FIG 45. A. Speed 1 Rev. per Second, Uniform effective pressure $p_m = p_s = 2$ lbs. per sq. in.
 A_1 unloaded no friction, A_2 unloaded and friction, A_3 loaded no friction, A_4 loaded and friction.

FIG 48. C. Speed 1 Rev. per Sec. $p_1 = 10$ lbs., $\tau = 12.8$, $p_2 = 0.516$ giving $p_m = p_s = 2$ lbs. per sq. in.
 C_1 unloaded no friction, C_2 unloaded and friction, C_3 loaded no friction, C_4 loaded and friction.

FIG 47. B. Speed 4 Rev. per Sec. Uniform effective pressure $p_m = p_3 = 2$ lbs. per sq. in.B₁ same as A₁, B₂ same as A₂, B₃ loaded no Friction, B₄ loaded and Friction.

FIG 49. D. Speed 4 Rev. per Sec., $p_1 = 10$ lbs., $r = 12.8$, $p_2 = 0.516$, giving $p_m - p_2 = 2$ lbs. per sq. in.

D_1 same as C_1

D_2 same as C_2

D_3 Loaded, no Friction; D_4 Loaded, and Friction.

FIG 50. E Speed 1 Rev. per Sec., $p_1 = 50$ lbs., $r = 5.93$, $p_2 = 3$ lbs., giving $p_1 - p_2 = 18.905$ lbs. per sq. in. E_1 unloaded, no Friction, E_2 unloaded and Friction.

FIG 51. E. Speed 1 Rev. per Sec., $p_1 = 50$ lbs., $r = 5.93$, $p_2 = 3$ lbs., giving $p_m - p_3 = 18.905$ lbs. per sq. in. E_3 loaded no friction, E_4 loaded and friction.

FIG 52. F. Speed 4 Rev. per Sec., $p_1 = 50$ lbs., $r = 5.93$, $p_3 = 3$ lbs., giving $p_2 - p_3$ 18.905 lbs. per sq. in. F_1 same as E_1 , F_2 same as E_2 , F_3 loaded no Friction, F_4 loaded and Friction.

FIG 53. G. Speed infinitely slow - $p = 50$ lbs, $r = 5.93$, $p_3 = 3$ lbs.
 $p_m - p_3 = 18.905$ lbs per Square inch

FIG 54.H. Distribution of Steam & Speed as for E; but connecting rod reduced to $3\frac{1}{2}$ times Stroke.

XXIX.—*On the Application of Graphic Methods to the Determination of the Efficiency of Machinery.* By PROFESSOR FLEEMING JENKIN. (Plates XXVI.—XXXIII.)

(Read 18th February 1878.)

PART II.—*The Horizontal Steam Engine.*

§ 31. In a previous paragraph (29) the loaded dynamic frame (fig. 38) for one position of a direct-acting horizontal steam-engine has been described, and the mode of drawing that frame explained, on the assumption that the loads L_a , L_b , L_c , L_d , are known: it was stated that these loads had been calculated for one particular engine. The present paper gives the varying effort which this engine is capable of exerting at each part of its stroke with given pressures (both constant and varying) in the cylinder, and given *constant* velocities of the crank-axle. The same calculations show the efficiency of the engine at each part of the stroke, and its total efficiency under the same circumstances. This problem has, it is believed, never hitherto been solved so as to take into account all the circumstances of mass, weight, and friction.

§ 32. No pains has been taken in the choice of the particular example. The object of the paper is not so much to draw general conclusions as to all steam-engines—which, indeed, differ too much in arrangement to make this feasible—as to show how, by a method of no great complexity, full information can be obtained as to any particular engine or class of engines.

The following are the particulars of the engine:—Stroke, 16 in.; diameter of cylinder, 8 in.; length of connecting-rod, 41 in.; centre of gravity of connecting-rod, 20 in. from the crank-pin; mass of connecting-rod, 34 lbs.; mass of piston and piston-rod, 46 lbs.; mass of balanced crank and that part of the fly-wheel which is borne by the main bearings, 200 lbs.; the mass of the frame fixed to the earth is indefinitely large; diameter of crank-shaft in bearings, $4\frac{1}{2}$ in.; diameter of crank-pin, 2·4"; diameter of pin at crosshead, 1·8". The position of the resistance overcome (fig. 17, Part I.) is that of a tangent to an imaginary circle of 18 in. radius, concentric with the crank-shaft, the tangent being so inclined as to cut the centre line of the engine 19·2" from the crank-shaft centre on the side towards the piston. This line may be regarded as the direction of the resistance exerted at the teeth of a spur-wheel. The position of this line exercises a very material influence on the results obtained, which must not, therefore, as was said before, be considered as applicable to engines generally. The coefficient of friction has been taken as $= \frac{1}{12}$.

§ 33. Eight distinct cases have been investigated. In these examples the action of the engine has been studied on four different assumptions—1st, neglecting weight, mass, and friction; 2d, neglecting weight and mass; 3d, neglecting friction, but taking mass and weight into account; 4d, taking mass, weight, and friction into account. A comparison of the results shows the influence of each element of the problem on the final result. These processes are obtained by what in the first part of the paper are called—1st, the dynamic frame; 2d, the dynamic frame with friction; 3d, the loaded dynamic frame; 4d, the loaded dynamic frame with friction. The examples will be called A, B, C, D, E, F, G, and H; the four assumptions will be indicated by the suffixes 1, 2, 3, 4, and it must be borne in mind that the suffix 4 corresponds to the complete solution, in which all the elements of the problem are taken into account. The results are graphically shown in a series of curves which will be called “Effort curves.” These were drawn to a very large scale and have been reduced by a photographic process.

§ 34. Example A.—Speed of engine, 1 revolution per second; effective pressure on cylinder, 2 lbs. per square inch—uniform throughout the stroke. This example has been chosen to indicate the effect of running the engine with a very low pressure and no expansion. The vertical ordinates of curve A_1 , fig. 45, Plate XXVI., indicate in pounds the total effort which the engine could exert along the assumed line of resistance, the total pressure on the cylinder being 100·5 lbs. The horizontal co-ordinates indicate the arc which the centre of the crank-pin has traversed, measured from the position in which the crank-pin is, at the end of its stroke, nearest to the cylinder. The co-ordinates of all the other curves have a similar signification. The ordinates for each curve have been calculated by separate figures for 24 evenly spaced positions of the crank, numbered as shown in fig. 46, when the piston and connecting rod lie to the left of the crank. The portion of the curve corresponding to the forward or front stroke, between the positions 0 and 12, will be called the front branch; the portion corresponding to the backward or back stroke, between 12 and 24, will be called the back branch. In A_1 the front and back branches are symmetrical relatively to ordinate 0 or 12. The relation of the ordinates of this curve to the total pressure 100·5 lbs. on the piston depends wholly on the relative velocity of the piston and a point on the circumference of a circle 3 feet diameter concentric with the crank-shaft. There are two dead points at 0 and 12, where, for an infinitely short arc, no resistance could be overcome.

In curve A_2 the front and back branches are sensibly symmetrical in the present example, although not rigidly so. Instead of a mere dead point we have now two dead arcs of different lengths, the longest being at the end of the front and beginning of the back strokes. These dead arcs together last for about 2·7 per cent. of each revolution. The negative ordinates throughout these dead arcs indicate that the engine, instead of driving, must be driven. The area enclosed

by the curves above the line OY measures the work which the steam can do. The area enclosed below the line OY in A_2 measures the work which must be done (not by the steam) to pull the engine through the dead arcs. This work, in practical cases, is done by the fly-wheel, which, if large enough, might do the necessary work without allowing the speed to fluctuate sensibly—a condition assumed throughout the calculations.

The whole work done by the steam is $100.5 \times 16 \times 12$, or 3216 inch lbs. The area of A_1 was found to be 3210 inch lbs., showing the error due to defective drawing to be very trifling; the area of A_2 (being the arithmetical difference between the positive and negative areas) is 2974 inch lbs. The efficiency, on assumption 2d, is therefore $\frac{2974}{3210} = 0.927$.

In curve A_3 the front and back branches are no longer even approximately symmetrical to one another. There is no dead point at the end of the front stroke, and there is a long dead arc at the end of the back stroke. The maximum effect is greater in the front than in the back stroke. These effects are due to the weight and mass of the connecting-rod. A counterbalance might be so placed as to make the front and back branches resemble one another much more closely. The area of the curve A_3 (calculated as for A_2) is 3212 inch lbs., the total error due to imperfect draughtsmanship being only 4 inch lbs.

Curve A_4 resembles A_3 in its general outline, and when these are compared with A_1 and A_2 they show the effect of the weight and mass of the parts in increasing friction. The increase in the loss due to friction is not even approximately constant throughout the stroke, but is much greater when the crank is nearly at right angles to the centre line of the engine. The loss is greatest during the back stroke, producing inequality between the useful work done during the back and front strokes. The causes of each departure from symmetry in all these curves can be followed on the diagrams of the dynamic frames, but this paper would be unduly extended if these were all to be printed. Moreover, such directions have already been given for drawing these as will enable any one to investigate the cause of each effect now described or shown on the curves. The area of A_4 is 2602 inch lbs., so that the true efficiency of the engine running at this speed, and with this low pressure of steam, is $\frac{2602}{3212} = 0.810$.

§ 35. Example B, fig. 47, Plate XXVII.—The effect of the resistance of the masses to acceleration as distinguished from the effect of the weights of the parts might have been exhibited by drawing effort curves—1st, on the assumption that although the parts resisted acceleration they had no weight; and, 2d, on the assumption that although the parts had weight they offered no resistance to acceleration, or were moving at an infinitely slow speed; then, comparing these curves with A_4 we should have seen the effect of each element of the problem. This, however, was thought unnecessary, because the effect of

the resistance of the masses to acceleration is strikingly shown by the curves B_3 and B_4 , being the effort curves of the same engine, with the same pressure in the cylinder, but running four times faster. When making one revolution per second, the resistances to acceleration are smaller than the weights of the elements in motion, as may be seen in Table I. of the Appendix, where these forces are given for each position. When the speed is increased to four revolutions per second, these forces are multiplied by sixteen, and their effect is then much greater than that of the weight of the parts and suffices completely to change the character of the effort curve. The negative portion of the curve lasts for nearly half the revolution of the crank, so that for nearly half of each revolution the fly-wheel would have to pull the engine round. This is true both for B_3 and B_4 for the curves without and with friction, and is simply due to the fact that during the first half of each stroke the reciprocating masses are being positively accelerated. The positive ordinates during the period when the engine is driving of course exceed those during which the engine is being driven; so that for curve B_3 the balance of positive area ought to be 3216 inch lbs. It actually is, on the drawing, 3256, the excess being due to small errors in drawing and computation. The area of B_4 is only 1744 inch lbs.

The efficiency, therefore, has sunk to 0.536; almost half the power of the engine is taken up in driving itself; the pressures on the joints caused by resistance to acceleration have at this high speed greatly increased the loss due to friction. The inequality between back and front strokes is also very great, the area of the front branch being 953 inch lbs., that of the back branch only 791. The loss due to friction sinks, however, almost to nothing at one point of the front stroke, very near the place where, in example A_4 , it was a maximum. This may serve as a warning against hasty generalisations. In curve B_4 there are sensible sudden changes of efficiency at points 0 and 12, due to a sudden change in the position of the points where the elements bear on one another.

§ 36. Example C, fig. 48, Plate XXVI.—Example C was selected with the object of ascertaining how far the efficiency is affected by using the steam expansively instead of admitting it throughout the stroke. There is a very general idea that the sudden shock, as it is called, of admitting steam at a much higher pressure for a short time at the beginning of the stroke must diminish the efficiency of an engine. This is not so in the present example. An imaginary indicator diagram was selected for example C, drawn on the supposition that the steam was first admitted at a constant pressure of 10 lbs. per square inch; that r , the reciprocal of the fraction of the stroke during which steam enters at a constant pressure, was 12.8; that the steam expanded, according to an adiabatic curve, and was suddenly released at the end of the stroke, and that during the return stroke the back pressure was 0.516. These data give a total effective initial pressure of 476.7 lbs.; a total effective final pressure of 3.7 lbs.; and a mean

effective intensity of pressure of 2 lbs. per square inch, as in examples A and B. The work done by the steam per stroke is, therefore, as before, 3212 inch lbs. The speed of the engine is taken, as for curve A, at one revolution per second. Curves C_3 and C_4 give the effects with and without friction; the area of C_3 is 3223 inch lbs., showing a very small error of execution; the area of C_4 is 2640; the ratio of these values gives 0.819 as the efficiency of the engine worked in this way. This efficiency is actually a little higher than that of curve A, which is repeated in this figure to allow it to be more readily compared with C. A similar result is to be observed in curve D, fig. 49, Plate XXVIII., constructed from the same data, but at the high speed of 4 revolutions per second. The area of D_3 is 3270 inch lbs., that of D_4 1965 inch lbs., giving an efficiency of 0.601—a value sensibly higher than that derived from curve B when the steam was admitted throughout the stroke. The errors due to imperfect drawing have generally the result of slightly increasing the effort, and the error in curve D_3 for this particular drawing (3270 over 3216) is nearly 1.7 per cent. The liability to error is much increased when, as here, a large portion of the area is negative. If this percentage were reckoned on the arithmetical sum of the areas, instead of on their difference, it would be insignificant. Notwithstanding this inevitable imperfection, there is every reason to expect that the errors in curves D_3 and D_4 resemble one another; and we have the less reason to suspect the accuracy of the conclusion, because we can see that since the tendency of resistance to acceleration during the beginning of each stroke is to diminish the effort, while that of a large initial pressure is to increase it, the two tendencies counteract one another without causing pressure on the main-bearings or crank-pin. Thus, in curves B_3 and B_4 we found that the loss due to friction at positions 5 and 18 was much reduced from this cause. In curves C_3 and C_4 we have this useful result of the mass of the moving parts without the reversal of the stresses which brings down the efficiency in B_3 and B_4 . Similarly, in curves D_3 and D_4 , the effect of the high initial pressure is to prevent the negative parts of the curve from falling so low as in example B. The extremely low efficiency of A_4 , B_4 , and C_4 illustrates the evil effect of using a large and heavy engine running with small mean pressures of steam. We not unfrequently see large engines ordered for factories, mines, or water-works to allow for subsequent extension, or for large variations in the work required at different times. The above cases show how very serious the loss may be when an engine is habitually worked much below its power. The cases are no doubt extreme, but they show the tendency of the practice.

§ 37. Example E, figs. 50 and 51, Plates XXIX. and XXX.—The four cases hitherto analysed are not, strictly speaking, practical cases: they were chosen so as to bring into prominence the effects of friction and high speed, separ-

ately and combined. These effects are obviously most marked when the engine is running lightly loaded. We will now consider a more practical example. The curves E correspond to a speed of 1 revolution per second, and to an indicator diagram drawn as follows:—Initial intensity of pressure 50 lbs. per square inch, $r=5.93$; back pressure, 3 lbs. per square inch; mean effective pressure, 18.905 lbs. per square inch; work done by steam, 30640 inch lbs. per revolution.

E_1 is the effort curve, neglecting mass, weight, and friction.

E_2 the effort curve, taking the friction into account which results from effect and resistance, but neglecting that due to weight and mass; the area of E_1 is 30590 inch lbs, the error being 50 inch lbs. The area of E_2 is 28400 inch lbs., making the efficiency 0.928 on this hypothesis.

E_3 is the effort curve, taking the effort mass and weight into account, but neglecting friction. The area is 30700, the error being 110 inch lbs. in the drawings.

E_4 is the effort curve, taking mass, weight, and friction into account—in other words, the true effort curve. Its area is 28030 inch lbs., giving an efficiency of 0.913, or a little less than that calculated for E_2 . Now that the steam pressures are large, the loss due to the weight and mass of the parts is at this speed comparatively insignificant. The total inevitable loss due to friction in the machine alone, even excluding the accidental friction due to tightness of piston and glands, and the power required to work the valve gear, is nearly 9 per cent. of the whole indicated horse-power. This example shows the complete fallacy of the experimental method sometimes adopted with the object of testing the efficiency of an engine. The engine is run with no resistance, and the indicated H.P. observed. This is assumed approximately to represent the loss due to the engine itself when doing useful work against a large resistance; but from example A we see that the power required to overcome the friction in the engine when a small resistance was being overcome was only 3210–2974 inch lbs., or 236 inch lbs.; whereas in example E the power required is 1190 inch lbs., or nearly five times as much. This ratio would be somewhat diminished by the constant accidental resistances in glands, &c., and by the power required for valves and pumps. It must, however, always remain very large.

In E_1 and E_2 the discontinuity of the curves at positions 0 and 12 is very marked. There must also be a slight break near positions 6 and 18, due to the change in the bearing-points on the crosshead pin; but the frictional loss at this point is so insignificant that the break in the curve is not sensible. There is very little difference in general character between curves E_1 E_2 and E_3 E_4 . Mass and weight play a very small part in the general result.

§ 38. Example F, fig. 52, Plate XXXI.—Example F is important and instructive, showing the result of running the engine at 4 revolutions per second

instead of 1 revolution per second, but maintaining the same indicator diagram as in example E, the mean effective pressure being 18·905 lbs. as before. We might expect this high speed to diminish the efficiency, as in examples B and D, but this is not the case. Curves F_1 and F_2 are identical with E_1 and E_2 and are therefore omitted; curves F_3 and F_4 are, however, very different in character from E_3 and E_4 . The resistance to acceleration causes the effort to be much more uniformly distributed. It greatly diminishes the maximum pressure, and largely increases the pressure during the second half of each stroke. The resistance to acceleration is not, however, so large relatively to the steam pressure as to produce negative ordinates, such as were found in cases B and D. The action of the reciprocating masses is, therefore, on the whole, beneficial, and we find the efficiency of the engine to be ·917, or somewhat higher than in example E. Much care has been taken in the computations to check this rather remarkable result, but there seems no reason to doubt its accuracy. The important conclusion to be drawn does not, however, depend on trifling differences—such as that between ·917 and ·913. It is that high speeds do not necessarily entail low efficiency, so that by judicious arrangement of the masses we can have small engines running rapidly with high expansion which, so far as frictional resistances are concerned, may be at least as efficient as the same engines running slowly, and necessarily more efficient than larger engines doing the same work.

§ 39. Examples G. and H, figs. 53 and 54, Plates XXXII. and XXXIII.—The unexpected results obtained rendered it desirable to investigate the case of infinitely slow speed, or that in which the effect of inertia was wholly disregarded. Curves G_3 and G_4 show the efforts with and without friction for the practical distribution of steam assumed in cases E and F. The total work done by the steam is 30770 inches, the useful work 20120, and the efficiency ·914—a result hardly differing sensibly from that obtained with the high speed of 4 revolutions per second.

§ 40. Example H.—Lastly, curves H_3 and H_4 show the efforts with and without friction when the engine is modified so as to make the connecting-rod only 28 in. long, or $3\frac{1}{2}$ times the stroke. The weight of the new rod is taken as 28 lbs., and its centre of gravity 14" from the crank end, and its radius of gyration about the cross-head 18·83 inches. The speed assumed was 1 revolution per second.* As calculated from the areas of the curves the whole work done by the steam is 30740 inch lbs., the useful work 27920 inch lbs., and the efficiency 0·91.

This result is of considerable practical value, showing that the efficiency of the engine is hardly diminished by shortening the connecting-rod to this extent.

* The initial steam pressure, the ratio of expansion, and the back pressure were assumed to be the same as in example E.

§ 41. *General Conclusions.*—The most important conclusion to be drawn from the foregoing examples is, that the investigation of the efficiency of any given direct-acting engine is rendered comparatively easy by the new method. Next in importance may be ranked the warning not to judge hastily as to the result of any given modification in proportions or speed. The differences introduced by a change of speed are especially remarkable, and show how futile reasoning must be as to relative efforts and resistances as to stresses on the various parts of the engine, and the efficiency of any design when the inertia of the masses is left out of the calculation.

Further conclusions may be drawn as follows:—1st, that high speeds do not necessarily involve small efficiency; 2d, that a short connecting-rod is not very disadvantageous; 3d, that expansive working, even when carried to great lengths, does not necessarily involve a loss of efficiency.

Table I. gives an abstract of the numerical values obtained with this particular engine, but the reader must be warned against considering these results as generally applicable to other engines of different proportions.

TABLE I.—*Abstract of Results.*

	Example.	Pressure.	Speed.	Efficiency.
Engine described in § 34.	$\left. \begin{array}{l} A_1 A_2 \\ A_3 A_4 \end{array} \right\}$	Uniform	Slow, 1 rev. per sec.	$\left\{ \begin{array}{l} \cdot 27 \\ \cdot 81 \end{array} \right\}$
	$\left. \begin{array}{l} B_1 B_2 \\ B_3 B_4 \end{array} \right\}$			$\left\{ \begin{array}{l} \cdot 927 \\ \cdot 536 \end{array} \right\}$
	$\left. \begin{array}{l} C_1 C_2 \\ C_3 C_4 \end{array} \right\}$	Expansion	Slow, 1 rev. „	$\left\{ \begin{array}{l} \cdot 907 \\ \cdot 819 \end{array} \right\}$
	$\left. \begin{array}{l} D_1 D_2 \\ D_3 D_4 \end{array} \right\}$			$\left\{ \begin{array}{l} \cdot 907 \\ \cdot 61 \end{array} \right\}$
	$\left. \begin{array}{l} E_1 E_2 \\ E_3 E_4 \end{array} \right\}$	Expansion	Slow, 1 rev. „	$\left\{ \begin{array}{l} \cdot 928 \\ \cdot 913 \end{array} \right\}$
	$\left. \begin{array}{l} F_1 F_2 \\ F_3 F_4 \end{array} \right\}$			$\left\{ \begin{array}{l} \cdot 928 \\ \cdot 917 \end{array} \right\}$
	$\left. \begin{array}{l} G_1 G_2 \\ G_3 G_4 \end{array} \right\}$			$\left\{ \begin{array}{l} \cdot 928 \\ \cdot 914 \end{array} \right\}$
Engine described in § 40.	$H_3 H_4$	Expansion	Slow, 1 rev. per sec.	$\cdot 908$

The necessary calculations and drawings for this paper have been made by my assistant Mr J. A. EWING, to whom I am much indebted both for the accuracy with which the work has been done, and for the interest he has shown in adopting the novel method of investigation.

The appendix contains data which will allow the reader to verify the results arrived at without going through all the calculations.

This appendix has been drawn up by Mr EWING.

APPENDIX TO PART II.

On the Application of Graphic Methods to the Determination of the Efficacy of Machinery.

To determine the forces required for the acceleration of the piston and connecting-rod in each position of the engine, we must know the acceleration of the piston, and the angular velocity and angular acceleration of the connecting-rod. If AC be the crank, and CB the connecting-rod, and if AB be called x , then $\frac{d^2x}{dt^2}$ is the acceleration of the piston; and if the angle CBA be called θ , $\frac{d\theta}{dt}$ is the angular velocity of the connecting-rod, and $\frac{d^2\theta}{dt^2}$ is its angular acceleration.

Fig. 55.

If the angle which the crank makes with AB be called α , $\frac{d\alpha}{dt}$ is the angular velocity of the crank.

Then, calling the crank radius r , and the length of the connecting-rod l , we have

$$\theta = \sin^{-1} \frac{CN}{CB} = \sin^{-1} \left(\frac{r \sin \alpha}{l} \right)$$

$$\frac{d\theta}{dt} = \frac{r \cos \alpha}{\sqrt{l^2 - r^2 \sin^2 \alpha}} \cdot \frac{d\alpha}{dt}$$

Differentiating again, and remembering that $\frac{d^2a}{dt^2} = 0$, we obtain finally

$$\frac{d^2\theta}{dt^2} = -\frac{r \sin \alpha (l^2 - r^2)}{(l^2 - r^2 \sin^2 \alpha)^{\frac{3}{2}}} \left(\frac{da}{dt}\right)^2.$$

Again

$$AB = AN + BN$$

$$x = r \cos \alpha + l \cos \theta$$

$$\frac{dx}{dt} = -r \sin \alpha \frac{da}{dt} - l \sin \theta \frac{d\theta}{dt},$$

$$\text{and} \quad \frac{d^2x}{dt^2} = -r \cos \alpha \left(\frac{da}{dt}\right)^2 - l \cos \theta \left(\frac{d\theta}{dt}\right)^2 - l \sin \theta \frac{d^2\theta}{dt^2}.$$

Substituting for $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ their values as determined above, and putting $r \sin \alpha$ for $l \sin \theta$, and $\sqrt{l^2 - r^2 \sin^2 \alpha}$ for $l \cos \theta$, we obtain finally

$$\frac{d^2x}{dt^2} = -r \left(\frac{da}{dt}\right)^2 \left\{ \cos \alpha + \frac{rl^2 \cos 2\alpha + r^3 \sin^4 \alpha}{(l^2 - r^2 \sin^2 \alpha)^{\frac{3}{2}}} \right\}$$

The forces required to accelerate the piston and the connecting-rod may now be calculated as follows :—

For the piston.—If M be the mass of the piston and piston-rod in lbs., the force in lbs. is

$$\frac{M}{g} \frac{d^2x}{dt^2}.$$

When this quantity is negative, the force acts towards the centre of the crank shaft.

For the connecting-rod.—The motion may be looked at as a translation of the whole rod in the direction of motion of the piston, combined with a rotation of the rod about the crosshead. Hence the force producing acceleration is the resultant of three components :— F_1 , the force required for the linear acceleration in the direction of motion of the piston ; F_2 , the force required for rotation about the crosshead at the angular velocity which the rod has at the instant under consideration. This acts towards the centre of rotation, and is equal and opposite to the so-called “centrifugal force ;” and, lastly, F_3 , the force required to give the rod the angular acceleration which it has at the given instant.

Fig. 56.

Let M' be the mass of the connecting-rod in lbs., and G its centre of mass,

(Note.—In figs. 55 and 56 the engine has been represented as seen from behind, if the engine in the previous figures be considered as viewed from the front.)

distant l_0 from the crosshead B; also let k be its radius of gyration about B. Then the first component mentioned above, or F_1 is

$$\frac{M'}{g} \frac{d^2x}{dt^2}$$

and acts through G parallel to the path of the piston.

The second component, F_2 , is

$$\frac{M'l_0}{g} \left(\frac{d\theta}{dt} \right)^2$$

and acts along the line of the rod towards B. The third component, F_3 , is

$$\frac{M'l_0}{g} \left(\frac{d^2\theta}{dt^2} \right)$$

and acts at right angles to the rod through the centre of the percussion H, which is at a distance $\frac{k^2}{l_0}$ from B.

These three forces may be most conveniently compounded by shifting F_3 to G, and introducing a couple whose moment is $F_3 \cdot HG$, or

$$\frac{M'(k^2 - l_0^2)}{g} \frac{d^2\theta}{dt^2}, \text{ or } \frac{I}{g} \frac{d^2\theta}{dt^2},$$

where I is the moment of inertia of the rod about G.

Forces equal and opposite to those components form the several parts of the whole resistance to acceleration, and when these are combined with the weight, the resultant is the whole load on the element. (See § 27, page 30, Trans. R.S.E., vol. xxviii.) This composition is effected graphically, and a single force is obtained acting through G, which has then to be shifted parallel to itself to such a distance as to give rise to the moment $\frac{I}{g} \frac{d^2\theta}{dt^2}$. This process gives a single force of determinate magnitude and position, as the load on the element in each position of the engine.

The following tables show the component parts of the acceleration and force in the cases which have been actually examined. In Table I. the connecting-rod is 41" long, and its mass is 34 lbs.; l_0 is 20", and $\frac{k^2}{l_0}$ is 34.08 inches. In Table II. the connecting-rod is 28" long, and its mass is 28 lbs.; l_0 is 14", and $\frac{k^2}{l_0}$ is 25.32 inches. In both cases the crank radius is 8", the mass of the piston and piston-rod 46 lbs., and the speed is one revolution per second, whence

$$\frac{d\alpha}{dt} = 2\pi.$$

The positions of the crank in column 1 are numbered thus:—Positions 0 and 24 are the same, and correspond to $\alpha = 0$. Position 12 corresponds to $\alpha = 180^\circ$. The movement in fig. 56 is contrary to that of the hands of a watch, and the interval between two successive positions is 15° . Looked at from the other side, as in earlier figures of the engine, the movement would be in the direction of the hands of a watch.

TABLE I.

Connecting-rod, 41". Crank, 8". Mass of connecting-rod, 34 lbs. Mass of piston and piston-rod, 46 lbs. Speed, one revolution per second.

$$l_0 = 21". \quad I = 9576 \text{ (inches and lbs.)}$$

Piston, mass = M.			Connecting Rod, mass = M'.					
Position of Crank.	$\frac{d^2x}{dt^2}$ feet and seconds.	$\frac{M}{g} \frac{d^2x}{dt^2}$ lbs.	$\frac{d\theta}{dt}$ radians and seconds.	$\frac{d^2\theta}{dt^2}$ radians and seconds.	F_1 or $\frac{M'}{g} \frac{d^2x}{dt^2}$ lbs.	F_2 or $\frac{M'l_0}{g} \left(\frac{d\theta}{dt}\right)^2$ lbs.	F_3 or $\frac{M'l_0}{g} \frac{d^2\theta}{dt^2}$ lbs.	Moment $\frac{I}{g} \frac{d^2\theta}{dt^2}$ lbs. and inches.
0	−31·45	−44·88	1·226	0	−33·21	2·78	0	0
1	−29·94	−42·77	1·186	−1·925	−31·61	2·60	−3·56	−47·7
2	−25·50	−36·46	1·067	−3·759	−26·93	2·10	−6·95	−93·2
3	−18·67	−26·66	0·876	−5·385	−19·71	1·42	−9·95	−133·5
4	−10·60	−15·13	0·622	−6·702	−10·69	0·71	−12·38	−166·1
5	−2·29	−3·27	0·323	−7·557	−2·42	0·19	−13·97	−187·3
6	5·24	7·48	0	−7·854	5·53	0	−14·51	−194·6
7	11·86	16·36	−0·323	−7·557	12·52	0·19	−13·97	−187·3
8	15·72	22·46	−0·622	−6·702	16·60	0·71	−12·38	−166·1
9	18·56	26·52	−0·876	−5·385	19·60	1·42	−9·95	−133·5
10	20·14	28·76	−1·067	−3·759	21·27	2·10	−6·95	−93·2
11	20·91	29·87	−1·186	−1·925	22·08	2·60	−3·56	−47·7
12	21·19	30·27	−1·226	0	22·37	2·78	0	0
13	20·91	29·87	−1·186	1·925	22·08	2·60	3·56	47·7
14	20·14	28·76	−1·067	3·759	21·27	2·10	6·95	93·2
15	18·56	26·52	−0·876	5·385	19·60	1·42	9·95	133·5
16	15·72	22·46	−0·622	6·702	16·60	0·71	12·38	166·1
17	11·86	16·36	−0·323	7·557	12·52	0·19	13·97	187·3
18	5·24	7·48	0	7·854	5·53	0	14·51	194·6
19	−2·29	−3·27	0·323	7·557	−2·42	0·19	13·97	187·3
20	−10·60	−15·13	0·622	6·702	−10·69	0·71	12·38	166·1
21	−18·67	−26·66	0·876	5·385	−19·71	1·42	9·95	133·5
22	−25·50	−36·46	1·067	3·759	−26·93	2·10	6·95	93·2
23	−29·94	−42·77	1·186	1·925	−31·61	2·60	3·56	47·7
24	−31·45	−44·88	1·226	0	−33·21	2·78	0	0

TABLE II.

Connecting-rod, 28". Crank, 8". Mass of connecting-rod, 28 lbs. Mass of piston and piston-rod = 46 lbs. Speed, one revolution per second.

$$l_0 = 14".$$

Piston, mass = M.			Connecting Rod, mass = M'.					
Position of Crank.	$\frac{d^2x}{dt^2}$ feet and seconds.	$\frac{M}{g} \frac{d^2x}{dt^2}$ lbs.	$\frac{d\theta}{dt}$ radians and seconds.	$\frac{d^2\theta}{dt^2}$ radians and seconds.	F_1 or $\frac{M'}{g} \frac{d^2x}{dt^2}$ lbs.	F_2 or $\frac{M'l_0}{g} \left(\frac{d\theta}{dt}\right)^2$ lbs.	F_3 or $\frac{M'l_0}{g} \frac{d^2\theta}{dt^2}$ lbs.	Moment $\frac{I}{g} \frac{d^2\theta}{dt^2}$ lbs. and inches.
0	-33.76	-48.23	1.795	0	-29.36	3.27	0	0
1	-31.93	-45.61	1.739	-2.703	-27.77	3.07	-2.75	-31.1
2	-26.70	-38.14	1.571	-5.342	-23.22	2.50	-5.42	-61.3
3	-18.81	-26.87	1.296	-7.797	-16.36	1.70	-7.91	-87.5
4	-9.41	-13.44	0.926	-9.863	-8.18	0.87	-10.01	-113.3
5	-0.08	-0.11	0.483	-11.269	-0.07	0.24	-11.43	-129.4
6	7.85	11.21	0	-11.770	6.83	0	-11.94	-135.2
7	13.54	19.34	-0.483	-11.269	11.77	0.24	-11.43	-129.4
8	16.91	24.16	-0.926	-9.863	14.70	0.87	-10.01	-113.3
9	18.41	26.30	-1.296	-7.797	16.01	1.70	-7.91	-87.5
10	18.88	26.97	-1.571	-5.342	16.42	2.50	-5.42	-61.3
11	18.91	27.01	-1.739	-2.703	16.44	3.07	-2.75	-31.1
12	18.80	26.86	-1.795	0	16.35	3.27	0	0
13	18.91	27.01	-1.739	2.703	16.44	3.07	2.75	31.1
14	18.88	26.97	-1.571	5.342	16.42	2.50	5.42	61.3
15	18.41	26.30	-1.296	7.797	16.01	1.70	7.91	87.5
16	16.91	24.16	-0.926	9.863	14.70	0.87	10.01	113.3
17	13.54	19.34	-0.483	11.269	11.77	0.24	11.43	129.4
18	7.85	11.21	0	11.770	6.83	0	11.94	135.2
19	-0.08	-0.11	0.483	11.269	-0.07	0.24	11.43	129.4
20	-9.41	-13.44	0.926	9.863	-8.18	0.87	10.01	113.3
21	-18.81	-26.87	1.296	7.797	-16.36	1.70	7.91	87.5
22	-26.70	-38.14	1.571	5.342	-23.22	2.50	5.42	61.3
23	-31.93	-45.61	1.739	2.703	-27.77	3.07	2.75	31.1
24	-33.76	-48.23	1.795	0	-29.36	3.27	0	0

The engine of Table I. was also examined when running at a speed of four revolutions per second. This, of course, had the effect of multiplying $\frac{d\theta}{dt}$ by four, and $\frac{d^2\theta}{dt^2}$ and $\frac{d^2x}{dt^2}$ by sixteen. The Table for this set of circumstances may, therefore, be deduced from Table I. by multiplying column 4 by four, and columns 2, 3, 5, 6, 7, 8, and 9 by sixteen.