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The bisector which cuts the axis of  $X$  outside the segment  $AB$  is

$$(x \sin \alpha - y \cos \alpha - p) - (x \sin \beta - y \cos \beta - q) = 0,$$

reducing to 
$$x \cos \frac{\alpha + \beta}{2} + y \sin \frac{\alpha + \beta}{2} = \frac{p - q}{2 \sin \frac{\alpha - \beta}{2}}$$

or 
$$x \sin \left( \frac{\pi + \alpha + \beta}{2} \right) - y \cos \left( \frac{\pi + \alpha + \beta}{2} \right) = \frac{p - q}{2 \sin \frac{\alpha - \beta}{2}}.$$

[The form  $x \sin \alpha - y \cos \alpha = p$  is used because it saves the necessity of considering the question of sign if  $\alpha$  or  $\beta$  or both are obtuse. It is in other ways a more convenient form than  $x \cos \alpha + y \sin \alpha = p$ .]

If the equations are of the form  $lx + my + n = 0$  and  $Lx + My + N = 0$ ,  $l$  and  $L$  being of the same sign; and if these lines cut  $OX$  in  $A$  and  $B$ , then the bisector which meets the axis of  $X$  between  $A$  and  $B$  is

$$\frac{lx + my + n}{\sqrt{l^2 + m^2}} + \frac{Lx + My + N}{\sqrt{L^2 + M^2}} = 0.$$

This follows immediately from the above.

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**602. [L<sup>1</sup>.14. a.]** A triangle is circumscribed about the parabola  $y^2 = 4ax$ , and two of its vertices move on the confocal  $y^2 = 4(a + \lambda)(x + \lambda)$ . To find the locus of the third vertex.

The pole of  $y_1, y_2$  with regard to  $y^2 = 4ax$  is  $\frac{y_1 y_2}{4a}, \frac{y_1 + y_2}{2}$ ;

$$\therefore \frac{(y_1 + y_2)^2}{4} = 4(a + \lambda) \left( \frac{y_1 y_2}{4a} + \lambda \right);$$

$$\therefore a(x_1 + x_2) = \frac{a + 2\lambda}{2a} y_1 y_2 + 4\lambda(a + \lambda),$$

or 
$$\frac{a + 2\lambda}{2a} y_1 \cdot y_2 = 2a[x_2 + (x_1 - \mu)],$$

where  $a\mu = 4\lambda(a + \lambda)$ .

Thus  $(x_1 - \mu, \frac{a + 2\lambda}{a} y_1)$  is on the tangent at  $y_2$ . Similarly it is on the tangent at  $y_3$ ;  $\therefore$  it is the third vertex, and its locus is

$$y^2 = \frac{(a + 2\lambda)^2}{a^2} y_1^2 = 4(a + \mu)x_1 = 4(a + \mu)(x + \mu).$$

A triangle is inscribed in the ellipse  $x^2/a^2 + y^2/b^2 - 1 = 0$ , and two of its sides touch the ellipse  $x^2/a'^2 + y^2/b'^2 - 1 = 0$ . To find the envelope of the third side.

Condition that a secant of the first ellipse should touch the second ellipse is

$$\frac{a'^2}{a^2} \cos^2 \frac{\alpha + \beta}{2} + \frac{b'^2}{b^2} \sin^2 \frac{\alpha + \beta}{2} = \cos^2 \frac{\alpha - \beta}{2},$$

i.e. 
$$p(1 + \cos \overline{\alpha + \beta}) + q(1 - \cos \overline{\alpha + \beta}) = 1 + \cos(\alpha - \beta) \quad \left( p \equiv \frac{a'^2}{a^2}, q \equiv \frac{b'^2}{b^2} \right),$$

i.e. 
$$(p + q - 1) + (p - q - 1) \cos \alpha \cos \beta - (p - q + 1) \sin \alpha \sin \gamma = 0.$$

Similarly

$$(p + q - 1) + (p - q - 1) \cos \alpha \cos \gamma - (p - q + 1) \sin \alpha \sin \gamma = 0.$$

These are the conditions that the points  $\beta$  and  $\gamma$  should be on the line

$$(p+q-1)+(p-q-1)\frac{x\cos a}{a}-(p-q+1)\frac{y\sin a}{b}=0.$$

This, then, is the equation of the third side, and its envelope is

$$\frac{x^2}{a^2}(p-q-1)^2+\frac{y^2}{b^2}(p-q+1)^2=(p+q-1)^2,$$

or 
$$(p^2-2pq+q^2-2p-2q+1)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+4pq\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0.$$

Similarly for the reciprocal problem, and any other similar problem in which the coordinates of a point on the first conic can be expressed in terms of a variable parameter.

[v. Smith's *Conics*, p. 327, for another solution; cf. Nos. 11, p. 433; 33, p. 439.] N. M. GIBBINS.

**603. [L<sup>1</sup>. 16. a.]** To find the other sides of a quadrilateral determined (1) by a point, line and conic, (2) two lines and conic.

(1) Equation of lines joining  $O(x', y')$  to the points  $A, B$ , in which the line  $L=0$  cuts the conic  $S=0$ , is

$$L^2S-2LL'P+L^2S'=0. \dots\dots\dots(i)$$

If these cut  $S$  again in  $C, D$ , equation of  $CD$  is  $LS'-2L'P=0$ .

If  $AB$  and  $DC$  meet in  $Q$ , equation of  $OQ$  is  $LS'-L'P=0$ .

Again,  $R(x_1y_1)$ , point of intersection of  $AC$  and  $BD$ , is the pole of  $OQ$ ;

$$\therefore \frac{ax_1+hy_1+g}{S'l-L'X'}=\frac{hx_1+by_1+f}{S'm-L'Y'}=\frac{gx_1+fy_1+c}{S'n-L'Z'};$$

$$\text{each ratio}=\frac{\Delta x_1}{S'(Al+Hm+G)-L'\Delta x'}=\text{two similar expressions,}$$

also 
$$=\frac{\Delta L_1}{S'\Sigma-L'^2\Delta}=\frac{P_1}{LS'-L'P}=\frac{S_1}{S'L_1}\equiv K.$$

Applying equation (i) to the point  $(x_1y_1)$ , the equation of  $AC$  and  $BD$  is

$$L^2S-2L_1LP_1+S_1L^2=0,$$

i.e. 
$$L_1S-2LP_1+S'KL^2, \text{ for } S_1=S'L_1K,$$

or 
$$\left(\frac{S'\Sigma}{\Delta}-L^2\right)S-2L(LS'-L'P)+L^2S'=0,$$

i.e. 
$$\frac{S'\Sigma}{\Delta}S=L^2S-2LL'P+L^2S'.$$

(2) Let now 
$$LS'-2L'P\equiv l'x+m'y+n'\equiv M,$$

so that 
$$2L'P=LS'-M \quad \text{and} \quad L'S'+M'=0.$$

Then

$$\frac{X'}{lS'-l'}=\frac{Y'}{mS'-m'}=\frac{Z'}{nS'-n'}=\frac{\Delta x'}{S'(Al+Hm+Gn)-(Al'+Hm'+Gn')}$$

= two similar expressions

$$=\frac{\Delta L'}{S'\Sigma-\Pi}=\frac{\Delta M'}{S'\Pi-\Sigma'}=\frac{\Delta(M'+L'S')}{S'^2\Sigma-\Sigma'}=\frac{P}{S'L-M}=\frac{1}{2L'}.$$

Lines  $AD$  and  $BC$  are  $L^2S+LM=0$  or  $(\Pi-S'\Sigma)S=2\Delta LM$ ,

and lines  $AC$  and  $BD$  are  $L^2S+LM=\frac{S'\Sigma}{\Delta}S$  or  $(\Pi+S'\Sigma)S=2\Delta LM$ .

Combined equation of the four lines is

$$(\Pi S-2\Delta LM)^2=\Sigma\Sigma'S^2, \text{ for } S'^2=\Sigma'/\Sigma.$$

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