ON CERTAIN CLASSES OF MATHIEU FUNCTIONS

By E. G. C. POOLE.

[Received March 3rd, 1921.—Read March 10th, 1921.]

1. *Mathieu's Equation.*

We shall be concerned with the well-known equations which arise when the equation of wave-motions in two dimensions is transformed to confocal coordinates. A full account of the equation and its history will be found in Whittaker and Watson's *Modern Analysis*, Ch. xix, so that the following brief recapitulation will be sufficient to remind the reader of the salient facts.

The equation
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + k^2 V = 0$$

is transformed by putting

$$x + iy = a \cosh (\xi + i\eta),$$

and gives
$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + k^2 a^2 (\cosh^2 \xi - \cos^2 \eta) V = 0.$$

Assuming a normal solution of type

$$V = F(\xi) G(\eta),$$

we see that the functions $F(\xi)$ and $G(\eta)$ must satisfy

$$\left. \begin{aligned} \frac{d^2 F}{d\xi^2} + (k^2 a^2 \cosh^2 \xi - p) F &= 0 \\ \frac{d^2 G}{d\eta^2} + (p - k^2 a^2 \cos^2 \eta) G &= 0 \end{aligned} \right\}. \quad (1.1)$$

These are reducible to the same form by interchanging the real and imaginary axes in one of them.

We may also put $\lambda = \cosh \xi$, $\mu = \cos \eta$,

which gives two equations of identical form :

$$\left. \begin{aligned} (\lambda^2 - 1) \frac{d^2 F}{d\lambda^2} + \lambda \frac{dF}{d\lambda} + (k^2 a^2 \lambda^2 - p) F &= 0 \\ (1 - \mu^2) \frac{d^2 G}{d\mu^2} - \mu \frac{dG}{d\mu} + (p - k^2 a^2 \mu^2) G &= 0 \end{aligned} \right\}. \quad (1.2)$$

We can express the original x, y in terms of λ, μ by the formulæ

$$x = a\lambda\mu, \quad y = a\sqrt{(\lambda^2 - 1)(1 - \mu^2)}.$$

A third form, employed by Lindemann, can be obtained by writing $\xi = \lambda^2$ (or μ^2),

$$4\xi(1 - \xi) \frac{d^2 u}{d\xi^2} + 2(1 - 2\xi) \frac{du}{d\xi} + (p - k^2 a^2 \xi) u = 0, \quad (1.3)$$

where u is written for F or G , as the case may be. Let us consider more particularly the form (1.2).

2. Group of the Equation.

Since the equations in λ, μ are identical, we may consider

$$(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + (p - k^2 a^2 x^2) y = 0.$$

This has regular singularities at $x = \pm 1$, the exponents at these being equal by symmetry. These are found to be 0 and $\frac{1}{2}$. The point at infinity is an irregular singularity of rank unity. We shall write the two fundamental solutions at $x = 1$ in the form

$$\left. \begin{aligned} F_1(1-x) &= 1 + a_1(1-x) + \dots + a_n(1-x)^n + \dots \\ F_2(1-x) &= \sqrt{1-x} [1 + b_1(1-x) + \dots + b_n(1-x)^n + \dots] \end{aligned} \right\}. \quad (2.1)$$

These series are convergent within a circle with centre at $x = 1$, and whose circumference grazes the nearest singularity $x = -1$. The radical is taken positively to begin with, at points on the real axis lying between the singular points.

By symmetry, the solutions at $x = -1$ can be expressed in the form $F_1(1+x), F_2(1+x)$, with a similar convention about the sign of $\sqrt{1+x}$.

Now at all points lying within the domain of convergency of *both* sets of solutions, we must be able to express these solutions in terms of two independent ones. Since F_1 and F_2 are independent, we have relations of the form

$$\left. \begin{aligned} F_1(1-x) &= \alpha F_1(1+x) + \beta F_2(1+x) \\ F_2(1-x) &= \gamma F_1(1+x) + \delta F_2(1+x) \end{aligned} \right\}, \quad (2.2)$$

provided that $|1-x| < 2, \quad |1+x| < 2.$

Changing the sign of x and repeating the substitution, we must have identically

$$\left. \begin{aligned} F_1(1-x) &\equiv (\alpha^2 + \beta\gamma) F_1(1-x) + \beta(\alpha + \delta) F_2(1-x) \\ F_2(1-x) &\equiv \gamma(\alpha + \delta) F_1(1-x) + (\beta\gamma + \delta^2) F_2(1-x) \end{aligned} \right\}. \quad (2.3)$$

Hence

$$\left. \begin{aligned} \alpha^2 + \beta\gamma &= \beta\gamma + \delta^2 = 1 \\ \beta(\alpha + \delta) &= \gamma(\alpha + \delta) = 0 \end{aligned} \right\}. \quad (2.4)$$

We must therefore have either

$$(i) \quad \alpha = \delta, \quad \beta = 0, \quad \gamma = 0, \quad \alpha^2 = 1,$$

or

$$(ii) \quad \alpha = -\delta, \quad \beta\gamma = 1 - \alpha^2.$$

We shall discuss these cases *seriatim*.

3. Discussion of Particular Cases.

$$(ia) \quad \alpha = \delta = +1, \quad \beta = \gamma = 0.$$

In this case

$$\left. \begin{aligned} F_1(1-x) &\equiv F_1(1+x) \\ F_2(1-x) &\equiv F_2(1+x) \end{aligned} \right\}, \quad (3.1)$$

provided $|1 \pm x| < 2$, which is certainly true if $|x| < 1$. Now the solutions F_1, F_2 being independent, these equations imply that there are two independent *even* solutions of the equation. The one, which is expressible in the form $F_1(1-x)$ near $x = 1$, and in the form $F_1(1+x)$ near $x = -1$, has no singularity in the finite part of the plane, and is therefore an even integral function. The other, by similar reasoning, takes the form

$$\sqrt{(1-x^2)} \times (\text{even integral function of } x).$$

Now, by considering the solutions near the origin, which is an ordinary

point, we see that there is only *one even* and *one odd* series in x satisfying the equation. Hence the present hypothesis, which requires the existence of *two independent even* solutions, must be rejected.

$$(ib) \alpha = \delta = -1, \beta = \gamma = 0.$$

$$\text{This gives } F_1(1-x) = -F_1(1+x), \quad F_2(1-x) = -F_2(1+x),$$

$$\text{provided that } |1 \pm x| < 2.$$

As above, we can show that this requires the existence of *two independent odd* solutions, which is again impossible.

Both sub-cases of (i) being excluded and no other sub-case being possible, we are left with case (ii), $\alpha + \delta = 0$, $\beta\gamma = 1 - \alpha^2$. This admits an infinity of possible solutions, some of which will be discussed in the next section. But the following particularly simple cases will at once occur to us.

$$(iia) \alpha = -\delta = \pm 1, \beta = 0.$$

$$\text{This gives } F_1(1-x) = \pm F_1(1+x), \quad (3.2)$$

$$\text{provided } |1 \pm x| < 2.$$

It follows that there is *one* solution, either even or odd, which has *no singularity in the finite part of the plane*. This solution is therefore an integral function in x . Putting $x = \cos \theta$, we can express it as a series of cosines of *either even or odd* integral multiples of θ , according as $\alpha = +1$ or $\alpha = -1$ respectively. The series will converge throughout the θ -plane except at infinity.

These functions form half the set discussed by Whittaker, and we may conveniently denote them by $C_{2n}(\theta)$, $C_{2n+1}(\theta)$, because they reduce to $\cos 2n\theta$, or $\cos(2n+1)\theta$, when we make $k^2 a^2 \rightarrow 0$ in Mathieu's equation. The condition that $\beta = 0$ implies that the parameter p must satisfy a certain transcendental equation, and in the limiting case $k^2 a^2 \rightarrow 0$, we have $p = m^2$, where m is an integer. A method of calculating p is given in Whittaker and Watson's treatise, and we shall return to this point below.

$$(iib) \alpha = \pm 1 = -\delta, \gamma = 0.$$

$$\text{Here } F_2(1-x) \equiv \mp F_2(1+x),$$

$$\text{provided } |1 \pm x| < 2. \quad (3.3)$$

In this case the solution which changes sign on describing a small circuit about $x = 1$, will also change sign on describing a small circuit about $x = -1$. Hence there is a solution of the form $\sqrt{1-x^2} \times$ (uniform function of x). Since *neither* solution tends to infinity at $x = \pm 1$, the uniform function of x will be finite everywhere except at infinity, and is therefore an *integral* function. The latter will be *even* if $\delta = +1 = -\alpha$, and *odd* if $\delta = -1 = -\alpha$. Putting $x = \cos \theta$, $\sqrt{1-x^2} = \sin \theta$, and expanding the integral function as a cosine series of integral multiples of θ , all odd or all even, we see that the solution can be written as a *sine series* of integral multiples of θ , all even or all odd respectively. These solutions are the other half of the set found by Whittaker, and we shall denote them by $S_{2n}(\theta)$, $S_{2n+1}(\theta)$, to show that they reduce to $\sin 2n\theta$, or $\sin (2n+1)\theta$, when $k^2\alpha^2 \rightarrow 0$. The values of p again tend to m^2 (m integer), but are not of the same form as those of the $C_m(\theta)$ functions.

Note.—If the two transcendental relations between p and $k^2\alpha^2$, viz. $\beta = 0$, $\gamma = 0$, were simultaneously true, then Mathieu's equation would admit two periodic solutions with period 2π . This could only occur if the value of $k^2\alpha^2$ were a zero of the p -eliminant of β and γ .

(iic) $\alpha = \delta = 0$, $\beta\gamma = 1$.

This interesting case, which appears to have escaped attention, gives

$$\left. \begin{aligned} F_1(1-x) &= \beta F_1(1+x) \\ F_2(1-x) &= \frac{1}{\beta} F_1(1+x) \end{aligned} \right\}, \quad (3.4)$$

provided $|1 \pm x| < 2$.

These relations are deducible from one another by a change in the sign of x .

These relations imply that there is *one* solution which is uniform in the vicinity of $x = 1$, and changes sign when it describes a circuit about $x = -1$; and there is another solution which is uniform at $x = -1$, and changes sign when it describes a circuit about $x = \pm 1$. These two solutions are of the form

$$\sqrt{1-x} \phi(x) \quad \text{and} \quad \sqrt{1+x} \phi(-x), \quad (3.5)$$

where ϕ is an integral function of x . Putting $x = \cos \theta$, the solutions take the form

$$\sin \frac{1}{2}\theta f(\theta), \quad \cos \frac{1}{2}\theta f(\pi - \theta),$$

where $f(\theta)$ is a cosine series converging throughout the θ -plane except at infinity, and proceeding by integral multiples of θ .

On rearranging we find series of the form

$$\sum_{(n)} a_n \cos(n + \tfrac{1}{2}) \theta, \quad \sum_{(n)} (-)^n a_n \sin(n + \tfrac{1}{2}) \theta.$$

We shall denote the solutions of this type by the symbols $C_{n+\frac{1}{2}}(\theta)$, $S_{n+\frac{1}{2}}(\theta)$ to indicate that they reduce to $\cos(n + \frac{1}{2})\theta$ and $\sin(n + \frac{1}{2})\theta$, when $k^2 a^2 \rightarrow 0$. The corresponding values of p tend to $(n + \frac{1}{2})^2$ when $k^2 a^2 \rightarrow 0$. In this case, we also note that the two solutions are *coexistent* for the same value of p , and if one is $F(\theta)$, the other is $F(\pi - \theta)$.

They admit the period 4π .

4. General Case of Periodic Solutions.

Having found that there are solutions admitting the period 4π , we are led to inquire whether there may not also be solutions admitting the period $2s\pi$, where s is any integer. Now when the angle θ increases by 2π , the variation in $x = \cos \theta$ is represented by a single circuit enclosing the points $x = \pm 1$. We therefore consider the effect of such a circuit on the fundamental solutions; we shall premise that in $F_2(1 \pm x)$, the radicals $\sqrt{1 \pm x}$ shall be *positive* at points on the real axis where $-1 < x < 1$.

Consider the solutions u , v , which are defined at A on the upper edge of a cut along the real axis between $x = +1$ and $x = -1$ as

$$u_A = F_1(1-x), \quad v_A = F_2(1-x). \quad (4.1)$$

Proceeding along the upper edge of the cut to B , we get

$$u_B = \alpha F_1(1+x) + \beta F_2(1+x), \quad v_B = \gamma F_1(1+x) + \delta F_2(1+x). \quad (4.2)$$

Now $F_2(1+x)$ changes sign as x describes the path BC . Hence

$$u_C = \alpha F_1(1+x) - \beta F_2(1+x), \quad v_C = \gamma F_1(1+x) - \delta F_2(1+x). \quad (4.3)$$

Returning to D along the lower edge of the cut

$$\left. \begin{aligned} u_D &= (\alpha^2 - \beta\gamma) F_1(1-x) + \beta(\alpha - \delta) F_2(1-x) \\ v_D &= \gamma(\alpha - \delta) F_1(1-x) + (\beta\gamma - \delta^2) F_2(1-x) \end{aligned} \right\}. \quad (4.4)$$

Finally describing the circuit DA , we return to the starting point with the values

$$\begin{aligned}\bar{u}_A &= (\alpha^2 - \beta\gamma) F_1(1-x) - \beta(\alpha - \delta) F_2(1-x) \\ \bar{v}_A &= \gamma(\alpha - \delta) F_1(1-x) + (\delta^2 - \beta\gamma) F_2(1-x)\end{aligned}$$

Now since $\alpha = -\delta$, and $\beta\gamma = 1 - \alpha^2$, we have

$$\begin{aligned}\bar{u} &= (2\alpha^2 - 1)u - 2\alpha\beta v \\ \bar{v} &= 2\alpha\gamma u + (2\alpha^2 - 1)v\end{aligned}\quad (4.5)$$

$$\beta\gamma = (1 - \alpha^2).$$

Let this substitution be reduced to its canonical form, by assuming

$$(A\bar{u} + B\bar{v}) = K(Au + Bv).$$

We shall find, on eliminating $A : B$, the following equation for K

$$\begin{aligned}(2\alpha^2 - 1 - K), \quad 2\alpha\gamma &= 0, \\ -2\alpha\beta, \quad (2\alpha^2 - 1 - K)\end{aligned}$$

that is to say $(2\alpha^2 - 1 - K)^2 + 4\alpha^2(1 - \alpha^2) = 0$,

because $\beta\gamma = 1 - \alpha^2$. This reduces to

$$K^2 + 2K(1 - 2\alpha^2) + 1 = 0.$$

It follows that the two values of K are *reciprocal*, and that their sum is

$$2(2\alpha^2 - 1).$$

Now, if $2\alpha^2 - 1 > 1$, i.e. $\alpha^2 > 1$, both values of K are *real*. But if $\alpha^2 < 1$, both are *complex* and of modulus unity.

If K is real, the two fundamental solutions of form $(Au + Bv)$, $(Cu + Dv)$ will become $K^n(Au + Bv)$ and $K^{-n}(Cu + Dv)$ after n circuits. Such solutions are *not periodic*.

But if K is complex and of modulus unity, it may happen that $K^n = 1$, for an *integer* value of n . In that case, both solutions will regain their initial value after n circuits, or, in terms of θ , they will admit the period $2n\pi$.

Since
$$K = e^{\pm 2\pi i/n},$$

we must have

$$(2\alpha^2 - 1) = \cos\left(\frac{2r\pi}{n}\right). \quad (4.6)$$

This is the transcendental equation which p must satisfy, and when it is satisfied, we shall have solutions of type

$$e^{i\theta/n} f(\theta), \quad e^{-i\theta/n} f(-\theta),$$

where f denotes a function of θ with the period 2π , which remains finite everywhere except at infinity.

If therefore
$$f(\theta) = \sum_{-\infty}^{\infty} c_m e^{m\theta i},$$

the coefficients c_m, c_{-m} must tend to zero like the coefficients of an integral function. This hypothesis will enable us to construct the corresponding solutions, and to obtain the transcendental equations for p .

5. Construction of Solutions.

We have
$$\frac{d^2 y}{d\theta^2} + \left\{ p - \frac{k^2 \alpha^2}{2} (1 + \cos 2\theta) \right\} y = 0, \quad (5.1)$$

where the value of p is at first undetermined. We shall proceed to construct a solution of the form

$$y = e^{i r \theta / s} \sum_{-\infty}^{\infty} c_n e^{i n \theta}, \quad (5.2)$$

on the hypothesis that (c_n) are the coefficients of an integral function, which is finite everywhere except at $\theta \rightarrow \infty$. On substituting the series (5.2) in the equation (5.1), the terms with even and odd suffixes fall into separate groups; since we may add or subtract s in assigning the value of r outside the sign of summation, there is no loss of generality in supposing all the n 's even. On substituting, and picking out the terms multiplying the same power of $e^{i\theta}$, we find that the (c_n) are determined by

$$\frac{k^2 \alpha^2}{4} (C_{2n+2} + C_{2n-2}) = \left\{ p - \frac{k^2 \alpha^2}{2} - \left(2n + \frac{r}{s} \right)^2 \right\} C_{2n}.$$

Let us put
$$L_n \equiv \frac{4}{k^2 \alpha^2} \left\{ \left(2n + \frac{r}{s} \right)^2 + \frac{k^2 \alpha^2}{2} - p \right\}. \quad (5.3)$$

Then
$$C_{2n+2} + L_n C_{2n} + C_{2n-2} = 0. \quad (5.4)$$

We shall solve on the hypothesis

$$\lim_{n \rightarrow +\infty} \frac{C_{2n+2}}{C_{2n}} \rightarrow 0, \quad \lim_{n \rightarrow +\infty} \frac{C_{-2n-2}}{C_{-2n}} \rightarrow 0.$$

We shall follow a method employed by Whittaker in a more restricted

case (see Whittaker and Watson, *Modern Analysis*, 19.52), and which is perhaps most beautifully illustrated in Hough's theory of the tides on a rotating globe (*Phil. Trans.*, A, Vols. 189, 191). It is not necessary to justify the method from a theoretical point of view here. It depends on a theorem in the convergency of continued fractions given by Poincaré, *Les Nouvelles Méthodes de la Mécanique Céleste*, Vol. 2, p. 257.

We divide the equations (5.4) into two groups, for which n is positive or negative respectively, reserving the equation for which $n = 0$ for future use. If we divide by C_{2n} , and solve the one set on the hypothesis

$$\frac{C_{2n+2}}{C_{2n}} \rightarrow 0,$$

and the other on the hypothesis

$$\frac{C_{-2n-2}}{C_{-2n}} \rightarrow 0,$$

(n positive), we have the formulæ

$$\left. \begin{aligned} \frac{C_{2n-2}}{C_{2n}} &= -L_n + \frac{1}{L_{n+1}} - \frac{1}{L_{n+2}} - \dots \text{to } \infty \\ \frac{C_{-2n+2}}{C_{-2n}} &= -L_{-n} + \frac{1}{L_{-n-1}} - \frac{1}{L_{-n-2}} - \dots \text{to } \infty \end{aligned} \right\}. \quad (5.5)$$

Since L_n is $O(n^2)$, these fractions will ultimately converge very rapidly. In the formulæ (5.5), we now put $n = 0$, and substitute in the equation

$$\frac{C_2}{C_0} + L_0 + \frac{C_{-2}}{C_0} = 0.$$

We now find that p must be so chosen that it verifies the relation

$$L_0 = \left\{ \frac{1}{L_1} - \frac{1}{L_2} - \frac{1}{L_3} - \dots \right\} + \left\{ \frac{1}{L_{-1}} - \frac{1}{L_{-2}} - \frac{1}{L_{-3}} - \dots \right\}, \quad (5.6)$$

which is the same as the infinite determinant

$$\begin{vmatrix} 0 & 0 & 1 & L_{-2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & L_{-1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L_0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & L_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & L_2 & 1 \end{vmatrix} = 0.$$

This defines a series of values of p , for which there exists a solution admitting the multiplier $e^{2\pi i/s}$ when θ increases by 2π . It can be proved by a discussion analogous to Hough's, or more directly by the "oscillation theorem," that the values of p will ultimately approximate to the roots of $L_n = 0$, as $n \rightarrow \infty$.

On separating real and imaginary parts, we have two solutions $C_{r/s+h}(\theta)$, $S_{r/s+h}(\theta)$, which reduce to $\cos(r/s+h)\theta$ and $\sin(r/s+h)\theta$ when $k^2\alpha^2 \rightarrow 0$. The index h corresponds to the different values of p satisfying the transcendental equation.

It would be possible to develop a theory of the Mathieu functions admitting the period $2s\pi$, and to show that these form a "complete system" for the expansion of certain classes of functions. The functions of a system would admit an increasing number of nodes and loops in the interval $(0, 2s\pi)$ as p increased, and would possess "orthogonal" properties over this range. The values of p are necessarily real. The discussion and proof of these properties would however exceed our limits, and we shall conclude by giving certain integral equations, some of them already known and others new, satisfied by functions of the types n , $n+\frac{1}{2}$, where n is an integer.

6. Integral Equations.

6.1. In the harmonic analysis, we have to form potential functions and wave functions adapted to many types of curvilinear coordinates. If we expand a function of one type in terms of those of other types, we obtain certain "addition theorems" or expansion formulæ, such as those of the Legendre or Bessel functions.

Now it is possible to expand certain large classes of two-dimensional wave functions in terms of Mathieu functions, the typical term of the expansion being $F(\xi)G(\eta)$, where F , G satisfy the equations (1.1). If the given function is everywhere finite except at infinity, and admits the periods $2s\pi$ for η , and $2s\pi i$ for ξ , it will in general be expansible in Mathieu functions of order n/s . We shall not attempt a rigorous proof of this theorem, but we shall refer the reader to treatises such as Kneser's or Hilbert's on integral equations, where analogous problems are rigorously treated.

Suppose now that our wave function is a *uniform* integral function of x , y . It will be expansible, if at all, in terms of the simplest class of Mathieu functions, those with period 2π (or $2i\pi$ in the case of the second solution).

Now consider the solutions of the equation of wave motion

$$e^{ikx}, \quad e^{iky}, \quad ye^{ikx}, \quad xe^{iky}. \quad (6.11)$$

These are integral functions of x and y . Hence on putting

$$x + iy = a \cosh (\xi + i\eta),$$

we shall look for expansions of the form $\Sigma F_n(\xi) G_n(\eta)$, where

$$F_n(\xi) \equiv G_n(i\xi),$$

and the $G_n(\eta)$ are periodic with period 2π .

If we proceed to determine the coefficients of the expansion

$$f(\xi, \eta) = \Sigma A_n G_n(i\xi) G_n(\eta), \quad (6.12)$$

by means of the well known orthogonal property, we find

$$\int_0^{2\pi} f(\xi, \eta) G(\eta) d\eta = A_n G_n(i\xi) \int_0^{2\pi} G_n^2(\eta) d\eta,$$

or say
$$G_n(z) = a_n \int_0^{2\pi} f(iz, \eta) G_n(\eta) d\eta, \quad (6.13)$$

for $G_n(-z) = \pm G_n(z)$, being either an odd or an even integral function of z . This will be an integral equation for $G_n(z)$.

We cannot however *define* $G_n(z)$ as a system of solutions of (6.13), for the function $f(iz, y)$ need not necessarily require *all* the G_n 's of the complete system in its expansion. With this proviso, we can however set up integral equations satisfied by *an infinite number* of the G_n 's. Our method will be to separate the real and imaginary parts of (6.11) and to replace ξ by $i\xi'$ in the result, to obtain symmetry in the two variables. We thus obtain eight *kernels* $K(\xi', \eta)$. Four of these give integral equations already given by Whittaker, the others appear to be new. Whittaker's kernels are marked with an asterisk in the following table. By considering the nature of the function, whether even or odd and whether proceeding by even or odd multiples of the variables, we can specify the nature of the functions $F_n(\theta)$ satisfying the equation

$$F_n(\theta) = a_n \int_0^{2\pi} K(\theta, \phi) F_n(\phi) d\phi. \quad (6.14)$$

$K(\theta, \phi)$	$F_n(\theta)$	
$\cos(ka \cos \theta \cos \phi)^*$		
$\cosh(ka \sin \theta \sin \phi)^*$	$C_{2n}(\theta)$	
$\sin(ka \cos \theta \cos \phi)^*$		
$\cos \theta \cos \phi \cosh(ka \sin \theta \sin \phi)$	$C_{2n+1}(\theta)$	(6.15)
$\sinh(ka \sin \theta \sin \phi)^*$		
$\sin \theta \sin \phi \cos(ka \cos \theta \cos \phi)$	$S_{2n+1}(\theta)$	
$\cos \theta \cos \phi \sinh(ka \sin \theta \sin \phi)$		
$\sin \theta \sin \phi \sin(ka \cos \theta \cos \phi)$	$S_{2n}(\theta)$	

We notice the appearance of hyperbolic functions in some of the kernels, as the result of putting $i \sin \phi$ for $\sinh \xi$.

6.2. We shall now attempt by similar methods to find an integral equation satisfied by the Mathieu functions with period 4π .

If we suppose p so chosen that the equation in η in (1.1) has a solution with period 4π , or the corresponding equation in μ in (1.2) has a solution of form $\sqrt{(1-\mu)} \phi(\mu)$, where ϕ is an integral function, let us consider the "element"

$$\sqrt{\{(\lambda-1)(1-\mu)\}} \phi(\lambda) \phi(\mu). \quad (6.21)$$

This is a wave function made up of two similar Mathieu functions, and by changing the sign of μ or λ , we can include the case where the second Mathieu function is taken instead of the first. Now in order that a function may be expansible in a convergent series of elements (6.21), we shall restrict ourselves to such functions as are of the form

$$\sqrt{\{(\lambda-1)(1-\mu)\}} \times (\text{symmetric integral function of } \lambda, \mu).$$

Now let us write

$$\xi = \sqrt{\{(\lambda+1)(1+\mu)\}}, \quad \eta = \sqrt{\{(\lambda-1)(1-\mu)\}}. \quad (6.22)$$

Then allowing for a change in sign of λ, μ , the functions required are

* Whittaker's types.

such as can be expanded as an integral function in ξ , η , *even in one variable and odd in the other*.

$$\text{Now we have } \left. \begin{aligned} a(\xi^2 - \eta^2) &= 2(x + a) \\ a\xi\eta &= y \end{aligned} \right\}, \quad (6.23)$$

$$\text{so that } (x + iy) = \frac{a}{2} (\xi + i\eta)^2 - a. \quad (6.24)$$

This change of variable reduces the equation of wave motions to

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + k^2 a^2 (\xi^2 + \eta^2) V = 0. \quad (6.25)$$

And the kernel we require must be an integral solution of this equation.

Now this equation is well known in the theory of the parabolic cylinder [Lamb, *Proceedings*, Vol. 4 (1907); Bateman, *Wave Motions*, p. 98; Whittaker and Watson, *Modern Analysis*, Ch. xvi]. We shall pick out the simplest solution, namely an *even* integral function of ξ , multiplying an *odd* integral function of η . If $V = L(\xi) M(\eta)$, we find

$$\frac{d^2 L}{d\xi^2} + (k^2 a^2 \xi^2 + p)L = 0, \quad \frac{d^2 M}{d\eta^2} + (k^2 a^2 \eta^2 - p)M = 0. \quad (6.26)$$

The choice of p being at our disposal, we make $p = -ika$. Then a solution of the first equation is

$$L(\xi) = e^{\frac{1}{2}ika\xi^2}. \quad (6.27)$$

Now the *even* solution of the second equation is clearly $M = e^{-ika\eta^2}$. This enables us to find the *odd* solution required by elementary reasoning, and we get

$$M(\eta) = e^{-\frac{1}{2}ika\eta^2} \int_0^\eta e^{ikat^2} dt. \quad (6.28)$$

Hence the kernel takes the form

$$e^{ik(x+a)} \int_0^\eta e^{ikat^2} dt.$$

Or, introducing the elliptic coordinates, which we may call θ and $\phi' = i\phi$, to distinguish from our auxiliary variable ξ , η , we have

$$e^{ika \cosh \phi \cos \theta} \int_0^{2 \sinh \frac{1}{2} \phi \sin \frac{1}{2} \theta} e^{ikat^2} dt = i e^{ika \cos \theta \cos \phi'} \int_0^{2 \sin \frac{1}{2} \theta \sin \frac{1}{2} \phi'} e^{-ikat^2} dt.$$

Neglecting constant factors, we shall write

$$\left. \begin{aligned} K_1(u, v) &\equiv e^{ika \cos u \cos v} \int_0^{2 \sin \frac{1}{2}u \sin \frac{1}{2}v} e^{-ikat^2} dt \\ K_2(u, v) &= K_1(u + \pi, v + \pi) = e^{ika \cos u \cos v} \int_0^{2 \cos \frac{1}{2}u \cos \frac{1}{2}v} e^{-ikat^2} dt \end{aligned} \right\}. \quad (6.29)$$

The kernel K_1 will give functions of the type $S_{m+\frac{1}{2}}$, and K_2 will give functions of type $C_{m+\frac{1}{2}}$. In fact, on expanding the kernels in series of the type

$$\left. \begin{aligned} K_1(u, v) &= \sum A_m S_{m+\frac{1}{2}}(u) S_{m+\frac{1}{2}}(v) \\ K_2(u, v) &= \sum B_m C_{m+\frac{1}{2}}(u) C_{m+\frac{1}{2}}(v) \end{aligned} \right\}, \quad (6.291)$$

we have by the usual method of determining the coefficients

$$\left. \begin{aligned} S_{m+\frac{1}{2}}(u) &= \alpha_m \int_0^{2\pi} K_1(u, v) S_{m+\frac{1}{2}}(v) dv \\ C_{m+\frac{1}{2}}(u) &= \beta_m \int_0^{2\pi} K_2(u, v) C_{m+\frac{1}{2}}(v) dv \end{aligned} \right\}, \quad (6.292)$$

these being deducible from one another, by changing u into $(\pi - u)$. We cannot, however, base the whole theory of the functions on the equations (6.292), because it has not been proved, and it appears improbable that it *can* be proved, that *every* function of the complete system is present in the expansion of these kernels.

To investigate the completeness of the system theoretically, we require to form the "Green's function" for the interval $(0, 2\pi)$ and to discuss the integral equation whose kernel is symmetric and continuous, but whose differential coefficient has a discontinuity at $u = v$. Into this question we cannot enter here.

7. Conclusion.

We may remark in conclusion that the methods of this paper can be applied with little more than verbal alterations to a large class of equations, which depend on the equation of wave motions in spheroidal coordinates.

In the case of the prolate spheroid, we put

$$\xi = a\lambda\mu, \quad \rho = a\sqrt{(\lambda^2 - 1)(1 - \mu^2)},$$

and retain the azimuth ϕ . A wave function of the form

$$V = e^{m\phi} F(\lambda) G(\mu),$$

gives us two similar equations in λ, μ for F, G of the form

$$\frac{d}{dx} \left\{ (1-x^2) \frac{dy}{dx} \right\} + \left(p - \frac{m^2}{1-x^2} - k^2 a^2 x^2 \right) y = 0. \quad (7.1)$$

In the case of the oblate spheroid, we put

$$\xi = a\lambda\mu, \quad \rho = a\sqrt{(\lambda^2+1)(1-\mu^2)},$$

and with similar assumptions, we find an equation in μ of the form

$$\frac{d}{dx} (1-x^2) \frac{dy}{dx} + \left(p - \frac{m^2}{1-x^2} + k^2 a^2 x^2 \right) y = 0, \quad (7.2)$$

only differing from the former in the sign of $k^2 a^2$. The corresponding equation in λ is found by putting $\lambda = ix$.

If m is *not an integer*, the theory of these equations presents the closest analogy with that of Mathieu's. They are in fact reducible to the canonical form

$$(1-x^2) \frac{d^2 y}{dx^2} - 2(1 \pm m)x \frac{dy}{dx} + (A + Bx^2) y = 0,$$

of which Mathieu's equation is a special case when $\pm m = -\frac{1}{2}$. The cases where m is an integer require different treatment, owing to the singularities at $x = \pm 1$ becoming logarithmic.