Deep Sea Ship-Waves. (*Continued from* Proc. R.S.E., January 23, 1905.) By Lord Kelvin.

(Read July 17, 1905.)

§ 65. Referring to § 63, we must, for the present, as time presses, leave detailed interpretation of the curves of fig. 17: merely remarking that, according to § 44, if $\delta = 0$, (which means that J is an integer), the disturbance, d, is infinitely great; of which the dynamical meaning is clear in (70) of § 39.

§ 66. Let us now find the depression of the water at distance x from the origin, when the disturbance is due to a single forcive, expressed by the formula *

$$\Pi(x) = g\frac{kb^2}{x^2 + b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95),$$

travelling uniformly with any velocity v. If this forcive were applied steadily to the surface of water at rest it would produce a steady depression $\frac{1}{g}\Pi(x)$, as we are taking the density of the water, unity. Thus the forcive $\Pi(x)$ would shape the water to an infinitely long trough, of cross-section shown in fig. 25, representing $z = kb^2/(x^2 + b^2)$ on the scale of $k = 10$ cm. and $b = 1$ cm.

Taking $\frac{1}{g}\int_0^x dx$ of (95) we find $\tan^{-1}(x/b).bk$. Hence the area of fig. 25 is $2\tan^{-1}8.bk$, or $\frac{166}{180}.\pi bk$, and the total area of the diagram extended to infinity on each side is πbk. Hence the area of fig. 25 is $\frac{166}{180}$, or ·92, of the total area. This total area, πbk, I call, for brevity, the forcive area; and πb, I call the mean breadth of the forcive area. The breadth of the forcive where $z = {\cdot}8k$ (as shown by the dotted line BB in the diagram) is b.

§ 67. Now let the forcive be suddenly set in motion, and kept moving uniformly with any velocity v in the rightward direction of our diagrams. This will produce a great commotion, settling

* What is denoted by x in this and following expressions, is the $(x - vt)$ of §§ 36 . . . . 40; the origin of co-ordinates being now fixed relatively to the travelling forcive.

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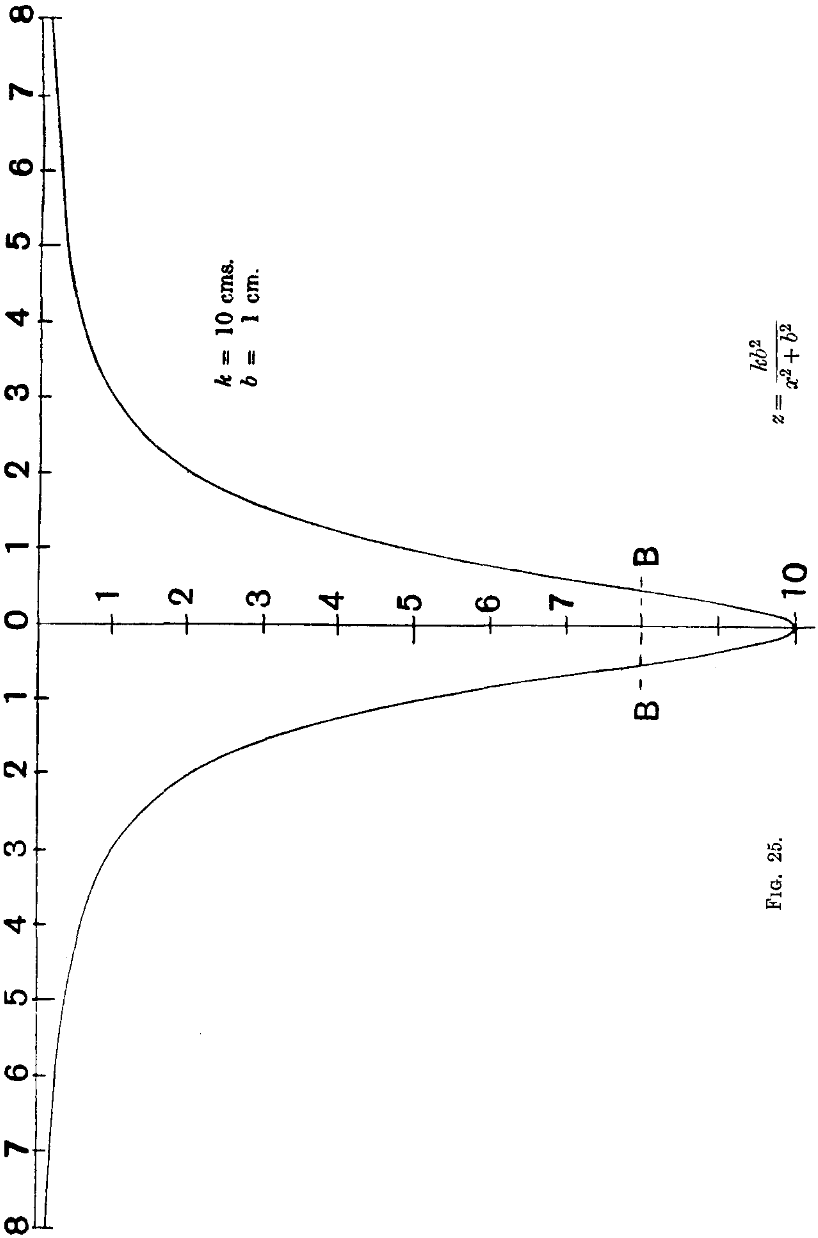

k = 10 cms.
b = 1 cm.
$z = \frac{kb^2}{x^2 + b^2}$
B
B
0 1 2 3 4 5 6 7 8
1 2 3 4 5 6 7 8
1 2 3 4 5 6 7 10

FIG. 25.

ultimately into more and more nearly steady motion through greater and greater distances from O. The investigation of §§ 1–10 above (Feb. 1904), and particularly the results described in §§ 5, 6, and illustrated in figs. 2, 3, show that in our present case the commotion, however violent, even if including *splashes*,* divides itself into two parts which travel away in the two directions from O, ultimately at wave-speed increasing in proportion to square root of distance (according to the law of falling bodies); and leaving in their rears, through ever broadening spaces, what would be more and more nearly absolute quiescence if the forcive were suddenly to cease after having acted for any time, long or short.

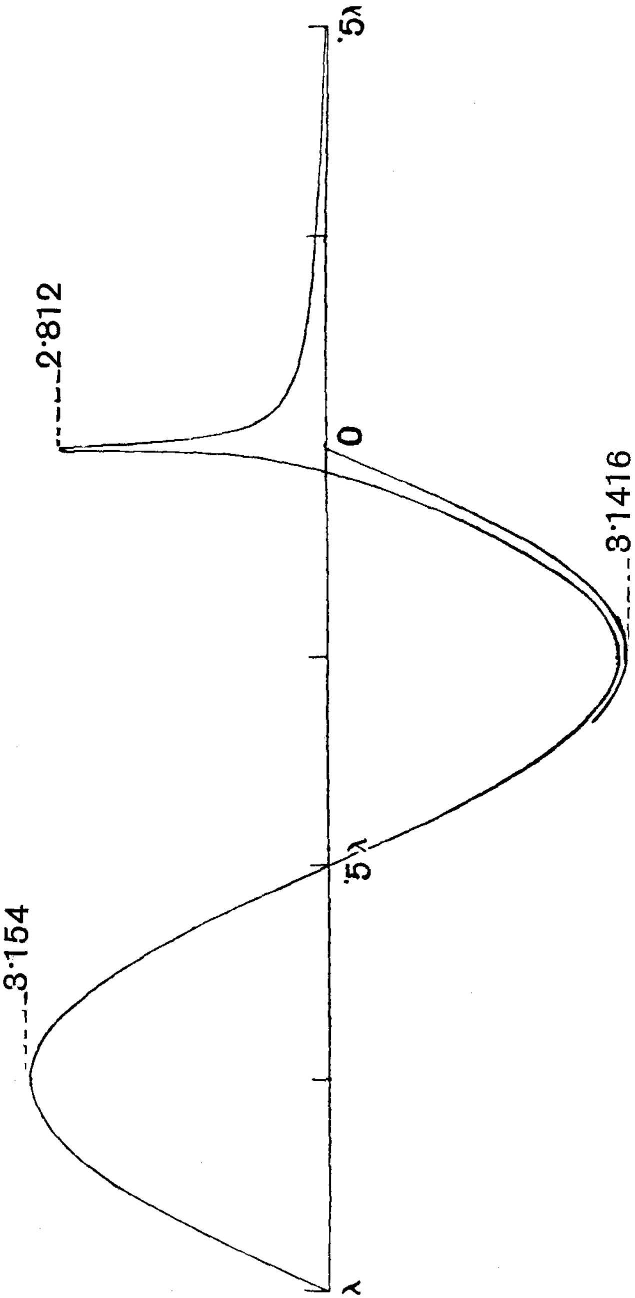
§ 68. But if the forcive continues acting, and travelling rightwards with constant speed, v, according to § 67, the travelling away of the two parts of the initial commotion in the two directions from O (itself merely a point of reference, moving uniformly rightwards), leaves the water, as shown by fig. 26, in a state of more and more nearly quite steady motion through an ever broadening space on the rear side of O, and through a small space in advance of O; provided certain moderating conditions are fulfilled in respect to k, b, v.

§ 69. To illustrate and prove § 68; first suppose v infinitely small. The water will be infinitely little disturbed from the static forcive-curve shown in fig. 25, and described in § 66. Small enough velocities will make very small disturbance with any finite value of k/b.

§ 70. But now go to the other extreme and let v be very great. It is clear, on dynamical principles without calculation, that v may be *great enough to make but very little disturbance* of the water-surface, however steep be the static forcive curve. A "skipping stone" and a ricochetting cannon shot, illustrate the application of the same dynamical principle in three-dimensional hydrokinetics. By mathematical calculation (§ 79 below) we shall see that, when v is great enough, we have

$$h \doteq 4\pi \frac{A}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97),$$

* However sudden and great the commotion is, the motion of the liquid is, and continues to be, irrotational throughout.



$$d(x) - \tfrac{1}{2}h \sin\frac{2\pi x}{\lambda}.$$

Fig. 26.

where h denotes the height of crests above mean water-level in the train of sinusoidal free waves left in the rear of the travelling forcive; A denotes the area of the forcive curve (fig. 25); being given in § 66 by the equation

$$A = \pi bk \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98):$$

and λ, given [§ 39, (71)] by

$$\lambda = 2\pi v^2/g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99),$$

denotes the wave-length of free waves travelling with velocity v.

§ 71. A very important theorem in respect to ship-waves is expressed by (97). Without calculation we see that, *if* λ *is very great* in comparison with πb, (the "mean breadth" of the forcive curve according to § 66), h must be simply proportional to A, for different forcives travelling at the same speed. This we see because, for the same value of b, h/k is the same, and because superposition of different forcives within any breadth small in comparison with λ, gives for h the sum of the values which they would give separately. Farther without calculation, we can see, by imagining altered the scale of our diagrams, that $h\lambda/A$ must be constant. But without calculation I do not see how we could find the factor 4π of (97), as in § 79 below.

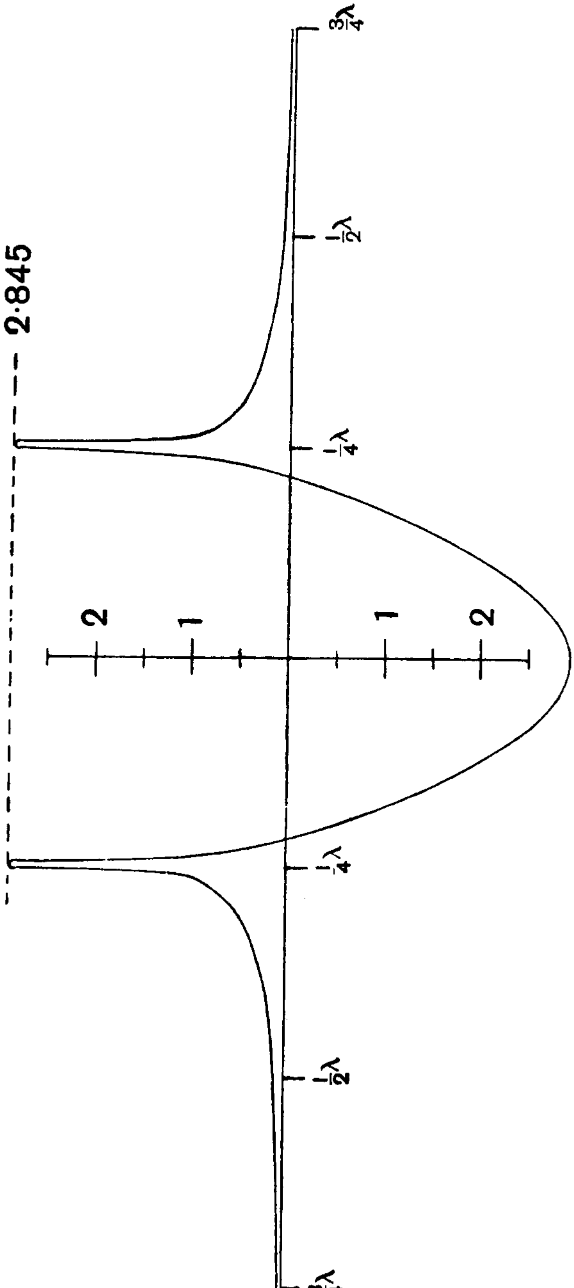
§ 72. The effect of the condition prescribed in § 71 is illustrated and explained by considering cases in which it is not fulfilled. For example, let two forcives be superposed with their middles at distance $\frac{1}{2}\lambda$; they will give $h=0$, that is to say no train of waves. The displaced water surface for this case is represented in fig. 27. Or let their distance be $\frac{1}{3}\lambda$ or $\frac{2}{3}\lambda$; the two will give the same value of h as that given by one only. Or let the two be at distance λ; they will make h twice as great as one forcive makes it.

§ 73. In figs. 26, 27, 29, 30, representing results of the calculations of §§ 78, 79 below, the abscissas are all marked according to wave-length. The scale of ordinates corresponds, in each of figs. 26, 27, 29, to $k = 243{\cdot}89$, and $\pi b = 1{\cdot}0251.10^{-3}.\lambda$. This makes by (98) and (97) $A = \frac{1}{4}\lambda$, and $h = \pi$. Fig. 30 represents the curve of fig. 29 at the maximum, in the neighbourhood, of O, on a greatly magnified scale: about 1720 times for the abscissas, and 39 times for the ordinates.

§ 74. Fig. 26 shows, on the right-hand side, the water slightly heaped up in front of the travelling forcive, which is a distribution of *downward* pressure whose middle is at O. On the left side of O, we see the water surface not differing perceptibly from a curve of sines beyond half a wave-length rearwards from O. A small portion of a wave-length of true curve of sines in the diagram shows how little the water's surface differs from the curve of sines at even so small a distance from O as a quarter wave-length.

It must be remembered that in reality the water surface is everywhere very nearly level; and in considering, as we shall have to do later, the work done by the forcive, we must interpret properly the enormous exaggeration of slopes shown in the diagrams. It is interesting to remark that the static *depression*, k, which the forcive if at rest would produce, is about 87 times the *elevation* actually produced above O by the forcive, travelling at the speed at which free waves, of the wave-length shown in the diagrams, travel. It is interesting also to remark that the limitation to very small slopes is not binding on the static forcive curve. Thus for example, a distribution of static pressure, everywhere perpendicular to the free surface, producing static depression exactly agreeing with fig. 25, would, if caused to travel at a speed for which the free-wave-length is very large in comparison with b, produce a disturbance, represented by fig. 26 with waves of moderate slopes: and, as said in § 69 above, would produce no disturbance at all if the speed of travelling were infinitely great.

§ 75. Fig. 27 is interesting as showing the waveless disturbance produced by two equal and similar forcives with their middles at distance equal to half the wave-length. This disturbance is essentially symmetrical in front and rear of the middle between the two forcives. By dynamical considerations of the equilibrium of downward pressures, we see that the area of fig. 27 (portion above line of abscissas being reckoned as negative) must be exactly equal to 2A, the sum of the areas of the two forcives, representing their integral amount of downward pressure. This area, being $2\pi bk$, with the numerical data of § 73, is numerically $\frac{1}{2}\lambda$; that is to say a rectangle whose length is $\frac{1}{2}\lambda$, and breadth the unit of our vertical scale. Approximate mensuration, with a very rough estimate of the area beyond the range of the diagram, continued to infinity on the two sides, verifies this conclusion.



$$d\left(x-\frac{\lambda}{4}\right)+d\left(x+\frac{\lambda}{4}\right).$$

FIG. 27.

§ 76. Fig. 28 is designed on the same plan as fig. 27 but with eleven half-wave-lengths as the distance between the two forcives instead of one half-wave-length. Like fig. 27, it is symmetrical on the two sides of the middle of the diagram; but, instead of being waveless, as is fig. 27, it shows four and a half waves, all very approximately sinusoidal, with two depressional halves of waves at their two ends, and elevations coming asymptotically to zero beyond the two ends of the diagram. The curve represented by fig. 26 is very accurately the right-hand extreme of fig. 28: and the same figure, turned right to left, is the left-hand extreme of fig. 28. If we commence with the water wholly at rest, and start the forcives at the proper speed, with force gradually (or somewhat suddenly) increasing up to the prescribed amount, the motion produced will be that represented by fig. 28, with, superimposed upon it, a disturbance quickly disappearing in ever lengthening waves of diminishing amplitude, travelling away in both directions from our field. If now, with the regular régime represented by fig. 28, we suddenly cease to apply the forcives, we have left a free procession of four and a half very approximately sinusoidal waves, between a front and a rear deviating from sinusoidality as shown in the diagram. From the instant of being left free, the front of this procession and its rear will rapidly become modified; while for three periods the central part of the procession will have travelled three wave-lengths, with very little deviation from sinusoidality. But, after four or five periods from the instant of being left free, the whole procession will have got into confusion. After twenty or thirty or forty periods, the water will be sensibly quiescent, not only through the whole space where the procession was, but through the whole space over which it would have travelled if its front and rear had been kept guarded by the continued action of the two travelling forcives. At no time after the cessation of the forcives can we reasonably or conveniently assign a "group velocity" to the group or procession of waves with which we are concerned. A prevalent idea is, I believe, that such a group of deep sea waves could be regarded as travelling with half the "wave-velocity" of waves of the length given in the original group. In § 30 above, reasons are given for accepting the theory of "group velocity" only in the case of mutually supporting

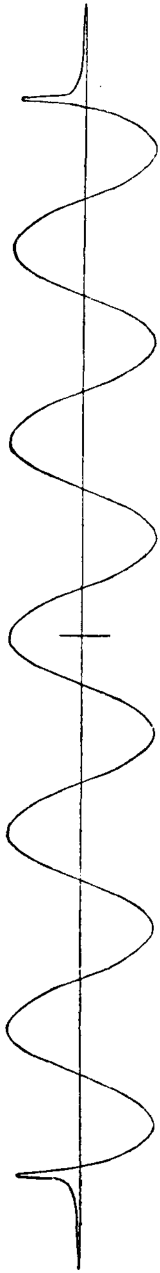


FIG. 28.

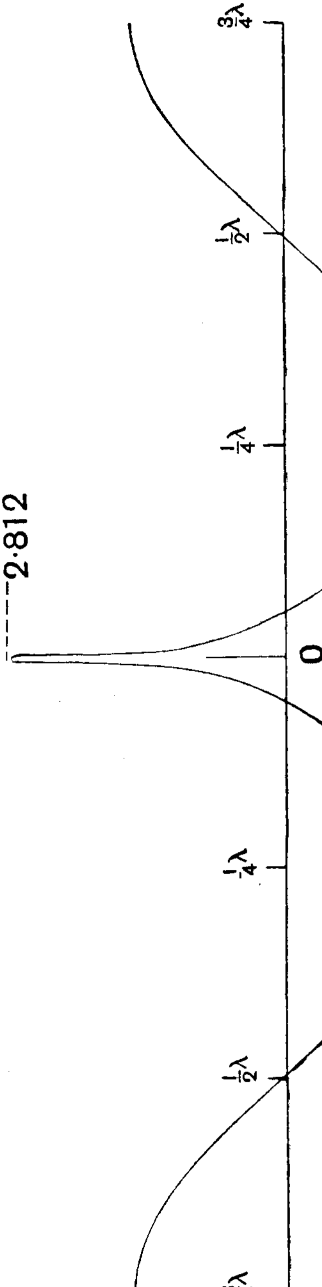
$$d\left(x-\frac{11\lambda}{4}\right)+d\left(x+\frac{11\lambda}{4}\right).$$

Scale of ordinates reduced to one quarter.

groups, given by Stokes in his Smith's Prize examination paper, published in the *Cambridge University Calendar* for 1876: and for rejecting it for the case of any single group of waves. In reality the front of a group, left to itself, travels with accelerated velocity exceeding the velocity of periodic waves of the given wave-length, instead of with half that velocity.

§ 77. Fig. 29 shows *the steady motion*, symmetrical in front and rear of a single travelling forcive, which is a solution of our problem; but it is an unstable solution (as probably are the solutions of the problem of § 45 above, shown in figs. 13, 14, 15). If any large finite portion of the water is given in motion according to fig. 29, say, for example, 50 wave-lengths preceding O (the forcive) and 50 wave-lengths following O, the front of the whole procession, to the right of O, will become dissipated into non-periodic waves travelling rightwards and leftwards with increasing wave-lengths and increasing velocities; and the approximately steady periodic portion of it will shrink backwards relatively to the forcive. Thus before the forcive has travelled fifty wave-lengths, the periodic waves in front of it are all gone: but there is still irregular disturbance both before and behind it. After the forcive has travelled a hundred wave-lengths, the whole motion in advance of it, and the motion for perhaps 30 wave-lengths or more in its rear, will have settled to nearly the condition represented by fig. 26, in which there is a small regular elevation in advance of the forcive, and a regular train of approximately sinusoidal waves in its rear; these waves being of double the wave-height given originally. This motion, as said above in § 68, will go on, leaving behind the forcive a train of steady periodic waves, increasing in number; and behind these an irregular train of waves, shorter and shorter, and less and less high the farther rearward we look for them (see R in fig. 10 of §§ 26, 27 above). It is an interesting, but not at all an easy problem, to investigate the extreme rear (with practically motionless water behind it) of the train of waves in the wake of a forcive travelling uniformly for ever. I hope to return to this subject when we come to consider the work done by the travelling forcive.

§ 78. Pass now to the investigation of the formulas by the calculation of which figs. 26, 27, 28, 29, 30 have been drawn, and



d(*x*).

FIG. 29.

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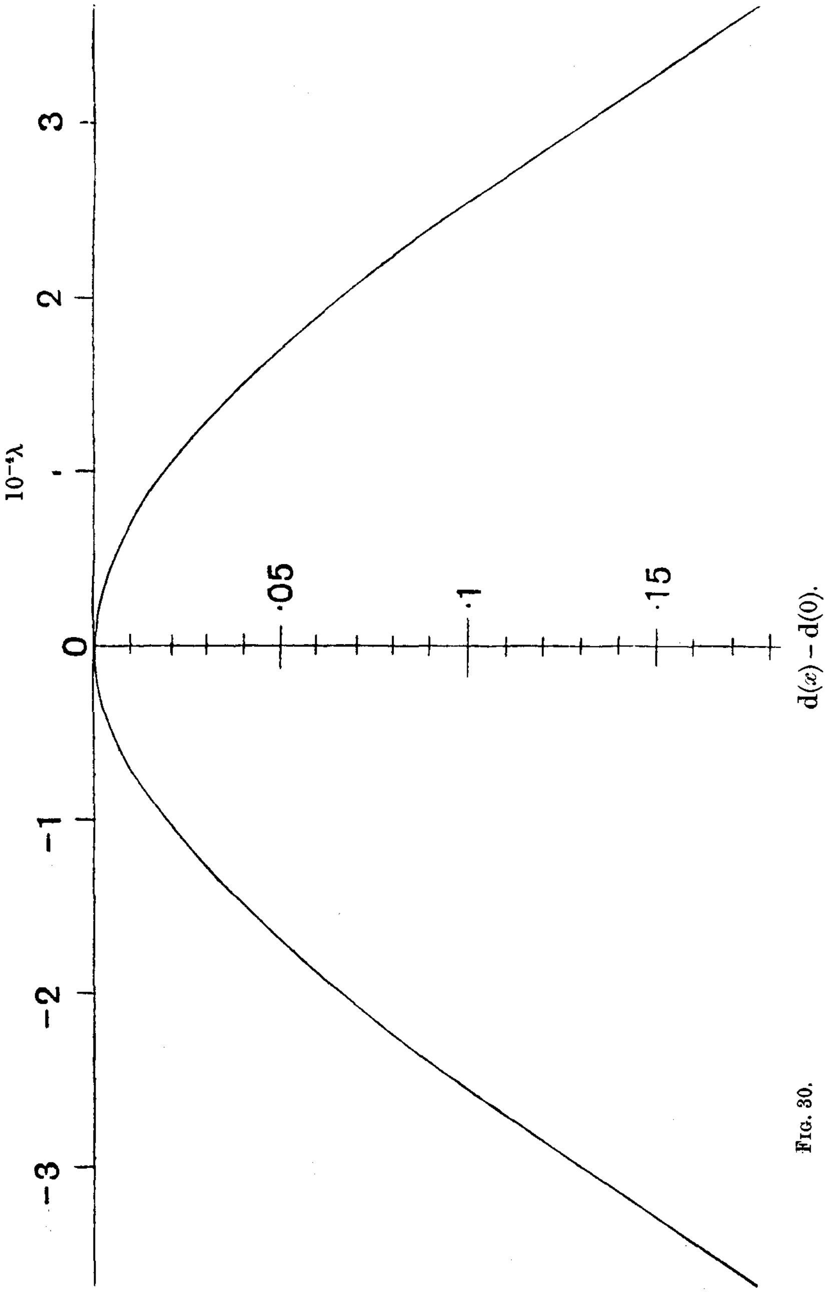

Fig. 30.

the theorem of (97) proved. Go back to the problem of § 41 above: but instead of taking $e = \cdot 9$, as in §§ 46–61, take $e = 1 - 10^{-4}$; and $c = 1/(2j+1)$. By (86) and (87) of § 45 we have the following solution

$$\mathrm{d} = \mathscr{I} + \mathscr{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

where

$$\mathscr{I} = e^{j+\frac{1}{2}}\left\{ \tfrac{1}{2}\sin(j+\tfrac{1}{2})\theta\tan^{-1}\frac{2\sqrt{e}\sin\frac{1}{2}\theta}{1-e} - \tfrac{1}{4}\cos(j+\tfrac{1}{2})\theta\log\frac{1+2\sqrt{e}\cos\frac{1}{2}\theta+e}{1-2\sqrt{e}\cos\frac{1}{2}\theta+e}\right\} \quad (101).$$

and

$$\mathscr{F} = \tfrac{1}{2}\cdot\frac{1}{2j+1} + \frac{e\cos\theta}{2j-1} + \frac{e^2\cos 2\theta}{2j-3} + \ldots\ldots + e^j\cos j\theta \qquad (102).$$

Fig. 29 has been calculated by putting $\theta = \frac{x}{\lambda}\cdot\frac{360^\circ}{j+\frac{1}{2}}$, and taking $j = 20$. The explanation is that, as we shall see by (78) of § 43 above, (100), (101), (102), express the water disturbance due to an infinite row of forcives at consecutive distances each equal to $(20\frac{1}{2})\,\lambda$; the expression for each forcive being

$$\frac{cba/2\pi}{b^2 + (x - na)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103),$$

where n is zero or any positive or negative integer; and by (79) we have

$$b = \frac{20\cdot 5.10^{-4}.\lambda}{2\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104).$$

Thus we see that the pressure at O due to each of the forcives next to O, on the two sides, is $1/\{1 + (2\pi.10^4)^2\}$ of the pressure due to the forcive whose centre is O. Thus we see that the pressures due to all the forcives, except the last mentioned, may be neglected through several wave-lengths on each side of O: and we conclude that (100), (101), (102) express, to a very high degree of approximation, the disturbance produced in the water by the single travelling forcive whose centre is at O.

§ 79. To prove (97) take $\theta = 180^\circ$ in (100), (101), (102); we thus find

$$(-1)^j\mathrm{d}(180^\circ) = e^j\left\{\frac{1}{2}\sqrt{e}\tan^{-1}\frac{2\sqrt{e}}{1-e} + 1 - \frac{e^{-1}}{3} + \frac{e^{-2}}{5}\ldots\ldots + \tfrac{1}{2}(-e)^{-j}\cdot\frac{1}{2j+1}\right\} \quad (105).$$

Instead now of taking $e = 1 - 10^{-4}$, as we took in our calculations for d(0), let us now take $e=1$. This reduces (105) to

$$(-1)^j \mathrm{d}(180°) = \frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5} \cdot \cdot \cdot \cdot + \frac{(-1)^{j-1}}{2j-1} + \frac{1}{2}\frac{(-1)^j}{2j+1} \quad (106).$$

Lastly take j an infinitely great odd or even integer, and we find

$$\mathrm{d}(180°) = (-1)^j \cdot \frac{\pi}{2} \quad . \quad . \quad . \quad . \quad . \quad (107).$$

Now fig. 26 is, as we have seen, found by superimposing on the motion represented by fig. 29 an infinite train of periodic waves represented by $-\frac{1}{2}\mathrm{h} \,.\, \sin\frac{2\pi x}{\lambda}$, and therefore $h=\pi$, which proves (97).

§ 80. To pass now from the two-dimensional problem of canal-ship-waves to the three-dimensional problem of sea-ship-waves, we shall use a synthetic method given by Rayleigh at the end of his paper on "The form of standing waves on the surface of running water," communicated to the London Mathematical Society in December 1883.* In an infinite plane expanse of water, consider two or more forcives, such as that represented by (95) of § 66, with their horizontal medial generating lines in different directions through one point O, travelling with uniform velocity, v, in any direction. The superposition of these forcives, and of the disturbances of the water which they produce, each calculated by an application of (100), (101), (102), gives us the solution of a three-dimensional wave problem; which becomes the ship-wave-problem if we make the constituents infinitely small and infinitely numerous. Rayleigh took each constituent forcive as confined to an infinitely narrow space, and combated the consequent troublesome infinity by introducing a resistance to be annulled in interpretation of results for points not infinitely near to O. I escape from the trouble in the two-dimensional system of waves, by taking (95) to express the distribution of pressure in the forcive, and making b as small as we please. Thus, as indicated in §§ 79, 73, 76, by taking $b=10\frac{1}{4}\lambda/(10^4.\pi)$ we calculated a finite value for d(0). But for values of x, considerably greater than half a wave-length, we were able to simplify the calculations by taking $b=0$.

* *Proc. L.M.S.*, 1883: republished in *Rayleigh's Scientific Papers*, vol. ii. art. 109.

§ 81. For the three-dimensional system let, in fig. 31, ψ be the inclination to O X of the rearward wave-normal of one of the constituent systems of waves. This is also the inclination to O Y of the medial line of the travelling forcive to which that set of waves is due. Take now for the forcive obtained by the super-

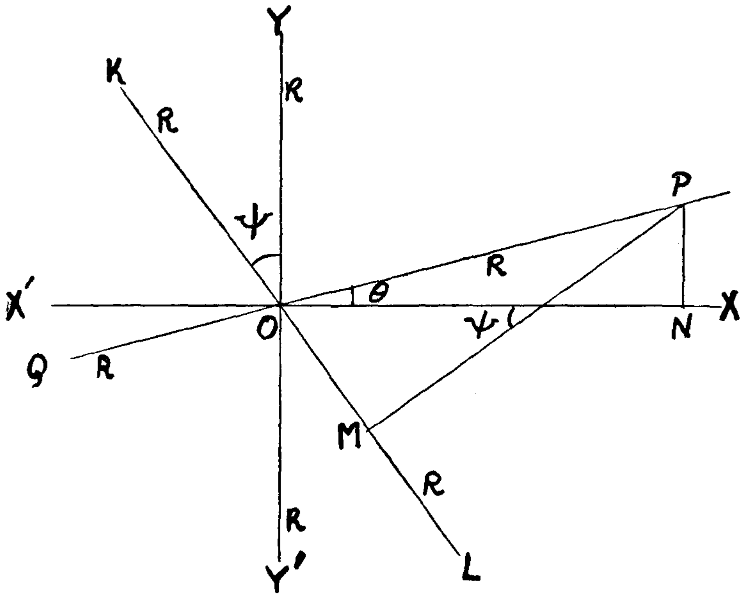

FIG. 31.

position of an infinite number of constituents, as described in § 80,

$$\frac{1}{g}\Pi(x, y) = \int_0^\pi d\psi \frac{b^2 k}{[(x\cos\psi + y\sin\psi)^2 + b^2]} \quad . \quad . \ (108),$$

where k may be a function of ψ, and b is the same for all values of ψ.

For the case of a circular forcive system we must take k constant; and we find

$$\frac{1}{g}\ \Pi(r) = \frac{\pi bk}{\sqrt{(r^2 + b^2)}} \text{ where } r^2 = x^2 + y^2 \quad . \quad . \ (109).$$

§ 82. Let now the forcive, whether circular or not, be kept travelling in the direction of x negative * with velocity v: and

* This is opposite to the direction of the motion of the forcive in fig. 26.

let λ denote the corresponding free wave-length given by the formula $2\pi v^2/g$. This is the wave-length of the constituent train of waves corresponding to $\psi = 0$. For the ψ-constituent, the component velocity perpendicular to the front is $v \cos \psi$, and the wave-length is $\lambda \cos^2\psi$. Looking now to fig. 26, with $\lambda \cos^2\psi$ instead of λ; and to fig. 31; and to equations (97), (98); we see that the portion of the depression at (x, y) due to the constituent of forcive shown under the integral in (108) is

$$\frac{4\pi^2 bk.d\psi}{\lambda \cos^2\psi} \sin \frac{2\pi(x \cos\psi + y \sin\psi)}{\lambda \cos^2\psi} \quad . \quad . \quad . \quad . \quad (110),$$

provided $x \cos\psi + y \sin\psi$ is considerably greater than $\frac{1}{2}\lambda \cos^2\psi$. Hence for the depression at (x, y) due to the whole travelling forcive, we have

$$d(x, y) = 4\pi^2 b \int_{-\left(\frac{\pi}{2}-\theta\right)}^{\frac{\pi}{2}} \frac{kd\psi}{\lambda \cos^2\psi} \sin \frac{2\pi(x \cos\psi + y \sin\psi)}{\lambda \cos^2\psi} \quad (111).$$

§ 83. The reason for choosing the limits $-\left(\frac{\pi}{2} - \theta\right)$ to $\frac{\pi}{2}$ is that each constituent forcive gives a train of sinusoidal waves in its rear, and no perceptible disturbance in its front at distances from it exceeding half a wave-length. Look now to fig. 31, and consider the infinite number of medial lines of the forcives included in the integrals (108), (111); all as lines passing through O. Four examples, QP, Y′Y, LK, XX′ of these lines are shown in the diagram: corresponding respectively to $\psi = -\left(\frac{\pi}{2} - \theta\right)$, $\psi = 0$, $\psi =$ any positive acute angle, $\psi = \frac{\pi}{2}$. On each of the first three of these lines RR indicates the rear. The fourth, XX′, is in the direction of the motion, and has neither front nor rear. The integral (111) must include all, and only all, the medial lines which have rears towards P. Hence QP is one limit of ψ in (111) because it passes through P; XX′ is the other limit because it has neither front nor rear. Thus all the lines included in the integral, lie in the obtuse angle POX′. Thus the integral (111) expresses the depression at P(x, y) due to the joint action of all the constituent forcives, because none except those whose medial lines lie in the angle POX′, contribute anything to the disturbance of the water at P.

§ 84. For interpreting and approximately evaluating the definite integral, we may conveniently put

$$r = \sqrt{x^2+y^2}, \quad \text{and} \quad u = \frac{\cos(\psi-\theta)}{\cos^2\psi} \quad . \quad . \quad . \quad (112),$$

and write (111) as follows:

$$\mathrm{d}(x, y) = 4\pi^2 b \int_{-\left(\frac{\pi}{2}-\theta\right)}^{\frac{\pi}{2}} \frac{kd\psi}{\lambda\cos^2\psi} \sin\frac{2\pi ru}{\lambda} \quad . \quad . \quad (113).$$

Now if we suppose r/λ very great, there will be exceedingly rapid transitions between equal positive and negative values of $\sin(2\pi ru/\lambda)$, *which will cause cancelling of all portions of the integral except those, if any there are, for which* $\mathrm{d}u/\mathrm{d}\psi$ *vanishes.* We shall see presently that *there are* two such values, ψ_1, ψ_2, both real if $\tan\theta < \sqrt{\frac{1}{8}}$; u being a maximum (u_1) for one of them, and a minimum (u_2) for the other; and that, when θ has any value between $\tan^{-1}\sqrt{\frac{1}{8}}$ and $2\pi - \tan^{-1}\sqrt{\frac{1}{8}}$, the values of ψ_1, ψ_2 are both imaginary. Consideration of this last-mentioned case shows that, in the whole area of sea in advance of two lines through the centre of the travelling forcive inclined at equal angles of $\tan^{-1}\sqrt{\frac{1}{8}}$, (or 19° 28′) on each side of the mid-wake, there is no perceptible disturbance at distances of much more than a half wave-length from the centre of the forcive. The main disturbance by ship-waves, therefore, lies in the rearward angular space between these two lines. It is illustrated by fig. 32, as we now proceed to prove by the proper interpretation of (113). Expanding the argument of the sin in (113) by Taylor's theorem for values of ψ differing from ψ_1 by small fractions of a radian, we find

$$\frac{2\pi ru}{\lambda} \doteq \frac{2\pi r}{\lambda}\left[u_1 + \tfrac{1}{2}\left(\frac{d^2u}{d\psi^2}\right)_1 (\psi-\psi_1)^2\right] = a_1 - q_1^2 \quad . \quad (114)$$

where

$$a_1 = \frac{2\pi ru_1}{\lambda}, \quad \text{and} \quad q_1 = (\psi-\psi_1)\sqrt{\frac{\pi r}{\lambda}\left(-\frac{d^2u}{d\psi^2}\right)_1} \quad . \quad (115).$$

From the second of (115) we find $d\psi = dq_1/(\beta_1\sqrt{\pi})$, where

$$\beta_1 = \sqrt{\frac{r}{\lambda}\left(-\frac{d^2u}{d\psi^2}\right)_1} \quad . \quad . \quad . \quad . \quad . \quad (116).$$

Dealing similarly in respect to ψ_2 and values of ψ differing but

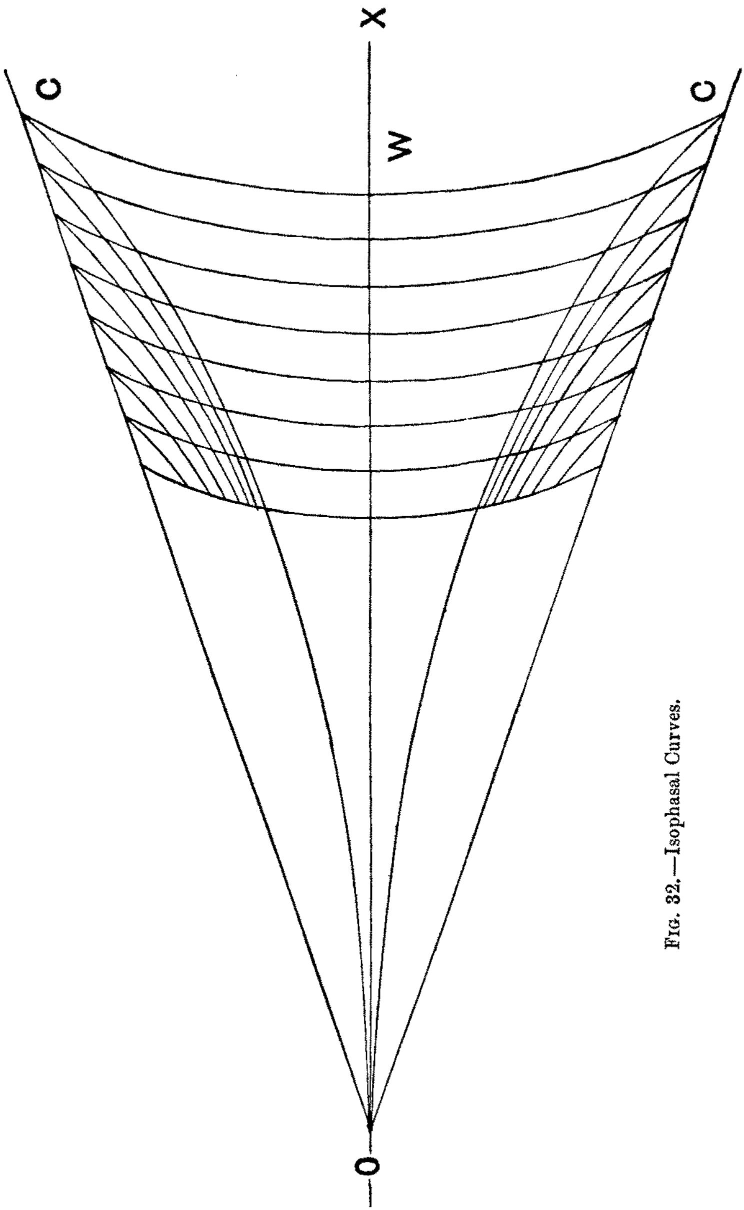

Fig. 32.—Isophasal Curves.

little from it, we take $+q_2^2$ instead of the $-q_1^2$ of (114), and $(d^2u/d\psi^2)_2$ instead of the $-(d^2u/d\psi^2)_1$ of (115); because u_1 is the maximum and u_2 the minimum. Calling k_1, k_2 the values of k corresponding to ψ_1, ψ_2, and using these expressions properly in (113), we find, for the depression of the water at (x, y),

$$\mathrm{d}(x, y) = \frac{4b\pi^{3/2}}{\lambda}\left[\frac{k_1}{\beta_1 \cos^2\psi_1}\int_{-\infty}^{\infty} dq_1 \sin(a_1 - q_1^2) + \frac{k_2}{\beta_2 \cos^2\psi_2}\int_{-\infty}^{\infty} dq_2 \sin(a_2 + q_2^2)\right]. \quad (117).$$

The limits ∞, $-\infty$ are assigned to the integrations relatively to q_1 and q_2 because the greatness of r/λ in (115) and corresponding formula relative to ψ_2, makes q_1 and q_2 each very great, (positive or negative,) for moderate properly small positive or negative values of $\psi - \psi_1$ and $\psi - \psi_2$. Now as discovered by Euler or Laplace (see *Gregory's Examples*, p. 479), we have

$\int_{-\infty}^{\infty} dq \sin q^2 = \int_{-\infty}^{\infty} dq \cos q^2 = \sqrt{\pi/2}$, and using these in (117) we find

$$\mathrm{d}(x, y) = \frac{2\sqrt{2}\pi^2 b}{\lambda}\left[\frac{k_1(\sin a_1 - \cos a_1)}{\beta_1 \cos^2\psi_1} + \frac{k_2(\sin a_2 + \cos a_2)}{\beta_2 \cos^2\psi_2}\right] \quad (118).$$

Substituting for a_1, a_2 values by (115) we find

$$\mathrm{d}(x, y) = \frac{4\pi^2 b}{\lambda}\left[\frac{k_1}{\beta_1 \cos^2\psi_1}\sin\frac{2\pi}{\lambda}\left(ru_1 - \frac{\lambda}{8}\right) + \frac{k_2}{\beta_2 \cos^2\psi_2}\sin\frac{2\pi}{\lambda}\left(ru_2 + \frac{\lambda}{8}\right)\right] \quad (118)'.$$

§ 85. To determine the quantities denoted by β_1, β_2 in (116) . . . . (118)′, we write (112) as follows:—

$$ru = (x + yt)\sqrt{1 + t^2}, \text{ where } t = \tan\psi \quad . \quad . \quad . \quad (119).$$

Hence, by differentiation on the supposition of x, y, r constant, we find

$$r\frac{du}{d\psi} = \left[xt + y(1 + 2t^2)\right]\sqrt{1 + t^2} \quad . \quad . \quad . \quad . \quad (120).$$

$$r\frac{d^2u}{d\psi^2}\left[x(1 + 2t^2) + yt(5 + 6t^2)\right]\sqrt{1 + t^2} \quad . \quad . \quad . \quad (121).$$

By (120) we find for t_m, which makes u a maximum or minimum,

$$xt_m + y(1 + 2t_m^2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (122);$$

a quadratic equation which, when $(y/x)^2 < \frac{1}{8}$, has real roots as follows,—

$$t_1 = -\frac{x}{4y} - \sqrt{\left[\left(\frac{x}{4y}\right)^2 - \frac{1}{2}\right]}, \; t_2 = -\frac{x}{4y} + \sqrt{\left[\left(\frac{x}{4y}\right)^2 - \frac{1}{2}\right]} \quad (123).$$

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And substituting t_m, (either of these,) for t in (121) we find

$$r\left(\frac{d^2u}{d\psi^2}\right)_m = \left[x(1-t_m^2)+2yt_m\right]\sqrt{1+t_m^2} \quad . \quad . \quad . \quad (124),$$

or with simplification by (119),

$$r\left(\frac{d^2u}{d\psi^2}\right)_m = 2ru_m - x(1+t_m^2)^{3/2}. \quad . \quad . \quad . \quad . \quad . \quad (124)'.$$

Eliminating t_m^2 from the first factor of (124) by (122) we find

$$r\left(\frac{d^2u}{d\psi^2}\right)_m = \left[\frac{3x}{2}+t_m\left(2y+\frac{x^2}{2y}\right)\right].\sqrt{1+t_m^2} \quad . \quad . \quad (124)'',$$

which, with $m=1$, and $m=2$, gives β_1 and β_2 by (116).

§ 86. Using (123) we see that $(d^2u/d\psi^2)_m$ vanishes when $x=y\sqrt{8}$, and that it is negative for t_1, and positive for t_2, when $x>y\sqrt{8}$. Hence t_1 makes $d_2u/d\psi^2$ negative. Therefore u_1 is the maximum; and t_2 makes it positive. Therefore u_2 is the minimum; and (119) gives for these maximum and minimum values

$$ru_1=(x+yt_1)\sqrt{1+t_1{}^2},\ ru_2=(x+yt_2)\sqrt{1+t_2{}^2} \quad . \ (125).$$

By (122), (123) we see that when $y/x=0$, we have $-t_1=+\infty$, and $-t_2=0$. If we increase y from 0 to $+x/\sqrt{8}$, $-t_1$ falls continuously from ∞ to $\sqrt{\frac{1}{2}}$, and $-t_2$ rises continuously from 0 to $\sqrt{\frac{1}{2}}$. Thus $-t_1$ and $-t_2$ become, each of them, $\sqrt{\frac{1}{2}}$; which is the tangent of 35° 16′.

§ 87. *Geometrical digression on a system of autotomic, monoparametric co-ordinates.** §§ 87–90.

In (119) put

$$ru=a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126),$$

where a denotes the parameter O W of the curve O C C, fig. 32, which we are about to describe; being the curve given intrinsically by (119) and (122) with suffix 'm' omitted from t. In the present paper these curves may be called isophasals, because the argument of the sine in (130) below is the same for all points on any one of them.

Solving (119) and (122) for x and y, we find

$$x=a\ \frac{1+2t^2}{(1+t^2)^{3/2}},\quad y=a\ \frac{-t}{(1+t^2)^{3/2}} \quad . \quad . \quad . \quad (127).$$

* Of this kind of co-ordinates in a plane, we have a well-known case in the elliptic co-ordinates consisting of confocal ellipses and hyperbolas.

The largest of the eight curves shown in fig. 32 has been described according to values of x, y calculated from these two equations, by giving to $-t$ values $\tan 0°$, $\tan 10°$, $\tan 20°$, ... $\tan 90°$. The seven other isophasals partially shown in fig. 32, all similar to the largest, have been drawn to correspond to seven equidifferent smaller values, 19λ, 18λ . . . . 13λ, of the parameter a, if we make the largest equal to 20λ.

§ 88. It is seen in the diagram that every two of these isophasals cut one another in two points, at equal distances on the two sides of O W. If we continue the system down to parameter 0, every point within the angle C O C is the intersection of two and only two of the curves given by (127), with two different values of the parameter a. If we are to complete each curve algebraically, we must duplicate our diagram by an equal and similar pattern on the left of O: and the doubled pattern, thus obtained, would show a system of waves, equal and similar in the front and rear, which (§ 77 above) is possible but instable. We are, however, at present only concerned with the stable ship-waves contained in the angle $\pm$ 19° 28′ on the two sides of the mid-wake; and we leave the algebraic extension with only the remark that all points in the angle C O C of the diagram, and the opposite angle leftward of O, can be specified by real values of the parameter a: while imaginary values of it would specify real points in the two obtuse angles.

§ 89. By differentiation of (127), we find

$$\frac{dx}{dy} = -t = -\tan\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (128);$$

which proves that $\tan^{-1}t$ is the angle measured anti-clockwise from O Y to the tangent to the curve at any point (x, y), in the lower half of the diagram. Elimination of t between the two equations of (127) gives, as the cartesian equation of our curve,

$$(x^2+y^2)^3 + a^2(8y^4 - 20x^2y^2 - x^4) + 16a^4y^2 = 0 \quad . \quad . \quad (129).$$

But the implicit equations (127) are much more convenient for all our uses. It is interesting to verify (129) for the case $-t = \pm\sqrt{\tfrac{1}{2}}$ in (127), corresponding to either of the two cusps shown in the diagram.

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§ 90. Going back now to § 86 and the continuous variations considered in it, we see that $-t_1$ and $-t_2$ are respectively the tangents of the inclinations, reckoned from O Y clockwise, of portions of the long arc O C and of the short arc W C, in the upper half of the diagram. Thus, if we carry a point from O to C in the long arc, and from C to W in the short arc, we have the change of inclinations to O Y represented continuously by the decrease of $\tan^{-1}(-t_1)$ from 90° to 35° 16′, while y increases from 0 to $x\sqrt{8}$; and the farther decrease of $\tan^{-1}(-t_2)$ from 35° 16′ to 0°, while y diminishes from $x\sqrt{8}$ to 0 again. The inclination to O Y of the two branches meeting in the cusp, C, is 35° 16′ (or $\tan^{-1}\sqrt{\frac{1}{2}}$). For any point in the short arc C W C of the curve u or $\cos(\psi-\theta)/\cos^2\psi$, is a minimum. In each of the long arcs u is a maximum. At every point of the curve the value of u, whether minimum or maximum, is a/r. Hence for different points of the curve, u is inversely proportional to the radius vector from O.

§ 91. Going back to (118)′ we now see that for all points on any one of our curves, ru_1 and ru_2 have both the same value, being the parameter O W of the curve. The first part of (118)′ is one constituent of the depression at any point on either of the long arcs; and the second part of (118)′ is one constituent of the depression at any point on the short arc. Taking for example the largest of the curves shown in fig. 32, we now see that for any point of either of its long arcs, the second constituent of the depression of the water is to be calculated from the second part of (118)′; while for any point of its short arc, the second constituent of the depression is to be calculated from the first part of (118)′.

§ 92. Explaining quite similarly the determination of $d(x, y)$ for every point of each of the smaller curves which we see in the diagram cutting the longer arcs of the largest curve, we arrive at the following conclusions as *the complete solution of our problem.*

The whole system of standing waves in the wake of the travelling forcive is given by the superposition of constituents calculated according to (127), with greater and smaller values of the parameter a with infinitely small successive differences. Hence, what we see in looking at the waves from above is exactly a system of crossing hills and valleys, with ridges and beds of hollows, all shaped according to the isophasal curves shown in fig. 32. Looking at any one of the short arc-ridges and following it through the cusps, we find it becoming the middle line of a valley in each of the long arcs of the curve. And following a short arc mid-valley through the cusps, we find, in the continuation of the curve, two long ridges. Every ridge, long or short, is furrowed by valleys. All the curved ridges and valleys are parts of one continuous system of curves, illustrated by fig. 32 and expressed by the algebraic equation (129).

With these explanations we may write (118)′ as follows:

$$\mathrm{d}(x, y) = \frac{4\pi^2 bk \sec^2\psi}{\lambda\beta} \sin\frac{2\pi}{\lambda}\left(ru - \frac{\lambda}{8}\right) \quad . \quad . \quad . \quad (130),$$

where

$$\beta = \sqrt{\frac{r}{\lambda}\left(\pm\frac{d^2u}{dx^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131).$$

§ 93. An important, perhaps the most important, feature of the wave-system which we actually see on the two sides of the mid-wake of a steamer travelling through smooth water at sea, or of a duckling* swimming as fast as it can in a pond, is the steepness of the waves in two lines which we know to be inclined at 19° 28′ to the mid-wake. The theory of this feature is expressed by the coefficient of the sine in (130), and is well illustrated by the calculation of $\sqrt{\frac{a}{\lambda}}\cdot\frac{\sec^2\psi}{\beta}$ for eleven points of any one of the curves of fig. 32, the results of which are shown in column 6 of the following table. They express the depression below, and elevation above mid-level, due to one constituent of the system of crossing hills and valleys described in § 92. Column 1 is $-\psi$. Columns 2, 3 are x/a and y/a, calculated from (127). Column 4 is u, calculated by (126) from columns 2, 3. Column 5 is $\frac{r}{a}\cdot d^2u/d\psi^2$, calculated from (124) and columns 2, 4. Column 6 is $\sqrt{\frac{a}{\lambda}}\cdot\frac{\sec^2\psi}{\beta}$, calculated from columns 1, 6. u, being, as we

* In the case of even the highest speed attained by a duckling, this angle is perhaps perceptibly greater than 19° 28′, because of the dynamic effect of the capillary surface tension of water. See *Baltimore Lectures*, p. 593 (letter to Professor Tait, of date 23rd Aug. 1871) and pp. 600, 601 (letter to William Froude, reprinted from *Nature* of 26th Oct. 1871).

have seen, a maximum for values of $-\psi$ from 0 to 35° 16′, and a minimum for values from this to 90°, we see that the proper suffix in columns 4, 6, for the first four lines of each column is 1, and for the last six lines is 2.

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.
$-\psi$	$\frac{x}{a}$	$\frac{y}{a}$	u	$\frac{r}{a}\frac{d^2u}{d\psi^2}$	$\sqrt{\frac{a}{\lambda}} \cdot \frac{\sec^2\psi}{\beta}$
0°	1·0000	0·0000	1·00000	1·00000	1·0000
10°	1·0145	·1685	·97239	·93782	1·0647
20°	1·0497	·3201	·91587	·73497	1·3210
30°	1·0825	·3750	·87290	·33333	2·3094
35° 16′	1·0887	·3849	·86602	0·00000	∞
40°	1·0826	·3773	·87225	− ·40830	2·6660
50°	1·0201	·3166	·93624	− 1·84070	1·7839
60°	·8750	·2165	1·10941	− 5·00003	1·7888
70°	·6441	·1100	1·53041	−14·0987	2·2793
80°	·3421	·0297	2·91222	−63·3341	4·1672
90°	0·0000	0 0000	∞	− ∞	∞

§ 94. In (130), k is generally a function of ψ; but if the forcive is circular, (§ 81 above) k is a constant, and for points on one of the isophasal curves (a = constant) the only variable coefficients of the sine are $\sec^2\psi$, and β^{-1}. But for different isophasal curves the coefficient in (130) expressing the magnitude of the range above and below mean level, varies inversely as $\sqrt{a}$. For mid-wake ($\psi = 0$) a is simply the distance from the forcive: and we conclude, not merely for our point-forcive, but for a great ship, that the waves at a very large number of wave-lengths right astern, are smaller in height inversely as the square root of the distance from the forcive or from the middle of the ship.

§ 95. The infinity for $\psi = \pm$ 35° 16′ represents a feature analogous to a caustic in optics. There is in nature no infinity for either case, if the source is finite and distributed, not infinitely intense and confined to an infinitely small space. According to the methods followed in §§ 1–72 above, we have in every case a finite intensity of source, or of forcive, except in § 80 where we have supposed b infinitely small, in comparison with λ, we avoid the infinity shown in column 6: and can, by great labour, calculate a table of mitigated numbers, rising to a very large maximum at $\psi = \pm 35° \ 16'$; but not to infinity; and so arrive mathematically at an expression for the very high waves seen on the two bounding lines of the wave-disturbance, inclined at 19° 28′ to the mid-wake. But it is interesting to remember that we see in reality a considerable number of white-capped waves (would-be infinities) before the well-known large glassy waves which form so interesting a feature of the wave-disturbances.

§§ 80–95 of the present paper are merely a working out of the simple problem of purely gravitational waves with no surface-tension on the principle given by Rayleigh* in 1883 for the much more complex problem of capillary waves in front, in which surface-tension is the chief constituent of the forcive, and waves in the rear, in which the chief constituent of the forcive is gravitational.

In all the work arithmetical, algebraic, graphic of §§ 32–95 above, I have had much valuable assistance from Mr J. de Graaff Hunter; who has just now been appointed to a post in the National Physical Laboratory.

* *Proc. Lond. Math. Soc.*, xv. pp. 69–78, 1883; reprinted in Lord Rayleigh's *Scientific Papers*, vol. ii. pp. 258–267.

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