly increases the difficulty of the problem. Its rigorous formal solution is given in Art. 34, and shows that the probability sought is, generally speaking, indefinite,—a result which agrees with the conclusions of Bishop TERROT, by whom the error to which attention has been directed was first pointed out, *Transactions of the Royal Society of Edinburgh*, vol. xxi., p. 369.

I trust that I have not in any way misrepresented Archbishop WHATELY's reasoning; and I am the more encouraged to believe that I have not, as a defence of it which appeared in the *United Church Journal* expressly proceeds upon the assumption that the probabilities with which we are concerned are probabilities that the authorship is *proved*. To this view Bishop TERROT justly demurred. Nor was its inconsistency materially diminished by assigning to proof a meaning less absolute than belongs to demonstration. For whatever degree of cogency,—of power to produce conviction,—we suppose to characterize proof, the thing itself belongs to consciousness, and the question whether given evidence is sufficient to convey proof to our minds or not, is a matter of knowledge, not of probability.