On Bicircular Quartics, with a Triple and a Double Focus, and Three Single Foci, all of them Collinear. By HENRY M. JEFFERY, F.R.S.

[*Read Nov.* 11*th*, 1880.]

1. Such a group of quartics with collinear foci may be thus expressed

$$\kappa = (1+dx)(x^2+y^2)+\lambda(x^2+y^2)^2.$$

These bicircular quartics satisfy the relation which connects a point with three single foci $\quad l\rho_1+m\rho_2+n\rho_3 = 0.$

This property defines l, m, n in terms of a, b, c, the distances of the three single foci from the triple focus or origin.

$$\frac{l\sqrt{a}}{b-c} = \frac{m\sqrt{b}}{c-a} = \frac{n\sqrt{c}}{a-b}.$$

These quartics, if non-singular, are of the eighth class, and have eight foci—a triple, a double, and three single foci. But if the quartics are singular and nodal, two single foci unite at a node and disappear, so that the quartics are of the sixth class.

In the limiting position, an acnode and a crunode unite to form a cusp in a quartic of the fifth class, in which a triple and a double focus remain. Hence, it appears that the preceding property of the focal distances is confined to the single foci, and is inapplicable to these bicircular quartics, if nodal.

2. All the nodal quartics of this group are either of the limaçon type—*i.e.*, are unifolium (defined § 14, and exhibited in Fig. 4) quartics in their nascent state; but do not comprise the true limaçon—or are lemniscatoid, and include the true lemniscata, being bifolium quartics in their nascent state. The companion curves in these cases are respectively unifolium and bifolium. The acnodal quartics are connected with the lemniscatoids by pairs of mere ovals, as the parameters gradually change their values. In all other cases the quartic is a mere oval. These varieties are determined by a mutual relation, which exists between the parameters κ and λ, when the quartics are nodal, and is here exhibited as a quintic discriminating curve. (Fig. 1.)

3. To complete the exposition, a second mutual relation between the parameters should be found, when there are in a group of quartics points of undulation or folium-points, at which quartics pass from

folium to non-folium quartics. This mutual relation is here exhibited as a second discriminating quintic. (Fig. 6.)

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4. Let the two forms of the quartic be compared, viz.,

$$\kappa = (1+dx)(x^2+y^2)+\lambda(x^2+y^2)^2 \quad \text{..................(A)},$$
$$l\rho_1+m\rho_2+n\rho_3 = 0,$$

where $\rho_1^2 = (x+a)^2+y^2$, and ρ_2, ρ_3 have like values.

The coefficients of x and x^2 in the latter must vanish. Hence

$$l\sqrt{a}+m\sqrt{b}+n\sqrt{c} = 0,$$
$$(l^2a-m^2b-n^2c)(l^2a^2-m^2b^2-n^2c^2)-2m^2n^2bc(b+c) = 0.$$

If l be eliminated,

$$(m\sqrt{b}+n\sqrt{c})\{m\sqrt{b}(a-b)+n\sqrt{c}(a-c)\} = 0.$$

The first factor gives $a = 0$, the triple focus at the origin, and is irrelevant; from the second, $\dfrac{l\sqrt{a}}{b-c} = \dfrac{m\sqrt{b}}{c-a} = \dfrac{n\sqrt{c}}{a-b}$.

5. If these values of l, m, n be substituted, the non-singular quartic is

$$\rho_1\frac{(b-c)}{\sqrt{a}}+\rho_2\frac{(c-a)}{\sqrt{b}}+\rho_3\frac{(a-b)}{\sqrt{c}} = 0,$$

where $\rho_1^2, \rho_2^2, \rho_3^2$ have the values $(x+a)^2+y^2$......

This may be expanded, and reduced to the form in which the coefficients are functions of the focal distances from the triple focus

$$(a^2+b^2+c^2-2bc-2ca-2ab)(x^2+y^2)^2-2abc(a+b+c+4x)(x^2+y^2)$$
$$+a^2b^2c^2 = 0 \quad \text{............(B)}.$$

Hence $\lambda = -\dfrac{\Sigma(a^2-2bc)}{2abc(a+b+c)}, \quad \kappa = \dfrac{abc}{2(a+b+c)}, \quad -d = \dfrac{4}{a+b+c}.$

COR.—If $b = c$, or the two foci unite in a double focus, $l = 0$, and the form fails; so that the fundamental property of bicircular quartics (§ 1) is inapplicable.

6. To determine the foci in bircular quartics of this group.

First, resume the equation

$$\kappa = (1+dx)(x^2+y^2)+\lambda(x^2+y^2)^2.$$

If S and T are the quartic and sextic invariants, which denote the envelopes of lines $(x\xi+y\eta-1 = 0)$ which cut the quartic equiharmonically and harmonically, the tangential equivalent to this quartic is

$$S^3-27T^2 = 0;$$

$$3S = (2\lambda+\xi^2+\eta^2+d\xi)^2 - \tfrac{3}{4}(\xi^2+\eta^2)\left[(\xi+d)^2+\eta^2\right] - 3\kappa\lambda(\xi^2+\eta^2)^2,$$

$$27T = -(2\lambda+\xi^2+\eta^2+d\xi)^3 + \tfrac{9}{8}(2\lambda+\xi^2+\eta^2+d\xi)(\xi^2+\eta^2)\left[(\xi+d)^2+\eta^2\right]$$
$$+9\kappa(\xi^2+\eta^2)\left\{\lambda^2+\frac{\lambda}{2}(\xi^2+\eta^2+d\xi)-\tfrac{3}{16}d^2\eta^2\right\}.$$

Now, if the terms be first considered which do not contain (κ), the expansion of $\left(\frac{S}{3}\right)^3 - T^2$ gives

$$\tfrac{27}{64}(\xi^2+\eta^2)^2\left[(\xi+d)^2+\eta^2\right]^2\left[4\lambda^2+4\lambda(\xi^2+\eta^2+d\xi)-d^2\eta^2\right].$$

Hence $(\xi^2+\eta^2)^2$ measures the equivalent tangential equation, which is thereby reduced to the eighth class, as was anticipated, since there are at infinity two double points, which will be shown (in § 17) to be flecnodes. The foci are obtained, by rejecting all terms in the equivalent equation, as above reduced, which contain $(\xi^2+\eta^2)$. The remaining terms are

$$\tfrac{27}{64}(d^2+2d\xi)^2(4\lambda^2+4\lambda d\xi-d^2\eta^2)-9\kappa\lambda(2\lambda+d\xi)^4$$
$$+18\kappa(2\lambda+d\xi)^3(\lambda^2+\tfrac{1}{2}\lambda d\xi-\tfrac{3}{16}d^2\eta^2).$$

If, further, $\xi^2-(\xi^2+\eta^2)$ be written for η^2 in this residuum, $(2\lambda+d\xi)^2$ is recognised as the common factor, which accordingly denotes the double focus. The other factor denotes the three single foci

$$(d^2+2d\xi)^2+8\kappa d^2\xi^2(2\lambda+d\xi)\ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}.$$

The origin is a triple focus ($1^3 = 0$), since the curve is a class-octavic, and must have eight foci.

The discriminant of (C) is the discriminant of (A) when $y = 0$, or two apses coincide, viz., the quintic

$$27\kappa d^4-4d^2(1+36\kappa\lambda)+16\lambda(1+4\kappa\lambda)^2=0\ldots\ldots\ldots\ldots\text{(D)}.$$

Hence it appears (1) that for a nodal curve two foci coincide, and (2) that two of the single foci may be imaginary on the axis of collinearity.

The equation (D) exhibits the mutual relation between the parameters when the two apses coincide at a node, and will be used to distinguish the various curves of the bicircular group. But it shall next be shown, by reference to the other quartic form (B), that two foci coincide at a node, and disappear; this is a theorem proved by Prof. Casey.

The double focus may be obtained in the usual way from the equation directly. It also represents the focal conic of Dr. Casey ("On Bicircular Quartics," § 14, Dublin, 1869.) Since the quartic is (B)

$$\frac{b-c}{\sqrt{a}}r_1+\frac{c-a}{\sqrt{b}}r_2+\frac{a-b}{\sqrt{c}}r_3=0,$$

the focal conic is

$$\frac{b-c}{\sqrt{a}}(1+a\xi)^{\frac{1}{2}}+\frac{c-a}{\sqrt{b}}(1+b\xi)^{\frac{1}{2}}+\frac{a-b}{\sqrt{c}}(1+c\xi)^{\frac{1}{2}}=0.$$

It has been shown (§ 5) that the quartic, when expanded, becomes

$$\lambda r^4 + (1+dx)\, r^2 = \kappa \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

So the equation to the focal conic is expanded

$$2\lambda + d\xi = 0.$$

7. Next, to determine the conditions of singularity in the quartic

$$\Sigma\,(a^2-2bc)\,(x^2+y^2)^2 - 2abc\,(a+b+c+4x)\,(x^2+y^2) + a^2b^2c^2 = 0 \ldots\ldots (B).$$

In this case two apses coincide, and $y = 0$.

The resulting quartic will have equal roots, if its quartic and sextic invariants are connected by the condition

$$\left(\frac{S}{3}\right)^3 - T^2 = 0,$$

$$[(a+b+c)^2-3bc]^3 - [(a+b+c)^3 - \tfrac{9}{2}\,\Sigma a^2b]^2 = \tfrac{27}{4}(b-c)^2(c-a)^2(a-b)^2 = 0.$$

Hence, for a critical quartic, two of the three focal distances (a, b, c) must coincide.

8. This condition incidentally gives a solution of the indeterminate equation $\qquad x^2+27y^2 = z^3$, or of $3x^2+y^2 = z^3$,
if $(a+b+c) = 3n$, unless a, b, c are in Arithmetical Progression, when $2\,(a+b+c)^3 = 9\Sigma a^2b$, and the form would fail.

But a direct solution is given by Euler ("Algebra," p. 399), of the general equation $x^2+my^2 = z^3$.

$$x = s\,(s^2-3mt^2), \quad y = (3s^2-mt^2)\,t.$$

This may be identified with the preceding solution, if

$$c-b = 3t+s, \quad b-a = 3t-s.$$

Euler has tabulated the lowest solutions of the particular form

$$3x^2+y^2 = z^3.$$

9. If the curve be nodal, the quartic (B) becomes

$$(a-4b)\,(x^2+y^2)^2 - 2b^2\,(a+2b+4x)\,(x^2+y^2) + ab^4 = 0.$$

Let the origin be transferred from the triple focus to the coincident foci ($b = c$), *i.e.*, let $x = x+b$; the transformed equation shows that the node is at this point

$$\left(\frac{a}{4}-b\right)(x^2+y^2)^2 + b\,[(a-2b)\,x - b^2]\,(x^2+y^2) + ab^2x^2 = 0 \ldots\ldots (E).$$

The several nodal quartics can be conveniently distinguished by this form, but as it does not embrace the companion-curves, which are non-singular, the quartic form (A) is resumed.

10. To exhibit the mutual relation which exists between the parameters κ and λ, when the bicircular quartics in the group under consideration are singular, as the first discriminating curve.

Resuming the quartic

$$\kappa = (1+dx)(x^2+y^2)+\lambda(x^2+y^2)^2 \text{ .... (A)},$$

the conditions of singularity are

$$(1)\ 4\kappa = 2x^2+dx^3,\ y=0,$$
$$(2)\ 4\lambda x^3+3dx+2 = 0.$$

Their eliminant is otherwise obtained in § 6,

$$27\kappa d^4-4d^2(1+36\kappa\lambda)+16\lambda(1+4\kappa\lambda)^2 = 0 \text{ .... (D)}.$$

But the curve is conveniently drawn, and its properties examined, by the implicit equations (1) and (2).

At a singular point on the quintic $\frac{d\kappa}{dx} = 0$, $\frac{d\lambda}{dx} = 0$; there are thus two cusps, one at infinity, when $x = 0$ at the extremity of the (λ) axis of co-ordinates; and another, when $4+3dx = 0$, $27d^2\kappa = 8$, $32\lambda = 9d^2$. The axes of co-ordinates are the asymptotes, and there are no points of inflexion.

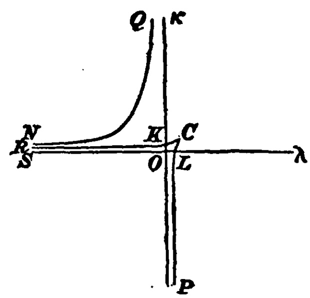

FIG. 1.

The locus of the cusp in all quintics (D) for a family of these groups of bicircular quartics is a rectangular hyperbola.

If d be eliminated from the equations

$$27d^2\kappa = 8,\ 32\lambda = 9d^2,$$

the eliminant is

$$12\kappa\lambda = 1.$$

11. By the aid of this discriminating quintic, all bicircular quartics of this group may be exhibited.

The general equation (E) to the nodal quartic in § 9 will be conveniently used, as well as (D). The constants in the two equations are thus related: $-d = \frac{4}{a+2b}$, $\kappa = \frac{ab^2}{2(a+2b)}$, $-\lambda = \frac{a-4b}{2b^2(a+2b)}$.

In the first quadrant (Fig. 1), if the point (κ, λ) lies on the portion KC, *i.e.*, if the parameters in the quartic (A) are so connected, the quartics are lemniscatoid; *i.e.*, are nodal, but not biflecnodal, while $a > b < 4b$; but if $a = 2b$ in (E), or if $d = \frac{1}{b}$, $\kappa = \frac{1}{4d^2}$, $\lambda = \frac{d^2}{4}$ in (A), the quartic is a lemniscata with a biflecnode

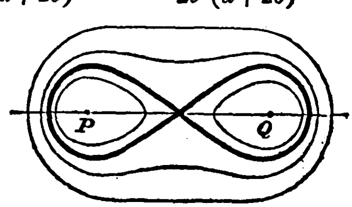

FIG. 2.

$$(x^2+y^2)^2 = 2b^2(x^2-y^2).$$

The companion-curves to the lemniscata, here drawn for higher values of κ, are a bifolium quartic, which passes through another with two folium points (§ 14, Fig. 6) into a pure oval, and for lower values of κ into two ovals, with or without folia, according as (κ, λ) lies beyond or within the second discriminating curve (Fig. 6).

If κ, λ be on the portion LO, and $a < b > 0$, the quartic has an acnode, and, for positions of (κ, λ) beyond the second curve (Fig. 6), becomes a mere oval. At the cusp O of the quintic, this acnode unites with the preceding crunode of a lemniscatoid to form a cusp in the quartic (Fig. 3). In (E), $a = b$, and the only cusped quartic is thus defined,

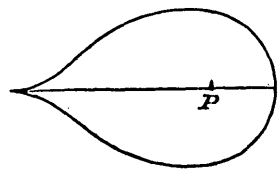

FIG. 3.

$$3(x^2+y^2)^2+4bx(x^2+y^2)+4b^2y^2 = 0.$$

If (κ, λ) is on the axis of κ, $\lambda = 0$, $a = 4b$, the quartic degenerates into the line at infinity and the cubic

$$\kappa = (1+dx)(x^2+y^2),$$

including the nodal curve of form (E), $(2x-b)(x^2+y^2)+4bx^2 = 0$.

These circular cubics are Newton's defective hyperbolæ, with a diameter, species 39, 41, 45; the ampullate cubic represents the folium of the unipartite quartic, and the campaniform the bipartite quartic. (Fig. 5.)

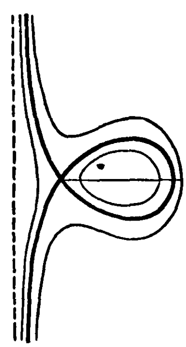

FIG 5.

In the second quadrant the bounding quintic has two branches. If $a+2b < 0$, no nodal quartic is possible. In other words, if (κ, λ) is on or beyond the outer branch NQ of the quintic, no quartic can be drawn. If $a > 4b$, or the point (κ, λ) is on the inner branch RK, the nodal curves are of the limaçon type. But they do not comprise the true limaçon, for the form (E) cannot assume the shape $(x^2+y^2+px)^2 = m^2(x^2+y^2)$.

One companion-curve is unifolium; the other consists of two non-folium ovals, as (κ, λ) lies above or below RK. If, (κ, λ) is on the axis of λ, $\kappa = 0$, the quartic degenerates into a circle and a point circle at the triple-focus. (Fig. 4.)

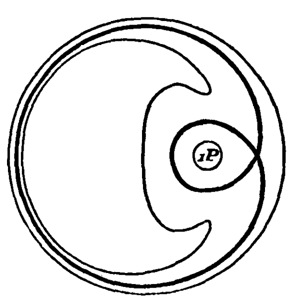

FIG. 4.

In the third quadrant, κ, λ are both negative, there is no critical value, and the quartics are non-folium single ovals. If (κ, λ) is on the axis of κ and negative, the quartic degenerates as before into a circular cubic, and the line at infinity, but the cubic is campaniform or conchoidal only (Newton's, Fig. 49).

In the fourth quadrant, if (κ, λ) lie between the part LP of the quintic and its asymptote, the quartic is a mere oval. If (κ, λ) is on LP, a, b have opposite signs, but $a < 2b$; then no quartic is possible, nor is any possible if (κ, λ) lie beyond LP. See below § 15.

12. If a bicircular quartic of this group be nodal, two foci, which unite at a node (§ 9), disappear. This proposition by Prof. Casey ("Bicircular Quartics," § 44) will be established for the special case of the lemniscata, and similarly proved to be generally true. In equation (E), if $a = b$, $(x^2+y^2)^2 = 2b^2\,(x^2-y^2)$.

S and T, defined in § 6, have these values:

$$12S = \left(\xi^2-\eta^2-\frac{2}{b^2}\right)^2,$$

$$216T = \left(\xi^2-\eta^2-\frac{2}{b^2}\right)^3+\frac{27}{b^2}(\xi^2+\eta^2)^2.$$

The tangential equation to the lemniscata is

$$\left(\frac{S}{3}\right)^3-T^2=0,$$

or

$$\left(\xi^2-\eta^2-\frac{2}{b^2}\right)^3+\frac{27}{2b^2}(\xi^2+\eta^2)^2=0.$$

Its geometrical interpretation is

$$(2qr+b^2)^3 = \tfrac{27}{2}\,b^4p^2,$$

if q, r, p are the perpendiculars on a tangent from the triple foci and node of the lemniscata.

There are therefore two triple foci $(\pm b, 0)$. One of these $(-b, 0)$ is the original triple-focus (since the origin was transferred in § 9), and at the other triple-focus the single-focus has united with the double-focus denoted by $(2\lambda+d\xi)^2$ in § 6. Hence the two single foci, which united at the node, have disappeared.

13. The cusped quartic of this group is a class-quintic, and also retains its triple and double foci. (Fig. 3.)

Its equation, as given in § 11, when the three single foci are made coincident, is $\quad 3\,(x^2+y^2)^2+4bx\,(x^2+y^2)+4b^2y^2=0.$

The invariants S and T have the subjoined values:

$$\frac{S}{12} = (1+b\xi)^2\left(1-\frac{b\xi}{3}\right)^2+\frac{2b^2}{3}(1+b\xi)(\xi^2+\eta^2),$$

$$\frac{T}{8} = (1+b\xi)^3\left(1-\frac{b\xi}{3}\right)^3+b^2(1+b\xi)^2\left(1-\frac{b\xi}{3}\right)(\xi^2+\eta^2)-\frac{b^4}{2}(\xi^2+\eta^2)^2.$$

For the equivalent tangential equation,

$$S^3 = 27T^2.$$

When reduced, this may assume the form

$$\frac{b^4}{4}(\xi^2+\eta^2)^2-\frac{b^2}{27}(35-b\xi)\ (1+b\xi)^2\ (\xi^2+\eta^2)-\tfrac{4}{9}(1+b\xi)^3\left(1-\frac{b\xi}{3}\right)^2=0.$$

The original origin $(-b, 0)$ is still a triple focus, and $\left(\frac{b}{3}, 0\right)$ is a double focus, as defined in § 6.

This is another proposition by Dr. Casey (§ 44), that if a bicircular quartic has a cusp, it arises from the union of three single foci.

Cor.—The inflexions in this pirum or cusped quartic may be thus determined.

When $S=0$, $T=0$, the tangent touches it at a point of inflexion,

$$9-3b\xi+4+4b\xi=0,\quad b\xi+13=0,$$
$$512=b^2(\xi^2+\eta^2),\quad b\eta=\pm 7\sqrt{7}.$$

14. To determine when folium points disappear in a group of quartics, three methods may be adopted.

Two points of inflexion, as well as a bitangent, characterise the depression in a curve, which Dr. Zeuthen (*Mathematische Annalen*, 1874) designates "*folium.*" Hence, first, the Hessian must touch the quartic at the folium-point, in which the two inflexions unite in the nascent state.

Ex.—The bifolium-companion to the lemniscata (Fig. 2)

$$(x^2+y^2)^2 = 2c^2(3x^2+y^2)+3c^4.$$

Its Hessian is

$$(x^2+y^2)^2(3x^2+y^2)+c^2(15x^4+30x^2y^2+7y^4)-3c^4(9x^2+11y^2)+9c^6=0.$$

Secondly, at folium-points, both $S=0$ and $T=0$.

Thirdly, $$\frac{d^2y}{dx^2}=0 \text{ and } \frac{d^3y}{dx^3}=0;$$

or if polar coordinates be used, and $ru=1$,

$$\frac{d^2u}{d\theta^2}+u=0,\quad \frac{d^3u}{d\theta^3}+\frac{du}{d\theta}=0.$$

The two curves touch at the points $(0, \pm c\sqrt{3})$.

15. To exhibit the mutual relation which exists between the parameters, when there are folium-points in this group of bicircular quartics, as a second discriminating curve. (Fig. 6.)

Let the general equation (A) be resumed,

$$\kappa = (1-dx)(x^2+y^2)+\lambda(x^2+y^2)^2.$$

Differentiate thrice; then the conditions of § 13 $\left(\frac{d^2y}{dx^2} = 0 = \frac{d^3y}{dx^3}\right)$ will yield the relations,

$$1-dx = 0,$$

$$d = 4\lambda\left(x+y\frac{dy}{dx}\right),$$

$$d^2 = 8\lambda^2(x^2+y^2)\left\{1+\left(\frac{dy}{dx}\right)^2\right\}.$$

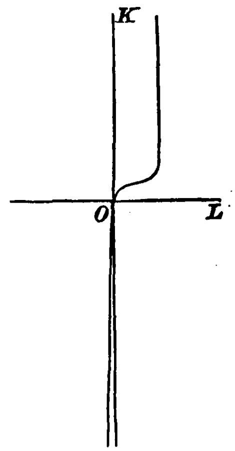

Fig. 6.

The eliminant is a quintic, which is the second discriminating curve.

$$\kappa(d^2+8\lambda)^2 = 4\lambda(8\kappa\lambda+1)^2.$$

If polar coordinates be used, the equation to the group is

$$\cos\theta = \lambda r+\frac{1}{r}-\frac{\kappa}{r^3}.$$

The conditions of § 13 yield the relations from which the quintic may be conveniently drawn as a unicursal curve,

$$\left(4\lambda-\frac{1}{r^3}\right)^2 = \frac{d^2}{r^2}-\frac{1}{r^4}, \quad \kappa = \lambda r^4.$$

If κ, λ are both negative, $\cos\theta$ cannot have a maximum or minimum value; hence, for values of (κ, λ) in the third quadrant, there is no folium.

The relation $(\kappa = \lambda r^4)$ shows that no values of (κ, λ) are in the second and fourth quadrants in Fig. 6.

This quintic has three inflexions, and the axis of (κ) for its asymptote. It intersects the first discriminating curve (Fig. 1, § 10), when $4\kappa d^2=1$, $4\lambda = d^2$; this occurs when the corresponding quartic is the lemniscata (§ 12). It has been shown to give real folium-points only when (λ, κ) is in the first quadrant.

16. Bicircular quartics alone of this group do not admit of four single collinear foci.

All these quartics with collinear foci satisfy the fundamental relation

$$l\rho_1+m\rho_2+n\rho_3 = 0,$$

where the focal distances are ρ_1, ρ_2, ρ_3, as in § 5.

The conditions are two for a fourth single focus or point-circle

$$\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2 = 0,$$

(1) of tangency $$\frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{n^2}{\nu}=0,$$

(2) the discriminant, which determines the focus as the intersection of two imaginary straight lines

$$\frac{(b-c)^2}{\lambda}+\frac{(c-a)^2}{\mu}+\frac{(a-b)^2}{\nu}=0.$$

These conditions suffice to determine a fourth focus, unless

$$\frac{l\sqrt{a}}{b-c}=\frac{m\sqrt{b}}{c-a}=\frac{n\sqrt{c}}{a-b}, \text{ as in § 4 };$$

whence $$\frac{\lambda a}{b-c}=\frac{\mu b}{c-a}=\frac{\nu c}{a-b}.$$

In this case, $$\lambda a+\mu b+\nu c=0,$$
$$\lambda a^2+\mu b^2+\nu c^2=0,$$

and the point-circle becomes the origin $(x^2+y^2=0)$, or is merged in the triple focus.

17. The triple and double foci in each quartic of the group indicate that the tangents at the two nodes at infinity are inflexional and ordinary; hence they are flecnodes; but the lemniscata is triply biflecnodal, since it has two triple foci.

18. The classification of this group would not be altered, if the parameters were differently chosen.

Let the group be otherwise denoted,

$$1=\mu\,(1+dx)\,(x^2+y^2)+\nu\,(x^2+y^2)^2.$$

By comparing the coefficients with those of the former equation,

$$\mu=\frac{1}{\kappa},\quad \nu=\frac{\lambda}{\kappa}.$$

These are Newton's formulæ for homographic transformation of a curve (*Principia*, Bk. I., Lemma 22). In the quintic thus transformed, the critic values would not be affected.

19. The fundamental property $(l\rho_1+m\rho_2+n\rho_3=0)$ of this group is equally applicable to the dual group of class-quartics with quadruple foci $$\kappa=(1+d\xi)\,(\xi^2+\eta^2)+\lambda\,(\xi^2+\eta^2)^2.$$

The distances ρ_1, ρ_2, ρ_3 must now be measured from the foot of the perpendicular on any tangent to three fixed points collinear with the quadruple focus.

In fact, this memoir was written while the author was studying the classification of class-quartics with quadruple foci, which, it is hoped, will shortly find a place in the *Quarterly Journal of Mathematics.*