On the Projective Differential Geometry of N -dimensional Spreads Generated by ∞^1 Flats.

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INTRODUCTION.

A ruled surface may be regarded as a two-dimensional spread generated by ∞^1 one-dimensional flats (straight lines). Among the most fundamental projective differential properties of ruled surfaces in three-dimensional space are (1) the distinction between skew and developable surfaces, (2) the relation which exists between a developable surface and its edge of regression, and (3) the incidence relations between a developable and its tangent planes. A space curve S_1 , its tangent developable S_2 , and the locus S_3 of its osculating planes form what I call an ascending series of spreads.

In this paper I have studied the analogous projective differential properties, in n -dimensional space, of m -dimensional spreads generated by ∞^1 ($m - 1$)-dimensional flats, or linear spaces, and of the ascending and descending series of spreads which they determine. The three-dimensional facts, however, are too simple to afford even a hint of the n -dimensional state of affairs. It is only when we reach four-dimensional space that some of the most characteristic features, for example branch series of spreads, begin to appear; and the interesting case in which a secondary series connecting two branch series exists does not occur in space of less than nine dimensions.

Only three writers, so far as I am aware, have ever touched upon this subject. In 1910 SEGRE published a paper in the *Rendiconti di Palermo*, volume 30, page 87, entitled *Preliminari di una teoria delle varietà luoghi di spazi*, in which he devoted four pages to *Varietà ∞^1 di spazi*. He there gave the first brief introduction to the general problem of this paper. A special case in which the given spread determines what I call a regular series of spreads had, however, been considered in 1901 by MORENO in a paper *On*

ruled loci in n -fold space, in the *Proceedings of the American Academy*, volume 37, page 121. In 1905 WILCZYŃSKI, in his book on *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, Teubner, while not considering this very problem except in the three-dimensional case (pages 130-131), set up the corresponding analytical apparatus for the n -dimensional case (pages 6 and 51-54). The system of linear homogeneous differential equations with which he starts is essentially the same as what I call the fundamental linear relations (*).

Homogeneous coordinates and parametric equations have been chosen as the most appropriate tools to use; and the results apply to a large class of non-analytic, as well as analytic, spreads.

PART I.

Elementary Theory, Including a Study of Ascending Branch Series.

THE DIRECTRICES OF A SPREAD.

1. Let the fundamental space within which we are operating be a linear space or flat, F_{n-1} , of $n - 1$ dimensions, which may be Euclidean or non-Euclidean, but is governed by the laws of projective geometry. Let F_{m-1} ($m < n$) be a linear space, or $m - 1$ flat, contained in F_{n-1} . Since F_{m-1} is determined by m independent points, we shall call m its *point-value* (**).

A simply infinite continuous system of $m - 1$ flats will generate an m -dimensional spread (space, variety) S_m , in general non-linear, which we shall call an *m-spread*. These m -spreads are the object of our investigation. For the sake of duality we include the case where $m = n - 1$, although S_m is then not a spread in the ordinary sense of the term.

(*) See also SEGRE, l. c., page 88, equations (4).

(**) Cf. SCHOUTE, *Mehr-dimensionale Geometrie*, volume 1 (1902), page 12.

2. Let A_{ij} ($i = 1, \dots, m$; $j = 1, \dots, n$) be the homogeneous coordinates of m independent points P_i lying in the generator F_{m-1} . If we regard A_{ij} as functions of a parameter ω and introduce the ratios $\alpha_1 : \dots : \alpha_m$ as additional parameters, we can write the parametric equations of S_m in the form

$$X_j = \sum_{i=1}^m \alpha_i A_{ij}(\omega), \quad \text{where } j = 1, \dots, n. \quad (1)$$

If in (1) we let ω be fixed, we obtain the equations of a generator F_{m-1} . If, on the other hand, we let $\alpha_1, \dots, \alpha_m$ be functions of ω , we obtain the equations of a curve enclosed in S_m and meeting each generator in at least one point. Every such curve we call a *directrix* of S_m . In particular, the loci A_i ($i = 1, \dots, m$) of the points P_i form a set of m independent *directrices*, whose equations are

$$X_i = A_{ij}(\omega), \quad j = 1, \dots, n.$$

If these m directrices are given and with them the point-correspondence established on them by the m sets of functions $A_{ij}(\omega)$, then S_m is determined; it is generated by the flats which connect corresponding points of the m curves. In this sense S_m is determined by the m directrices, and we can use the notation

$$S_m = [A_1, \dots, A_m]. \quad (2)$$

The most general transformation to a new parameter $\bar{\omega}$ and to a new set of independent directrices $\bar{A}_1, \dots, \bar{A}_m$ is evidently given by the equations

$$\bar{\omega} = f(\omega) \quad (a)$$

$$A_{kj}(\omega) = \sum_{i=1}^m \alpha_{ki}(\omega) \cdot A_{ij}(\omega) \quad (k = 1, \dots, m; j = 1, \dots, n), \quad (b)$$

where $f(\omega)$ and $\alpha_{ki}(\omega)$ are arbitrary functions of ω , such that the determinant $|\alpha_{ki}(\omega)|$ does not vanish. In particular, if the directrix A_i is left unchanged by the transformation, that is, if $\bar{A}_i = A_i$, then

$$\bar{A}_{ij}(\omega) = \alpha_{ii}(\omega) \cdot A_{ij}(\omega) \quad (j = 1, \dots, n). \quad (c)$$

3. In all that follows we shall assume that the functions $A_{ij}(\omega)$ are continuous and possess a certain finite number of successive derivatives depending on circumstances (See § 44). We denote derivatives by primes.

If on a directrix A_i a point P_i corresponds to a given value of ω , we define the *derivative* (*) of P_i as a point P'_i , whose coordinates are the values of the derivatives of the coordinates of P_i for the same value of ω . The tangent to A_i at the point P_i evidently passes thru P'_i . The position of P'_i on the tangent is obviously changed if A_i is transformed into itself by (c), but not by (a).

TANGENT FLATS.

4. Let us now consider the flat F determined by the m points P_i and their derivatives P'_i . F is the connecting flat of the tangents to the directrices A_i at their corresponding points P_i , and may also be regarded as the connecting flat of the generator F_{m-1} and its consecutive generator. Hence we shall call F the *tangent flat* to S_m along the generator F_{m-1} . It is determined by S_m and F_{m-1} , and is therefore not changed when S_m is transformed into itself by (a) or (b).

5. The point-value of F , being not less than m , shall be denoted by $m+r$. It is evident that $m+r$ is equal to the rank of the matrix

$$M' = \left\| \begin{array}{cccccc} A_{11} & \dots & A_{m1} & A'_{11} & \dots & A'_{m1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{1n} & \dots & A_{mn} & A'_{1n} & \dots & A'_{mn} \end{array} \right\|,$$

of which the first m columns constitute a partial matrix M of rank m . The rank of M' is easily seen to be invariant under the transformations (a) and (b).

The positive integer r is the difference between the rank of M' and the rank of M . We shall call it the *range* of the spread S_m along the generator F_{m-1} , because it increases with the dimensionality $m+r-1$ of the connecting flat F of F_{m-1} and its consecutive generator.

It satisfies the inequalities

$$0 \leq r \leq m, \quad r \leq n - m.$$

(*) See SEGRE, l. c., page 88. WILCZYŃSKI also uses the same concept, l. c., page 54; but applies the term *derivative* to a somewhat different, though related, concept, page 147.

One extreme case, $r = m$, arises when the $2m$ points P_i, P'_i are independent and the tangent flat F is of point-value $2m$; the opposite extreme, $r = 0$, occurs when the points P'_i all lie in F_{m-1} , and the latter may be called a stationary generator, coinciding with its consecutive generator.

THE TANGENT SPREAD.

6. We define the *derivative of a directrix* A as a curve A' generated by the derivatives of the points of A . Let us consider the spread

$$S = [A_1, \dots, A_m, A'_1, \dots, A'_m], \quad (3)$$

determined by *the directrices of S_m and their derivatives*. S is obviously the locus of the tangent flats to S_m along its generators, and so shall be called the *tangent spread of S_m* . Its dimensionality is equal to the maximum point-value $m + r$ of its generators and therefore also equal to the maximum value of the rank of the matrix $M'(\omega)$, which is obtained from M' by regarding its elements as functions of the parameter ω . So we write $S = S_{m+r}$. We call the $2m$ curves A_i, A'_i a *system of fundamental directrices* of S_{m+r} (altho only $m + r$ of them are independent), because every directrix of S_{m+r} is linearly dependent on them.

THE RANGE OF A SPREAD.

7. The quantity r is then the maximum, or general (*), value of the range of S_m along its different generators, and we shall call it simply the *range of S_m* . To denote this, we write $S_m = S_m^*$. So *the range of a spread generated by ∞^1 flats is the difference between its dimensionality and that of its tangent spread*. It depends only on the spread itself and not on the form of its equations.

(*) The term *general* is used, because for the simpler classes of spreads the value of the range is less than its maximum value only along a finite number of generators.

8. If the range of a spread along any particular generator falls below the general value, that generator is *singular*. Hence a singular generator is one along which the tangent flat is of lower point-value than usual. A singular generator along which the range is zero is *stationary*.

9. If an m -spread is of range zero along all its generators, they are all stationary, that is, the spread consists of a single (fixed) $m - 1$ flat; and being $(m - 1)$ -dimensional instead of m -dimensional, it is not, strictly speaking, an m -spread at all. Nevertheless, in order to establish a more complete agreement between the analysis and the nomenclature, we extend the definition of m -spreads to cover this case. Hence we define an m -spread of range zero to be an $m - 1$ flat; in symbols, $S_m^0 = F_{m-1}$.

FOCAL FLATS.

10. Let us now consider those particular directrices of S_m^r , if any, whose derivatives are also directrices of S_m^r . On any such directrix let P be a point corresponding to a particular value of ω and therefore lying in the corresponding generator F_{m-1} . The derivative P' of P will also lie in F_{m-1} and P will lie in the flat of intersection F of F_{m-1} and its consecutive generator G_{m-1} . Conversely, the derivative of every point of F lies in F_{m-1} .

11. We call F the *focal flat* (*) of S_m^r in the generator F_{m-1} . It is invariant under the transformations (a) and (b), § 2. If F_{m-1} is a non-singular generator, the connecting flat of F_{m-1} and G_{m-1} , as we have seen (§§ 4, 5), is of point-value $m + r$, and their flat of intersection is therefore (**) of point-value $m - r$; that is, $F = F_{m-r-1}$. On the other hand, if F_{m-1} is a singular generator, the point-value of the corresponding focal flat F is greater than $m - r$.

THE FOCAL SPREAD.

12. The number of independent *directrices of S_m^r whose derivatives are also directrices of S_m^r* we now see to be $m - r$. They determine a spread S_{m-r} , which we call the *focal spread of S_m^r* , because among its generators are the

(*) This is the *spazio singolare* of SEGRE, l. c., page 89.

(**) See SCHOUTE, l. c., page 14; and BERTINI, *Geometria Proiettiva degli Iperspazi* (1907), page 9, § 11.

focal flats of S_m^r in the non-singular generators of the latter. The focal spread of a developable surface S_2^1 is evidently its edge of regression.

13. As a parallel theorem to that of § 7 we see that *the range of a spread is the difference between its dimensionality and that of its focal spread*. It is clear that the operation of descending from a spread S_m to its focal spread is dual to the operation of ascending from S_m to its tangent spread, while the concept of range is self-dual. Thus if S_m^r and S_{n-m}^{r-m} are correlative spreads, the focal spread, S_{m-r}^r of the first and the tangent spread S_{n-m+r}^r of the second are correlative spreads of the same range.

14. A spread of range zero obviously coincides with its tangent spread and with its focal spread. An m -spread of range m has no focal spread; its consecutive generators do not meet, in general. For the sake of duality, however, we shall indicate this fact by saying that the focal spread of an S_m^m is $S_0^0 = F_{-1}$, just as the tangent spread of an S_{n-m}^{n-m} is $S_n^0 = F_{n-1}$. A simple self-dual example in three-dimensional space is a skew ruled surface S_2^2 , whose tangent spread is S_1^1 and whose focal spread is S_0^0 .

THE EQUATIONS OF THE FOCAL SPREAD.

15. We now proceed to find the equations of the directrices of the focal spread S_{m-r}^r . Since among the $2m$ columns of the matrix $M'(\omega)$ just $m+r$ are linearly independent, there must exist $m-r$ linearly independent relations, identities in ω , of the form

$$\sum_{i=1}^m l_{ki}(\omega) A'_{ij}(\omega) + \sum_{i=1}^m h_{ki}(\omega) A_{ij}(\omega) = 0 \quad \left(\begin{matrix} j=1, \dots, n; \\ k=1, \dots, m-r \end{matrix} \right), \quad (4)$$

which may be written more briefly in the form (*)

$$\sum_{i=1}^m l_{ki} A'_i + \sum_{i=1}^m h_{ki} A_i = 0 \quad (k=1, \dots, m-r), \quad (5)$$

(*) The equations (5) may be regarded as a system of linear differential equations in which ω is the independent variable and $A_1, \dots, A_m$ are the dependent variables. The equations (4) then express the fact that the system (5) possesses the n solutions

$$A_{1j}, \dots, A_{mj} \quad (j=1, \dots, n).$$

and also in the form

$$\frac{d}{d\omega} \sum_{i=1}^m l_{ki} A_i = \sum_{i=1}^m (l'_{ki} - h_{ki}) A_i \quad (k = 1, \dots, m - r). \quad (6)$$

This shows that the directrices of S_m^r represented by the $m - r$ symbols

$$L_k = \sum_{i=1}^m l_{ki} A_i \quad (7)$$

are such that their derivatives are also directrices of S_m^r . Moreover, they are independent, that is, the matrix (l_{ki}) , which has m columns and $m - r$ rows, is of rank $m - r$ for general values of ω ; for if not, it would be possible to eliminate $A'_1, \dots, A'_m$ from the equations (5) and obtain an identical relation among the directrices $A_1, \dots, A_m$, which would contradict the hypothesis that these are independent.

16. Hence the required directrices of the focal spread S_{m-r} are given by (7); in other words,

$$S_{m-r} = [L_1, \dots, L_{m-r}]. \quad (8)$$

Comparing (5) and (7), we see that *if the independent directrices A_i of an m -spread and their derivatives are connected by exactly $m - r$ independent linear relations of the form*

$$f_k (A'_1, \dots, A'_m, A_1, \dots, A_m) = 0,$$

its focal spread is determined by the $m - r$ independent directrices

$$f_k (A_1, \dots, A_m, 0, \dots, 0).$$

ONE SPREAD ENCLOSING ANOTHER.

17. A spread S_m shall be said to be *enclosed* in a spread S_m' , if every generator of S_m lies in, or coincides with, the corresponding generator of S_m' ; every directrix of S_m is then included among the directrices of S_m' . Any spread S_m^r is obviously enclosed in its tangent spread S_{m+r} and encloses its focal spread S_{m-r} . Moreover, it follows from the definitions (§§ 6, 12) that S_m^r

is enclosed in the focal spread S_{m+r-s} of its tangent spread S_{m+r}^* , and dually that S_m^* encloses the tangent spread S_{m-r+t} of its focal spread S_{m-r}^* (*).

18. Hence the range r of a spread S_m^* is equal to, or greater than, the range s of its tangent spread S_{m+r}^* and is also equal to, or greater than, the range t of its focal spread S_{m-r}^* . If $r > s$, S_{m+r-s} is of higher dimensionality than S_m^* , while if $r = s$, the two spreads coincide. Dually, if $r > t$, S_{m-r+t} is of lower dimensionality than S_m^* , while if $r = t$, these two spreads coincide. Consequently, if the first of two spreads of the same range is the tangent spread of the second, the second is the focal spread of the first, and conversely. In this case the two spreads form a regular descending series (See § 24).

THE CONDITIONS OF TANGENCY.

19. We now consider certain relations that may exist between the four spreads

$$\left. \begin{array}{ll} T_1, & T, \\ S, & S^1, \end{array} \right\} \quad (9)$$

where T_1 is the focal spread of T and S^1 is the tangent spread of S . In the first place, if S is enclosed in T_1 , S^1 is enclosed in T , and conversely. For it is evident that either of these conditions is equivalent to saying:

(a) that the directrices of S and their derivatives are directrices of T .

Hence we have the two correlative theorems:

If the tangent spread of a spread S is enclosed in a spread T , then S is enclosed in the focal spread of T .		If the focal spread of a spread T encloses a spread S , then T encloses the tangent spread of S .
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20. Next, suppose that S^1 encloses T . This is equivalent to saying:

(b) that every directrix of T is linearly dependent on the directrices of S and their derivatives.

Finally, suppose that S encloses T_1 . This is equivalent to saying:

(b') that every directrix of T , whose derivative is also a directrix of T , is a directrix of S .

(*) See SEGRE, I. c., Art. 4, page 90.

The conditions (b) and (b') are geometrically, although not analytically, correlative. They are obviously independent of one another. If conditions (a) and (b) are both satisfied, T coincides with S^1 , and conversely. If (a) and (b') are both satisfied, S coincides with T_1 , and conversely. Hence:

<p>Conditions (a) and (b) together constitute a necessary and sufficient condition that T is the tangent spread of S.</p>		<p>Conditions (a) and (b') together constitute a necessary and sufficient condition that S is the focal spread of T.</p>
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Combining these two theorems, we see that T and S form a regular descending series, if and only if the three conditions (a), (b) and (b') are all satisfied.

21. If the range of S is equal to the difference between the dimensionalities of T and S , then either (a) or (b) is alone sufficient to make T the tangent spread of S .

Dually, if the range of T is equal to the difference between the dimensionalities of T and S , either (a) or (b') is sufficient to make S the focal spread of T .

If the ranges of T and S are both equal to the difference between their dimensionalities, and if any one of the three conditions (a), (b), (b') is satisfied, then T and S form a regular descending series.

ASCENDING AND DESCENDING SERIES.

22. We define the *2nd tangent spread* of a spread S as the tangent spread of the tangent spread of S , and set up corresponding definitions for the λ th *tangent spread* and the λ th *focal spread*. The 2nd tangent spread of S_m^r is evidently an S_{m+r+s} ($s \leq r$) and the 2nd focal spread is an S_{m-r-t} ($t \leq r$).

23. A spread and its first, second, etc., tangent spreads form an *ascending series* G of spreads, whose ranges form a non-increasing series of integers. If the series G is continued far enough, a spread of range zero will be reached, namely a fixed flat $S_{m_1}^0 = F_{m_1-1}$ ($m_1 \leq n$), within which all the other spreads of the series are enclosed.

Dually, a spread and its first, second, etc., focal spreads form a *descending series* H of spreads, whose ranges again form a non-increasing series of

integers (*). If continued far enough, H will include a spread of range zero, say $S_{m_2}^0 = F_{m_2-1}$, where $m_2 \geq 0$. If $m_2 > 0$, every other spread in the series is a conical spread, whose generators all enclose the fixed flat F_{m_2-1} ; if $m_2 = 0$, the generators have no fixed flat in common.

REGULAR SERIES.

24. If an ascending series determined by a spread S and ending with a spread T is also a descending series determined by T , that is, if every term of the series is the tangent spread of the preceding term and the focal spread of the succeeding term, the series is said to be *regular* (**). All the spreads of a regular series are of the same range, and their dimensionalities form an arithmetical progression. The whole series is determined by any one of its terms. A simple example of a regular series in three-dimensional space is one that consists of a twisted curve S_1^1 , its tangent surface S_2^1 , and the locus S_3^1 of its osculating planes. An ascending (or descending) series of spreads of the same range is obviously regular; and if the dimensionalities of the i spreads of an ascending (or descending) series form an arithmetical progression, the first $i - 1$ spreads of the series are certainly regular.

THE CONDITIONS FOR AN i TH TANGENT OR FOCAL SPREAD.

25. The theorems of §§ 19-21 can easily be generalized to apply to i th tangent spreads and i th focal spreads. Consider the two series of spreads, one descending and the other ascending:

$$\begin{array}{l} T_i, \quad T_{i-1}, \dots, T_1, \quad T, \quad \} \\ S, \quad S^1, \dots, S^{i-1}, \quad S^i, \quad \} \end{array} \quad (10)$$

(*) The equivalent of this theorem was proved by SEGRE, l. c., p. 91.

(**) Regular series were studied by MORENO in the paper referred to above.

where T_λ is the λ th focal spread of T and S^λ is the λ th tangent spread of S . If S is enclosed in T_i , S' is enclosed in T , and conversely. For either of these conditions is equivalent to saying:

(a) that the directrices of S and their 1st, ..., i th derivatives are all directrices of T .

Hence:

If the i th tangent spread of S is enclosed in T , S is enclosed in the i th focal spread of T .		If the i th focal spread of T encloses S , T encloses the i th tangent spread of S .
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26. To say that S' encloses T is equivalent to imposing the condition:

(b) that every directrix of T is linearly dependent on the directrices of S and their 1st, ..., i th derivatives.

Dually, to say that S encloses T_i is equivalent to the condition:

(b') that every directrix of T , whose 1st, ..., i th derivatives are all directrices of T , is a directrix of S .

<i>Conditions (a) and (b) together constitute a necessary and sufficient condition that T is the ith tangent spread of S.</i>		<i>Conditions (a) and (b') together constitute a necessary and sufficient condition that S is the ith focal spread of T.</i>
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Conditions (a), (b) and (b') together constitute a necessary and sufficient condition that S and T are the first and last terms of a regular ascending series of $i + 1$ terms.

27. If S' and T have equal dimensionalities, either (a) or (b) is alone sufficient to make T the i th tangent spread of S .		If S and T_i have equal dimensionalities, either (a) or (b') is alone sufficient to make S the i th focal spread of T .
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If S and S' have the same dimensionalities as T_i and T , respectively, then any one of the three conditions (a), (b), (b') is alone sufficient to make S and T the first and last terms of a regular ascending series of $i + 1$ terms.

THE TREE DETERMINED BY A SPREAD.

28. If we take the i th tangent spread of the j th focal spread of the k th tangent spread, etc., of a given spread S , and let i, j, k , etc., vary in all possible ways, we shall obtain a finite number of distinct spreads derived from S .

This system of spreads, together with S itself, we call the *tree* determined by S . It includes the ascending series G and the descending series H determined by S , the various *descending branch series* determined by the spreads of G , the *ascending branch series* determined by the spreads of H , the secondary branch series determined by the spreads of the latter, and so on.

In the study of the projective differential properties of spreads generated by a single infinity of flats, it is clear that the structure of the trees determined by them is of fundamental importance. Hence we shall try to investigate the morphology of these trees.

29. Let S^λ denote the λ th tangent spread of S , $S^{\lambda\mu}$ the μ th focal spread of S^λ , $S^{\lambda\mu\nu}$ the ν th tangent spread of $S^{\lambda\mu}$, etc.; similarly, let S_α denote the α th focal spread of S , $S_{\alpha\beta}$ the β th tangent spread of S_α , etc.; further, let $S^{0\mu} = S_\mu$, $S_{0\beta} = S^\beta$, $S_{\alpha 0} = S_\alpha$ (*), etc. It is clear that if a spread S encloses a spread T , $S^{\lambda\mu\nu}$ will enclose $T^{\lambda\mu\nu}$, $S_{\alpha\beta\gamma}$ will enclose $T_{\alpha\beta\gamma}$, and indeed, *every spread of the tree determined by S will enclose the corresponding spread of the tree determined by T .*

30. It has been shown in § 17 that S ($= S_{00}$) encloses $S_{1,0}$ and $S_{1,1}$ and that $S_{1,0}$ encloses $S_{2,0}$ and $S_{2,1}$. By the remark just made $S_{1,1}$, the tangent spread of $S_{1,0}$, encloses $S_{2,2}$, the tangent spread of $S_{2,1}$. Hence S_{00} encloses all the spreads $S_{1,0}$, $S_{1,1}$, $S_{2,0}$, $S_{2,1}$, and $S_{2,2}$. Similarly S_{01} encloses $S_{1,2}$ and $S_{2,3}$, besides all the others just mentioned.

By an easy generalization and the use of duality it follows that

$S_{\lambda\mu}$ encloses every spread S_{ij} for which the conditions $\lambda \leq i$ and $\lambda - \mu \leq i - j$ are satisfied, and that S encloses every spread S_{ij} for which $j \leq i$. In particular, S_{ij} encloses every spread $S_{i+k, j+k}$ ($k > 0$), S_{ii} encloses every spread $S_{i+k, i+k}$, and S encloses every spread S_{ii} .	$S^{\lambda\mu}$ is enclosed in every spread S^{ij} for which the conditions $\lambda \leq i$ and $\lambda - \mu \leq i - j$ are satisfied, and that S is enclosed in every spread S^{ij} for which $j \leq i$. In particular, S^{ij} is enclosed in every spread $S^{i+k, j+k}$ ($k > 0$), S^{ii} is enclosed in every spread $S^{i+k, i+k}$, and S is enclosed in every spread S^{ii} .
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(*) Where there is any danger of confusion, we shall use the symbol S_{a0} for the a th focal spread of S and reserve S_a for a spread of dimensionality a .

THE i TH SPECIAL ENCLOSED (ENCLOSING) SPREAD.

31. In view of the last clause we define
 the i th special enclosed spread of a given spread S as the i th tangent spread S_{ii} of the i th focal spread of S . | the i th special enclosing spread of a given spread S as the i th focal spread S^{ii} of the i th tangent spread of S .

If S coincides with its i th special enclosed (enclosing) spread, it evidently coincides with every intermediate special enclosed (enclosing) spread.

32. It is also easy to see that if the conditions $\lambda \leq i$, $\lambda - \mu \leq i - j$, $\lambda - \mu + \nu \leq i - j + k$ are satisfied, then

$S_{\lambda\mu\nu}$ will enclose S_{ijk} , | $S^{\lambda\mu\nu}$ will be enclosed in S^{ijk} .

Putting $\lambda = k = 0$ in the left-hand theorem, we find that if $\nu - \mu \leq i - j$, $S^{\mu\nu}$ will enclose S_{ij} . Again, if $\mu \geq j - i$ and $\mu - \nu \geq j - i - k$, then $S^{\mu\nu}$ will enclose S_{ijk} , | $S_{\mu\nu}$ will be enclosed in S^{ijk} .

33. As a special case of the last theorem we see that $S^{i,i+j}$, which is the same as $(S^{ii})_j$, always encloses $S_{j,j+i,j+i}$, and that $S_{j,j+i}$, which is the same as $(S_{jj})^i$, is always enclosed in $S^{i,i+j,i+j}$. Hence:

The j th focal spread of the i th special enclosing spread of S encloses the $(i+j)$ th special enclosing spread of the j th focal spread of S . | The i th tangent spread of the j th special enclosed spread of S is enclosed in the $(i+j)$ th special enclosed spread of the i th tangent spread of S .

In other words $S^{i,i+j}$ and $S_{j,j+i}$ are so related that the $(i+j)$ th tangent spread of the first encloses the second, and dually the first encloses the $(i+j)$ th focal spread of the second.

THE EQUATIONS OF THE SECOND TANGENT SPREAD.

34. Returning to the spread $S = S_m^r$, as defined in §§ 1-18, we easily see that its second tangent spread is given by the symbolic equation

$$S^2 = S_{m+r+s} = [A_i, A'_i, A''_i (i = 1, \dots, m)], \quad (11)$$

where all of the directrices A_i , r of the directrices A'_i , and s of the directrices A''_i , are independent; the $3m$ fundamental directrices are connected

by the $m - r$ relations (5), by $m - r$ relations obtained from (5) by differentiation as to ω , and by $r - s$ additional relations due to the fact (if $r > s$) that $S^1 = S_{m+r}^*$ is of lower range than S_m^* . In a similar manner we can write the equations of the third and higher tangent spreads of S_m^* . If the j th tangent spread is of range zero, the $(j + 1)$ st derivatives of all the directrices A_i are linearly dependent on the derivatives of lower order and on the directrices A_i themselves.

THE EQUATIONS OF THE SECOND FOCAL SPREAD.

35. To find the equations of the second and higher *focal* spreads of S_m^* is more difficult. The duality between tangent spreads and focal spreads applies only to the geometrical results and not to the analytical methods we are employing. We could, to be sure, give a dualistic interpretation to our coordinates, letting A_{ij} ($j = 1, \dots, n$) be the coordinates of an $n - 2$ flat instead of a point, and so express all our pairs of correlative theorems in the same analytic form. The equations of the second focal spread $S_{2,0} = S_{m-r-t}$ would then appear in a form similar to (11). But we shall need its equations in point coordinates, and we can find them by treating the equation (8) and the directrices L_k somewhat as the equation (2) and the directrices A_i were treated in §§ 15, 16.

36. To do this we consider the matrix

$$\bar{L} = \left\| \begin{array}{c|c} l_{11} \dots \dots \dots l_{1m} & 0 \\ \dots \dots \dots & \cdot \\ \dots \dots \dots & \cdot \\ l_{m-r,1} \dots \dots \dots l_{m-r,m} & 0 \\ \hline l'_{11} - h_{11} \dots \dots \dots l'_{1m} - h_{1m} & L_1 \\ \dots \dots \dots & \cdot \\ \dots \dots \dots & \cdot \\ l'_{m-r,1} - h_{m-r,1} \dots \dots \dots l'_{m-r,m} - h_{m-r,m} & L_{m-r} \end{array} \right\|$$

and let L be the matrix obtained from $\bar{L}$ by omitting the last column. The first $m - r$ rows of L form a matrix of rank $m - r$, as we saw in § 15, and

L itself is of rank $m - r + t$. Hence the range t of any spread S_{m-r}^t , whose fundamental directrices L_k are independent linear functions, satisfying (7) and (6), of a set of independent directrices A_i is equal to the rank of the matrix L minus the dimensionality of the spread.

37. The $2(m - r)$ rows of L are therefore connected by $m - r - t$ linearly independent relations of the form

$$\sum_{k=1}^{m-r} f_{jk} (l'_{ki} - h_{ki}) + \sum_{k=1}^{m-r} g_{jk} l_{ki} = 0 \quad \begin{pmatrix} i = 1, \dots, m; \\ j = 1, \dots, m - r - t \end{pmatrix}, \quad (12)$$

identities in ω , in which all the quantities involved are functions of ω . If we multiply (12) by A_i and take the summation for $i = 1, \dots, m$, we can then by means of (7) and (6) throw the result into the simple form

$$\sum_{k=1}^{m-r} f_{jk} L'_k + \sum_{k=1}^{m-r} g_{jk} L_k = 0 \quad (j = 1, \dots, m - r - t), \quad (13)$$

which gives a set of independent linear relations between the fundamental directrices of S_{m-r}^t and their derivatives. Accordingly, by the principle of § 16, we see that the required independent directrices of the second focal spread S_{m-r-t} are

$$\sum_{k=1}^{m-r} f_{jk} L_k \quad (j = 1, \dots, m - r - t). \quad (14)$$

38. A somewhat different set of fundamental directrices, expressed as determinants, can be found as follows. The matrix $\bar{L}$ is of rank $m - r + t + 1$, and all its non-vanishing determinants of order $m - r + t + 1$ involve elements taken from the last column and are therefore linear homogeneous functions of $L_1, \dots, L_{m-r}$. It is easy to see that these determinants represent directrices of S_{m-r-t} and that they are precisely sufficient to determine it, $m - r - t$ of them being independent.

A NORMAL SYSTEM OF FUNDAMENTAL DIRECTRICES.

39. In order to penetrate more deeply into the theory of the successive focal spreads of $S_m^r (= S)$ and of the ascending branch series determined by them, it will be necessary to throw the equations of S_m^r into a simple form by finding a *normal system of fundamental directrices*.

Let $t+1$ be the number of spreads in the complete descending series determined by S_m^r , the lowest term of which is of range zero (See § 23). We denote the spreads of the series by

$$S_m^r (= S_{m_t}^{r_t}), S_{m_{t-1}}^{r_{t-1}}, \dots, S_{m_1}^{r_1}, S_{m_0}^{r_0} (= S_{m_0}^0 = F_{m_0-1}), \quad (15)$$

where

$$r_0 (= 0) < r_1 \leq r_2 \leq \dots \leq r_t (= r) \quad (16)$$

and

$$m_0 < m_1 < m_2 < \dots < m_t (= m). \quad (17)$$

$S_{m_t}^{r_t}$ is then the $(t-i)$ th focal spread of S_m^r .

40. If $m_0 > 0$, $S_{m_0}^0$ is a fixed $m_0 - 1$ flat and is therefore determined by m_0 independent fixed points $C_1, \dots, C_{m_0}$, which may be regarded as its fundamental directrices. Symbolically,

$$S_{m_0}^0 = [C_1, \dots, C_{m_0}]. \quad (18)$$

Since $S_{m_0}^0$ is the focal spread of $S_{m_1}^{r_1}$, the theorem of § 13 shows that $m_1 = m_0 + r_1$; and we can choose as independent directrices of $S_{m_1}^{r_1}$ the m_0 fixed points determining $S_{m_0}^0$ and r_1 curves, which we denote by $A_1, \dots, A_{r_1}$. Thus we have

$$S_{m_1}^{r_1} = [C_1, \dots, C_{m_0}; A_1, \dots, A_{r_1}]. \quad (19)$$

If $m_0 = 0$, the fixed points C_λ are absent and $S_{m_1}^{r_1}$ is determined by the directrices A_λ alone.

41. Since the tangent spread $S_{m_1+r_1}$ of $S_{m_1}^{r_1}$ is determined by the directrices of $S_{m_1}^{r_1}$ and their derivatives and since the derivative of C_λ coincides with C_λ , we have (denoting derivatives by upper indices)

$$S_{m_1+r_1} = [C_\lambda (\lambda = 1, \dots, m_0); A_\mu, A_\mu^1 (\mu = 1, \dots, r_1)]. \quad (20)$$

But the number of directrices indicated in this equation, namely $m_0 + 2r_1 = m_1 + r_1$, is exactly equal to the dimensionality of the spread; hence they are all independent. Assuming $t > 1$, we have the equations $m_2 = m_1 + r_2 = m_0 + r_1 + r_2$, $r_2 \geq r_1$; and $S_{m_1+r_1}$ is the first special enclosed spread of $S_{m_2}^{r_2}$. Therefore $S_{m_2}^{r_2}$ is determined by the directrices of $S_{m_1+r_1}$ together with $r_2 - r_1$ additional independent directrices (if $r_2 > r_1$), which we denote by $A_{r_1+1}^1, \dots, A_{r_2}^1$. We write these with upper indices, not because they are the derivatives of other curves with which we are concerned, but merely

for the sake of simplicity in the statement of later theorems (See §§ 68-93). Symbolically, we have

$$S_{m_2}^{r_2} = [C_\lambda (\lambda = 1, \dots, m_0); A_{\mu_1} (\mu_1 = 1, \dots, r_1); A_{\mu_2}^1 (\mu_2 = 1, \dots, r_2)]. \quad (21)$$

If $r_2 = r_1$, $S_{m_2}^{r_2}$ coincides with its first special enclosed spread $S_{m_1+r_1}$ and forms with its first focal spread $S_{m_1}^{r_1}$ a regular series.

42. Ascending further to the tangent spread $S_{m_2+r_2}$ of $S_{m_2}^{r_2}$, which is determined by the directrices indicated in (21) and their derivatives, we find that these derivatives include exactly r_2 additional directrices, namely

$$A_{\mu_2}^2 (\mu_2 = 1, \dots, r_2), \quad (22)$$

which are therefore independent of each other and of the rest. But the tangent spread of $S_{m_1+r_1}$ is similarly determined by adjoining to $S_{m_1+r_1}$ the r_1 additional directrices $A_{\mu_1}^2 (\mu_1 = 1, \dots, r_1)$; and the latter are independent, being included among the directrices (22). Hence $S_{m_1+r_1}$ is of range r_1 and its tangent spread is an $S_{m_1+2r_1}^{r_1}$; that is, $S_{m_1}^{r_1}$ and $S_{m_1+r_1}^{r_1}$ form a regular series, whether $r_2 = r_1$ or $r_2 > r_1$. Assuming $t > 2$, $S_{m_2+r_2}$ and $S_{m_1+2r_1}$ are the first and second special enclosed spreads, respectively, of $S_{m_3}^{r_3}$. As independent directrices of the latter spread we choose those already found for $S_{m_2+r_2}$ together with $r_3 - r_2$ additional curves (if $r_3 > r_2$), which we denote by

$$A_{\mu}^2 (\mu = r_2 + 1, \dots, r_3),$$

and so obtain the symbolic equation

$$S_{m_3}^{r_3} = [C_\lambda (\lambda = 1, \dots, m_0); A_{\mu_1}, A_{\mu_2}^1, A_{\mu_3}^2 (\mu_v = 1, \dots, r_v; v = 1, 2, 3)]. \quad (23)$$

Hence $m_3 = m_0 + r_1 + r_2 + r_3$.

43. Continuing as before, we easily see that $S_{m_2+r_2}$ is of range r_2 and $S_{m_1+2r_1}$ of range r_1 . It follows that $S_{m_2}^{r_2}$ and its tangent spread $S_{m_2+r_2}^{r_2}$ form a regular series of two terms, and that $S_{m_1}^{r_1}$ and its first and second tangent spreads $S_{m_1+r_1}^{r_1}$ and $S_{m_1+2r_1}^{r_1}$ form a regular series of three terms. If $r_3 = r_2 = r_1$, $S_{m_3}^{r_3}$ coincides with its first and second special enclosed spreads, and forms a regular series with its first and second focal spreads.

44. By means of a simple induction proof, which is here omitted, the preceding line of argument can evidently be continued until the original spread S_m^r is reached. Thus we find that any spread $S_m^r (= S)$, whose descending series of focal spreads is given by (15), is determined by a normal

system of independent directrices of the form

$$\left. \begin{array}{l} A_1^{t-1} . . . A_{r_1}^{t-1} . . . A_{r_2}^{t-1} A_{r'_{t-1}}^{t-1} . . . A_{r'_t}^{t-1} \\ A_1^{t-2} . . . A_{r_1}^{t-2} . . . A_{r_2}^{t-2} A_{r'_{t-1}}^{t-2} \\ \\ \\ \\ A_1^1 A_{r_1}^1 A_{r_2}^1 \\ A_1 A_{r_1} \\ C_1 C_{m_o}, \end{array} \right\} \quad (24)$$

The $(t - i)$ th focal spread S_m^r of S is determined by the directrices whose symbols are written in the $i + 1$ bottom rows of this array, and is therefore given by the symbolic equation

$$S_{t-i} = S_{m_i}^{n_i} = [C_\lambda (\lambda = 1, \dots, m_0); A_{\mu_\nu}^{\nu-1} (\mu_\nu = 1, \dots, r_\nu; \nu = 1, \dots, i)] \quad (0 \leq i \leq t), \quad (25)$$

where $A_{\mu}^0 = A_{\mu}$, and where

$$m_i = m_{i-1} + r_i = m_0 + r_1 + \cdots + r_i \quad (i = 1, \dots, t), \quad (26)$$

$$m = m_{t-1} + r = m_0 + r_1 + \cdots + r_t, \quad (27)$$

and

$$m + r \leq n. \quad (28)$$

The equations (18), (19), (21), and (23) are special cases of (25), and the equation of $\mathcal{S}_m^r$ itself is another special case, in which $i = t$.

45. When we ascend from $S_{m_i}^{r_i}$ to its first $t-i$ successive tangent spreads, we merely adjoin to its fundamental directrices their successive derivatives, which are confined to the r_i left-hand columns of the array (24). Thus we find that the j th tangent spread of $S_{m_i}^{r_i}$ ($j \leq t-i$) is of range r_i and is given by the equation

$$\left. \begin{aligned} S_{i-i,j} = S_{m_i+jr_i}^{r_i} = & [C_\lambda(\lambda = 1, \dots, m_0); A_{\nu}^{\nu-1}(\mu_\nu = 1, \dots, r_\nu; \nu = 1, \dots, i); \\ & A_u^{\nu-1}(\mu = 1, \dots, r_i; \nu = i+1, \dots, i+j)]. \end{aligned} \right\} \quad (29)$$

That is, its fundamental directrices are those contained in a rectangle situated in the lower left-hand corner of the array (24) and having r_i columns and $i+j+1$ rows, including, however, the entire bottom row.

In particular, putting $j = t - i$, we obtain the $(t - i)$ -th special enclosed spread $S_{t-i, t-i}$ of S , and we see that its determining directrices are simply all those contained in the r_i left-hand columns of the array, again including the entire bottom row.

46. The tangent spread S_{m+r} of S_m^r is obtained by adjoining to the array (24) an additional row at the top, consisting of the directrices of the t th order $A_1^t, \dots, A_r^t$. The $m + r$ directrices of the augmented array so formed are all independent.

Similarly, the tangent spread $S_{t-i, t-i+1}$ of the $(t - i)$ -th special enclosed spread of S_m^r is determined by the r_i left-hand columns of the augmented array, and is therefore not itself enclosed in S_m^r , unless $r_i = 0$ (Cf. § 30).

Dually, the focal spread $S^{i, i+1}$ of the i th special enclosing spread S^{ii} of a given spread S does not itself enclose S , unless the range of S^{ii} is zero.

GENERAL THEOREMS.

47. By the use of the normal system of directrices just explained (§§ 39-46) we can immediately derive the following theorems (§§ 47-49).

The focal spread S_{10} of a given spread S is also the focal spread of its own tangent spread S_{11} . That is, a focal spread and its tangent spread are of the same range. Hence a given spread and its first special enclosed spread have the same focal spread.

48. The i th focal spread S_{i0} of a given spread S is also the j th focal spread of its own j th tangent spread, provided $j \leq i$.

An i th focal (tangent) spread and its 1st, 2nd, ..., i th tangent (focal) spreads are all of the same range. A given spread and its i th special enclosed (enclosing) spread have the same i th focal (tangent) spread.

49. *The last spread of a descending (ascending) series G of $i + 1$ terms determines an ascending (descending) series whose first $i + 1$ terms form a regular series H . If G is itself regular, it coincides with H . If the last j terms*

The tangent spread S^{10} of a given spread S is also the tangent spread of its own focal spread S^{11} . That is, a tangent spread and its focal spread are of the same range. Hence a given spread and its first special enclosing spread have the same tangent spread.

The i th tangent spread S^{i0} of a given spread S is also the j th tangent spread of its own j th focal spread, provided $j \leq i$.

of G are all of the same range, they are common to G and H . If any term of G is of lower range than the preceding term, it is the starting-point of a regular ascending (descending) *branch series*.

Hence a descending (ascending) series has as many distinct ascending (descending) branch series, all regular, as there are reductions in range in its pairs of successive terms.

If a regular descending (ascending) series of i spreads is continued downward (upward) by a spread S of lower range than the rest, S determines a regular ascending (descending) branch series of $i + 1$ spreads.

SPREADS WHOSE SUCCESSIVE FOCAL SPREADS ARE OF GIVEN RANGE.

50. Given a set of positive integers r_i, m_i, n and t , satisfying the conditions (16), (26), (27) and (28), it is always possible to find in $(n - 1)$ dimensional space an infinite number of spreads S_m^r each having a descending series of $t + 1$ focal spreads of the form (15) (*). For we have only to choose the directrices $A_1, \dots, A_{r_1}$ arbitrarily, except that they and their 1st, ..., t th derivatives are independent, then to choose the next set of directrices $A_{r_1+1}^1, \dots, A_{r_2}^1$ arbitrarily, except that they and their 1st, ..., $(t - 1)$ -st derivatives are independent of each other and of the preceding directrices and their 1st, ..., t th derivatives, and so on. But the total number $m + r$ of directrices and derivatives so obtained is not greater than n ; so their independence is always obtainable. Hence S_m^r can always be found and its determination depends on the choice of r arbitrary directrices.

Dually, it is always possible to find an infinite number of spreads each having an *ascending* series of tangent spreads of given range.

SPREADS HAVING A GIVEN i TH FOCAL SPREAD.

51. On the other hand, the question whether it is always possible to find a spread having a given spread as its focal (or tangent) spread is to be answered in the negative. For if a given spread S_m^r is a focal spread, its

(*) Cf. SEGRE, l. c., page 91.

normal directrices together with the first and also the second derivatives of those of highest order must form an independent system, and this condition is not always satisfied. If it is satisfied, S_m^r is the focal spread of its own tangent spread S_{m+r}^r and also, if $m + 2r \leq n$, of an infinite number of spreads $S_{m+r'}^{r'}$ of range $r' > r$. The determination of the latter depends on the choice of $r' - r$ directrices, which are arbitrary except that they and their first derivatives are independent of one another and of the directrices of the 2nd tangent spread S_{m+2r}^r of S_m^r .

Similar results hold for the more general case in which S_m^r is an i th focal spread.

52. Hence a necessary and sufficient condition that a given spread S_m , of range r , is an i th focal spread is that its i th tangent spread be also of range r , that is, be an S_{m+ir}^r . When this condition is satisfied, S_m^r is the i th focal spread of only one spread of the minimum dimensionality $m + ir$, namely S_{m+ir}^r itself, and of an infinite number of spreads of higher dimensionality m' , and therefore of higher range r' (provided $m' + r' \leq n$), the determination of which depends on the choice of $r' - r$ arbitrary directrices.

Dually, a necessary and sufficient condition that a given spread S_m of range r is an i th tangent spread is that its i th focal spread be also of range r , that is, be an S_{m-ir}^r . When this condition is satisfied, S_m^r is the i th tangent spread of only one spread of the maximum dimensionality $m - ir$, namely S_{m-ir}^r itself, and of an infinite number of spreads of lower dimensionality m' , and therefore of higher range r' , provided $m' - r' \geq 0$.

PART II.

Advanced Theory, Including a Study of the Relations between Ascending and Descending Branch Series.

53. We now proceed to derive the parametric equations of the spreads $S^{\lambda\mu}$ belonging to the various descending branch series determined by the tangent spreads of S . They will help us to discover some of the relations which exist between the spreads $S^{\lambda\mu}$ on the one hand and the spreads $S_{\alpha\beta}$ on the other,

and also to gain additional information about the spreads $S^{\lambda\mu\nu}$ and $S_{\alpha\beta\gamma}$. We shall assume throughout that the equations of S are expressed in terms of the normal system of independent directrices (24).

54. It is obvious that a system of fundamental directrices of S' is obtained by adjoining to the array (24) i additional rows at the top, containing the 1st, 2nd, ..., and i th derivatives of the directrices in the upper row. The directrices of the augmented array so obtained are not necessarily independent, unless $i = 1$ (See § 46). They are independent, if S'^{-1} is of the same range as S ; while if S'^{-1} is of lower range than S , the fundamental directrices of S' are connected by certain identical linear relations. In terms of the coefficients involved in these relations we can write, as we shall see, the equations of the successive focal spreads of S'^{-1} .

Since the treatment of the general case is somewhat complicated by reason of the numerous suffixes that seem to be required, it will tend to clarify the general line of argument, if we give a preliminary discussion of a special case.

ILLUSTRATIVE EXAMPLE, §§ 55-67.

55. Let S be a six-dimensional spread (generated by ∞^1 5-flats) of range 3, whose first and second focal spreads are of ranges 2 and 1, respectively, and whose first and second tangent spreads are of ranges 3 and 2, respectively. In the notation of §§ 39-45, $t = 3$, $r_1 = 1$, $r_2 = 2$, $r_3 = r = 3$, $m_0 = 0$, $m_1 = 1$, $m_2 = 3$, $m_3 = m = 6$. Hence

$$S = S_6^3 = [A_1; A_1^1, A_2^1; A_1^2, A_2^2, A_3^2], \quad (30)$$

and its normal directrices are those indicated in the three bottom rows of the array

$$\left. \begin{array}{ccc} A_1^5 & A_2^5 & A_3^5 \\ A_1^4 & A_2^4 & A_3^4 \\ A_1^3 & A_2^3 & A_3^3 \\ A_1^2 & A_2^2 & A_3^2 \\ A_1^1 & A_2^1 & \\ A_1 & & \end{array} \right\} \quad (31)$$

Then

$$S_{1,0} = S_3^2 = [A_1; A_1^1, A_2^1] \quad \text{and} \quad S_{2,0} = S_1^1 = [A_1]. \quad (32)$$

Moreover

$$\left. \begin{aligned} S^{1,0} = S_9^3 &= [A_1; A_1^1, A_2^1; A_\lambda^2, A_\lambda^3 (\lambda = 1, 2, 3)], \\ S^{2,0} = S_{12}^2 &= [A_1; A_1^1, A_2^1; A_\lambda^2, A_\lambda^3, A_\lambda^4 (\lambda = 1, 2, 3)], \\ S^{3,0} = S_{14} &= [A_1; A_1^1, A_2^1; A_\lambda^2, A_\lambda^3, A_\lambda^4, A_\lambda^5 (\lambda = 1, 2, 3)]. \end{aligned} \right\} \quad (33)$$

The twelve fundamental directrices of S_{12}^2 are those written in the five bottom rows of the array (31) and are independent, while the fifteen fundamental directrices of S_{14} are those of the entire array and are connected by one linear relation.

The dimensionality $n - 1$ of the fundamental flat F_{n-1} within which S is enclosed must be at least equal to 13; if it equals 13, S_{14} is of range zero and coincides with F_{n-1} .

TRANSFORMATION OF THE LINEAR RELATION.

56. The linear relation just mentioned we write in the form

$$\sum h_{0\lambda} A_\lambda^5 + \sum h_{1\lambda} A_\lambda^4 + \sum h_{2\lambda} A_\lambda^3 + \sum h_{3\lambda} A_\lambda^2 + (h_{41} A_1^1 + h_{42} A_2^1) + h_{51} A_1 = 0, \quad (34)$$

where the summations hold for $\lambda = 1, 2, 3$. The coefficients $h_{\beta\lambda}$ as well as the quantities A_λ^5 are all functions of ω , and the equation is an identity in ω . Upper indices, as before, denote differentiation as to ω . Obviously the three leading coefficients h_{01} , h_{02} , h_{03} cannot all be identically zero.

It will be convenient for our purpose to transform that part of (34) which involves the second and higher derivatives, and write the equation in the form

$$\left. \begin{aligned} \frac{d^3}{d\omega^3} \sum l_{0\lambda} A_\lambda^3 + \frac{d^2}{d\omega^2} \sum l_{1\lambda} A_\lambda^2 + \frac{d}{d\omega} \sum l_{2\lambda} A_\lambda^2 + \sum l_{3\lambda} A_\lambda^2 \\ + (h_{41} A_1^1 + h_{42} A_2^1) + h_{51} A_1 = 0. \end{aligned} \right\} \quad (35)$$

57. We wish to show that this is always possible and to find the equations of transformation. That is, we wish to show that twelve quantities $l_{\rho\lambda}$, functions of ω , can always be found to satisfy the equation

$$\sum_{\rho=0}^3 \left(\sum_{\lambda=1}^3 l_{\rho\lambda} A_\lambda^3 \right)^{3-\rho} \equiv \sum_{\beta=0}^3 \sum_{\lambda=1}^3 h_{\beta\lambda} A_\lambda^5 \omega^{-\beta}, \quad (36)$$

which is an identity in the twelve quantities A_λ^δ ($\lambda = 1, 2, 3$; $\delta = 2, 3, 4, 5$) as well as in ω .

The left-hand member, by means of LEIBNITZ'S theorem on the derivatives of a product, becomes

$$\begin{aligned} \Sigma l_{0\lambda} A_\lambda^3 + \binom{3}{1} \Sigma l_{0\lambda}^1 A_\lambda^4 + \binom{3}{2} \Sigma l_{0\lambda}^2 A_\lambda^5 + \Sigma l_{0\lambda}^3 A_\lambda^6 \\ + \Sigma l_{1\lambda} A_\lambda^4 + \binom{2}{1} \Sigma l_{1\lambda}^1 A_\lambda^5 + \Sigma l_{1\lambda}^2 A_\lambda^6 \\ + \Sigma l_{2\lambda} A_\lambda^5 + \Sigma l_{2\lambda}^1 A_\lambda^6 \\ + \Sigma l_{3\lambda} A_\lambda^6. \end{aligned}$$

Hence, by equating the coefficients of the quantities A_λ^δ , we derive the equations

$$\left. \begin{aligned} l_{0\lambda} &= h_{0\lambda}, \\ \binom{3}{1} l_{0\lambda}^1 + l_{1\lambda} &= h_{1\lambda}, \\ \binom{3}{2} l_{0\lambda}^2 + \binom{2}{1} l_{1\lambda}^1 + l_{2\lambda} &= h_{2\lambda}, \\ l_{0\lambda}^3 + l_{1\lambda}^2 + l_{2\lambda}^1 + l_{3\lambda} &= h_{3\lambda}, \end{aligned} \right\} \quad (\lambda = 1, 2, 3) \quad (37)$$

which may be written compactly in the form

$$\sum_{\rho=0}^{\beta} \binom{3-\rho}{\beta-\rho} l_{0\lambda}^{\rho} = h_{\beta\lambda} \quad (\lambda = 1, 2, 3; \beta = 0, 1, 2, 3). \quad (37')$$

Regarding the quantities $l_{\rho\lambda}$ as unknown, we easily show that these equations, which are differential equations in form, but algebraic in essence, have one and only one solution, namely

$$\left. \begin{aligned} l_{0\lambda} &= h_{0\lambda}, \\ l_{1\lambda} &= -\binom{3}{1} h_{0\lambda}^1 + h_{1\lambda}, \\ l_{2\lambda} &= \binom{3}{2} h_{0\lambda}^2 - \binom{2}{1} h_{1\lambda}^1 + h_{2\lambda}, \\ l_{3\lambda} &= -h_{0\lambda}^3 + h_{1\lambda}^2 - h_{2\lambda}^1 + h_{3\lambda}, \end{aligned} \right\} \quad (\lambda = 1, 2, 3) \quad (38)$$

which can be written

$$l_{\rho\lambda} = \sum_{\gamma=0}^{\rho} (-1)^{\rho-\gamma} \binom{3-\gamma}{\rho-\gamma} h_{\rho\lambda}^{\rho-\gamma} \quad (\lambda = 1, 2, 3; \rho = 0, 1, 2, 3). \quad (38')$$

58. We wish to verify this fact by a method which is sufficiently general to apply without essential change to the general case to be considered later (§ 75), and which will make a restatement of the argument unnecessary at that point.

Taking the $(\beta - \rho)$ -th derivative of both sides of (38') and substituting the value of $l_{\rho\lambda}^{\rho-\rho}$ so found in (37'), we obtain

$$h_{\rho\lambda} = \sum_{\rho=0}^{\beta} \sum_{\gamma=0}^{\rho} (-1)^{\rho-\gamma} \binom{3-\gamma}{\rho-\gamma} \binom{3-\rho}{\beta-\rho} h_{\rho\lambda}^{\rho-\gamma} \quad (\lambda = 1, 2, 3; \beta = 0, 1, 2, 3). \quad (39)$$

Here $\rho \geq \gamma$, and when $\rho = \gamma$, γ varies from zero to β ; hence we can rearrange the terms and write (39) in the form

$$h_{\rho\lambda} = \sum_{\gamma=0}^{\beta} \left[\sum_{\rho=\gamma}^{\beta} (-1)^{\rho-\gamma} \binom{3-\gamma}{\rho-\gamma} \binom{3-\rho}{\beta-\rho} \right] h_{\rho\lambda}^{\rho-\gamma}. \quad (40)$$

But (40) is identically true, because by a well-known combinatory formula (*) the coefficient of $h_{\rho\lambda}^{\rho-\gamma}$ in the right-hand member vanishes, except when $\gamma = \beta$.

Consequently the equations (38') constitute a solution (and evidently the only one) of the equations (37'), and therefore also of the equation (36); and by their means every linear relation (34) can be written in the form (35).

THE SUCCESSIVE FOCAL SPREADS OF $S^{2,0}$.

59. We now introduce three auxiliary quantities

$$\left. \begin{aligned} L_{22} &= (\sum l_{0\lambda} A_{\lambda}^2)^2 + (\sum l_{1\lambda} A_{\lambda}^2)' + \sum l_{2\lambda} A_{\lambda}^2, \\ L_{12} &= (\sum l_{0\lambda} A_{\lambda}^2)' + \sum l_{1\lambda} A_{\lambda}^2, \\ L_{02} &= \sum l_{0\lambda} A_{\lambda}^2, \end{aligned} \right\} \quad (41)$$

where the summations are again for $\lambda = 1, 2, 3$.

(*) See NETTO, *Combinatorik* (1901), page 16, second formula, putting $n = 3 - \gamma$ and $m = \beta - \gamma$.

These will enable us to write the equations of the 1st, 2nd and 3rd focal spreads of $S^{2,0}$, whose dimensionalities and ranges, as determined by the theorems of §§ 48 and 49, are indicated by the notation $S^{2,1} = S_{10}^2$, $S^{2,2} = S_8^2$, $S^{2,3} = S_6$; the range of S_6 is undetermined, but must be ≤ 2 .

We wish, namely, to prove that their equations are as follows ($S^{2,0}$ is included for the sake of completeness):

$$\left. \begin{aligned} S^{2,0} = S_{12}^2 &= [A_1, A_1^1, A_2^1; A_2^2, A_2^3, A_2^4 \ (\lambda = 1, 2, 3)], \\ S^{2,1} = S_{10}^2 &= [A_1, A_1^1, A_2^1; A_2^2, A_2^3 \ (\lambda = 1, 2, 3); L_{02}^2], \\ S^{2,2} = S_8^2 &= [A_1, A_1^1, A_2^1; A_2^2 \ (\lambda = 1, 2, 3); L_{02}^1, L_{12}^1], \\ S^{2,3} = S_6 &= [A_1, A_1^1, A_2^1; L_{02}, L_{12}, L_{22}]. \end{aligned} \right\} \quad (42)$$

60. *Proof.* Provisionally, let us denote the spreads defined by the 2nd, 3rd and 4th systems of curves of (42) by the symbols (42_2) , (42_3) and (42_4) . Their fundamental directrices, as so given, are independent, because the fundamental directrices of $S^{2,0}$ are independent, and the coefficients l_{01} , l_{02} , l_{03} cannot all vanish. Accordingly, the dimensionalities of (42_2) , (42_3) and (42_4) are equal to 10, 8, and 6, respectively.

It will be sufficient, therefore, in view of the first theorem (right-hand) of § 27, to prove that the condition (a) of § 25 is satisfied in all three cases. Indeed, if it is satisfied in the case of the spread (42_4) , it obviously will be in the other two cases. Hence we merely have to show that the fundamental directrices of (42_4) and their 1st, 2nd and 3rd derivatives are directrices of $S^{2,0}$. Part of this statement is immediately evident and the rest becomes so by means of the relation (35). For instance, L_{12} and L_{12}^2 are directrices of $S^{2,0}$, because they involve no derivatives of the quantities A_λ higher than the 4th; while L_{12}^3 is expressible, by means of (35), in terms of derivatives not higher than the 3rd; and L_{12}^4 , therefore, is expressible in terms of derivatives not higher than the 4th. A similar argument holds for the other fundamental directrices of (42_4) , and the proof is complete.

THE SUCCESSIVE TANGENT SPREADS OF $S_{1,0}$ AND OF $S_{2,0}$.

61. Since the entire tree determined by S is also determined by $S^{1,0}$ (although the corresponding spreads are not the same), therefore the ascending branch series determined by $S_{1,0}$, which is in this case the 2nd focal spread

of $S^{1,0}$, is regular for at least three terms. The first four terms of this series are given by the equations

$$\left. \begin{aligned} S_{1,0} &= S_2^2 = [A_1; A_\lambda^1 (\lambda = 1, 2)], \\ S_{1,1} &= S_5^2 = [A_1; A_\lambda^1, A_\lambda^2 (\lambda = 1, 2)], \\ S_{1,2} &= S_7^2 = [A_1; A_\lambda^1, A_\lambda^2, A_\lambda^3 (\lambda = 1, 2)], \\ S_{1,3} &= S_9 = [A_1; A_\lambda^1, A_\lambda^2, A_\lambda^3, A_\lambda^4 (\lambda = 1, 2)]; \end{aligned} \right\} \quad (43)$$

their fundamental directrices all lie in the first two columns of the array (31).

The range of $S_{1,3}$ may be 2 or 1, but not 0. For if it were 0, the directrices of the first two columns of (31) would be connected by two independent linear relations; whereas the directrices of the entire array are connected by only one relation, namely (35). Hence $S_{1,3}$ is of range 2, unless the condition

$$l_{03} = l_{13} = l_{23} = l_{33} = 0 \quad (44)$$

is satisfied, in which case it is of range 1.

62. Similarly, the ascending branch series determined by $S_{2,0}$ is regular for at least four terms, and its first five terms are given by the equations

$$\left. \begin{aligned} S_{2,0} &= S_1^1 = [A_1], \\ S_{2,1} &= S_2^1 = [A_1, A_1^1], \\ S_{2,2} &= S_3^1 = [A_1, A_1^1, A_1^2], \\ S_{2,3} &= S_4^1 = [A_1, A_1^1, A_1^2, A_1^3], \\ S_{2,4} &= S_5 = [A_1, A_1^1, A_1^2, A_1^3, A_1^4]. \end{aligned} \right\} \quad (45)$$

S_5 is of range 1, unless the identical relation (34) or (35) involves only the elements of the first column of (31), in which case it is of range 0.

63. We have now found all the spreads, seventeen in number, of the tree determined by S , so far as they can be found merely by the given definition (§ 55). Their dimensionalities are indicated schematically in the adjoining table. The remaining spreads of the tree depend for their number and distribution on the further specialization of S ; but the amount of variation in the different distributions is narrowly restricted, as we shall see.

CONNECTING SERIES.

64. Taking up the case in which (44) is satisfied and $S_{1,3}$ is of range 1 ($=S_9^1$), we see that $S_{1,3}$ will then determine a secondary descending branch series of which the first four terms are regular. Their equations can obviously be obtained by the method just explained (§ 59) for the descending series determined by $S^{2,0}$, and are as follows:

$$\left. \begin{aligned} S_{1,3} &= S_9^1 = [A_1; A_\lambda^1, A_\lambda^2, A_\lambda^3, A_\lambda^4 \ (\lambda = 1, 2)], \\ S_{1,3,1} &= S_8^1 = [A_1; A_\lambda^1, A_\lambda^2, A_\lambda^3 \ (\lambda = 1, 2); L_{02}^2], \\ S_{1,3,2} &= S_7^1 = [A_1; A_\lambda^1, A_\lambda^2 \ (\lambda = 1, 2); L_{02}^1, L_{12}^1], \\ S_{1,3,3} &= S_6^1 = [A_1; A_\lambda^1 \ (\lambda = 1, 2); L_{02}, L_{12}, L_{22}]. \end{aligned} \right\} \quad (46)$$

In the definition (41) of the quantities L_{02} , L_{12} and L_{22} involved in (42) and (46) it is to be observed that in view of (44) the summations can now be limited to $\lambda = 1, 2$.

Comparing (42) and (46) we discover the remarkable fact that $S^{2,3}$ coincides with $S_{1,3,3}$ and is therefore of range 1 ($=S_6^1$). Hence $S^{2,3}$ and $S_{1,3}$ ($=S_9^1$), which belong to the two distinct branch series (42) and (43), respectively, are in this case connected by still another series (46), which is regular and determined by both spreads, and which we may call their *connecting series*. Consequently $S^{2,3}$ is the the 3rd special enclosing spread of $S_{1,0}$ ($=S_3^2$), and their common 3rd tangent spread is $S_{1,3}$; dually, $S_{1,3}$ is the 3rd special enclosed spread of $S^{2,0}$ ($=S_{12}^2$), and their common 3rd focal spread is $S^{2,3}$.

65. If, however, $S_{1,3}$ is of range 2, the principle of duality shows that $S^{2,3}$ is also of range 2. Hence $S^{2,3}$, like $S_{1,3}$, may be of range 2 or 1, but not 0; and the two spreads are either both of range 2, or both of range 1.

To find additional terms of the three ascending series (33), (43) and (45), when the ranges of their respective spreads $S^{3,0}$, $S_{1,3}$ and $S_{2,4}$ are known, is a simple matter, requiring no further mention.

To find additional terms of the descending series (42) is more difficult and requires, in general, the use of the principle of § 16, somewhat as was done in §§ 36-38.

66. Thus, to find the equations of $S^{2,4}$, we are led to consider the

matrix

$$\bar{L} = \begin{vmatrix} A_1 & A_1^1 & A_2^1 & A_1^2 & A_2^2 & A_3^2 & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & A_1^1 \\ 0 & 0 & 0 & 0 & 1 & 0 & A_2^1 \\ 0 & 0 & 0 & l_{01} & l_{02} & l_{03} & 0 \\ 0 & 0 & 0 & l_{11} & l_{12} & l_{13} & -L_{02} \\ 0 & 0 & 0 & l_{21} & l_{22} & l_{23} & -L_{12} \\ h_{31} & h_{41} & h_{42} & l_{31} & l_{32} & l_{33} & -L_{22} \end{vmatrix}$$

and the partial matrix L formed by the first six columns of $\bar{L}$. Letting ρ be the range of $S^{2,3}$, we see that $4 + \rho$ and $5 + \rho$ are the ranks of the matrices L and $\bar{L}$, respectively. The non-vanishing determinants of order $5 + \rho$ of $\bar{L}$ are directrices of $S^{2,4}$ and $5 - \rho$ of them are independent; together with A_1 they form a system of fundamental directrices of $S^{2,4}$.

For instance, if $\rho = 2$, we easily find that

$$S^{2,4} = S_4 = [A_1, M_0, M_1, M_2], \quad (47)$$

where

$$\left. \begin{aligned} M_0 &= l_{03} L_{02} + (l_{03} l_{11} - l_{13} l_{01}) A_1^1 + (l_{03} l_{12} - l_{13} l_{02}) A_2^1, \\ M_1 &= l_{03} L_{12} + (l_{03} l_{21} - l_{23} l_{01}) A_1^1 + (l_{03} l_{22} - l_{23} l_{02}) A_2^1, \\ M_2 &= l_{03} L_{22} + (l_{03} l_{31} - l_{33} l_{01}) A_1^1 + (l_{03} l_{32} - l_{33} l_{02}) A_2^1. \end{aligned} \right\} \quad (48)$$

ANOTHER TRANSFORMATION OF THE LINEAR RELATION.

67. On the other hand, if $S^{2,3}$ is of range 1, we have also at our disposal an entirely different and more advantageous method of finding $S^{2,4}$ (in this case an S_5), in which the identical relation (34) is treated in a new way. We now write (34) in the form

$$\sum_{\rho=0}^4 \left(\sum_{\lambda=1}^2 b_{\rho\lambda} A_\lambda^1 \right)^{4-\rho} + h_{31} A_1 = 0 \quad (49)$$

instead of the form (35). That is, we employ the derivatives of linear homogeneous functions of A_1^1 and A_2^1 instead of the derivatives of linear homogeneous functions of A_1^2 , A_2^2 and A_3^2 . This is now possible, because the condition $h_{03} = h_{13} = h_{23} = h_{33} = 0$, which is equivalent to (44), is satisfied. The equations of transformation, analogous to (38), could easily be written. Introducing the quantities

$$B_{\nu i} = \sum_{\rho=0}^{\nu} \left(\sum_{\lambda=1}^2 b_{\rho\lambda} A_{\lambda}^1 \right)^{\nu-\rho} \quad (\nu = 0, 1, 2, 3), \quad (50)$$

we can now rewrite the equations of the spreads of the two descending series (42) and (46) by substituting for the quantities L_{02} , L_{12} , L_{22} and their derivatives the corresponding quantities B_{01}^1 , B_{11}^1 , B_{21}^1 and their derivatives. In particular, for their common spread we write

$$S^{2,3} = S_6^1 = [A_1, A_1^1, A_2^1; B_{01}^1, B_{11}^1, B_{21}^1]. \quad (51)$$

This method of choosing the fundamental directrices of $S^{2,3}$ immediately enables us to descend to the required spread

$$S^{2,4} = S_5 = [A_1; B_{01}, B_{11}, B_{21}, B_{31}]. \quad (52)$$

By another application of the method of § 66 we could easily descend one step further to the spread $S^{2,5}$.

In the general discussion, to which we now return, we shall use a method which can be applied, according to circumstances, in any one of a number of ways, analogous to the two ways just described of writing the relation (34).

ASCENDING SERIES AND SUCCESSIVE REDUCTIONS IN RANGE.

68. As in §§ 53, 54 let S be the general spread S_m^r given by the directrices (24), § 44. Let the ascending series determined by S consist of the $u+1$ distinct spreads

$$S^{i,0} = S_{m_{t+i}}^{r_{t+i}} \quad (i = 0, 1, \dots, u),$$

the highest of which, $S^{u,0} (= S_{m_{t+u}}^{r_{t+u}} = S_{m_{t+u}}^0 = F_{m_{t+u}-1})$, is a fixed flat within which all the spreads of the tree determined by S are enclosed. We can obviously write

$$S^{i,0} = [S; A_{\lambda}^j (\lambda = 1, \dots, r; j = t, \dots, t+i-1)] \quad (i = 1, \dots, u), \quad (53)$$

where S denotes the system of curves (24). The conditions

$$r_{t+u} = 0, \quad r_{t+i} \leq r_{t+i-1} \quad (i = 1, \dots, u), \quad (54)$$

$$m_{t+u} \leq n, \quad (55)$$

$$\left. \begin{aligned} m_{t+i} &= m_{t+i-1} + r_{t+i-1} \\ &= m + r_t + r_{t+1} + \dots + r_{t+i-1} \end{aligned} \right\} \quad (i = 1, \dots, u), \quad (56)$$

are evidently satisfied.

Let s_i denote the *reduction in range* that takes place in ascending from $S^{i-1,0}$ to $S^{i,0}$. Then the conditions

$$r_{t+i} = r_{t+i-1} - s_i \quad (i = 1, \dots, u), \quad (57)$$

$$r = s_1 + \dots + s_u, \quad (58)$$

$$\left. \begin{aligned} r_{t+i} &= r - (s_1 + \dots + s_i) \\ &= s_{t+1} + \dots + s_u \end{aligned} \right\} \quad (i = 1, \dots, u-1), \quad (59)$$

$$\left. \begin{aligned} m_{t+i} &= m + i r - \sum_{\delta=1}^{i-1} (i - \delta) s_\delta \\ &= m + i r_{t+i} + \sum_{\delta=1}^i \delta s_\delta \end{aligned} \right\} \quad (i = 1, \dots, u), \quad (60)$$

$$m_{t+u} = m + \sum_{\delta=1}^u \delta s_\delta \quad (61)$$

are also satisfied.

THE FUNDAMENTAL LINEAR RELATIONS.

69. Since the range r_{t+1} of $S^{1,0}$ is s_1 units less than the range of S , the fundamental directrices of $S^{2,0}$ must be connected, if $s_1 > 0$, by s_1 independent linear relations, or differential equations, of the form

$$\sum_{\beta=0}^{t+1} \sum_{\lambda=1}^r h_{\beta\lambda\mu 1} A_\lambda^{t+1-\beta} + \sum_{\lambda=1}^{m_0} h_{\lambda\mu 1} C_\lambda = 0 \quad (\mu = 1, \dots, s_1). \quad (62)$$

In these and all similar equations every quantity A_λ^j not already defined in § 44 is assumed to have the value zero. That is, $A_\lambda^j = 0$, when $j < t-1$ and $\lambda > r_{j+1}$.

Since the left-hand members of the equations (62) are independent linear functions of $A_1^{t+1}, \dots, A_r^{t+1}$, the matrix of the coefficients of these quantities, namely

$$\| h_{0\lambda\mu 1} \| = \left\| \begin{array}{cccccc} h_{0111} & \cdot & \cdot & \cdot & \cdot & h_{0r11} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{01s_1 1} & \cdot & \cdot & \cdot & \cdot & h_{0rs_1 1} \end{array} \right\|,$$

must be of rank s_1 for general values of ω .

Ascending one step further, we find the fundamental directrices of $S^{3,0}$ connected by $2s_1 + s_2$ independent linear relations, namely (a) the s_1 equations (62), (b) s_1 equations of the form

$$\sum_{\lambda=1}^r h_{0\lambda\mu 1} A_\lambda^{t+2} + \dots = 0, \quad (63)$$

obtained from (62) by differentiation as to ω , and (c) s_2 additional equations (if $s_2 > 0$) of the form

$$\sum_{\beta=0}^{t+2} \sum_{\lambda=1}^r h_{\beta\lambda\mu 2} A_\lambda^{t+2-\beta} + \sum_{\lambda=1}^{m_0} h_{\lambda\mu 2} C_\lambda = 0 \quad (\mu = 1, \dots, s_2). \quad (64)$$

Since the left-hand members of the $s_1 + s_2$ equations (63), (64) are independent linear functions of $A_1^{t+2}, \dots, A_r^{t+2}$, the matrix

$$\left\| \begin{array}{c} h_{0\lambda\mu 1} \\ h_{0\lambda\mu 2} \end{array} \right\|,$$

whose elements are the coefficients of $A_1^{t+2}, \dots, A_r^{t+2}$ in these equations, must be of rank $s_1 + s_2$.

70. Continuing in this way, we see that the fundamental directrices of $S^{i+1,0}$, $m + (i+1)r$ in number, are connected by $\sum_{\delta=1}^i (i - \delta + 1) s_\delta$ independent linear relations, of which the following:

$$\sum_{\beta=0}^{t+\varepsilon} \sum_{\lambda=1}^r h_{\beta\lambda\mu\varepsilon} A_\lambda^{t+\varepsilon-\beta} + \sum_{\lambda=1}^{m_0} h_{\lambda\mu\varepsilon} C_\lambda = 0 \quad (\mu = 1, \dots, s_\varepsilon; \varepsilon = 1, \dots, i), \quad (65)$$

$s_1 + \dots + s_i$ in number, may be regarded as *fundamental*, and the rest are obtained from them by differentiation as to ω . The matrix

$$\left\| \begin{array}{c} h_{0\lambda\mu 1} \\ \cdot \\ \cdot \\ \cdot \\ h_{0\lambda\mu i} \end{array} \right\|,$$

whose elements are the coefficients of the highest derivatives of the quantities A_λ in the equations (65), and which has r columns and $s_1 + \dots + s_i$ rows, is evidently of rank $s_1 + \dots + s_i$.

The relations (62) and (64) are included among the relations (65), and correspond to the values $\varepsilon = 1$ and $\varepsilon = 2$. Putting $i = u$ in (65), we obtain the complete set of fundamental relations, $s_1 + \dots + s_u (=r)$ in number, in virtue of which $S^{u+1,0}$ coincides with $S^{u,0}$ and is of range zero.

ASCENDING BRANCH SERIES AND SUCCESSIVE REDUCTIONS IN RANGE.

71. We next consider the ascending series determined by S_{jj} , whose range is r_{t-j} and whose equation was found in § 45. For its i th tangent spread we have the equation

$$S_{j,j+i} = [S_{jj}; A_\lambda^i (\lambda = 1, \dots, r_{t-j}; j = t, \dots, t+i-1)]. \quad (66)$$

Let the successive reductions in range at the different steps of the ascent from S_{jj} to $S_{j,j+i}$ be $s_{1j}, \dots, s_{ij}$; then the range of $S_{j,j+i}$ will be

$$r_{t-j} - (s_{1j} + \dots + s_{ij}), \quad (67)$$

and the inequality

$$s_{1j} + \dots + s_{ij} \leq r_{t-j} \quad (68)$$

must hold.

72. Every reduction in range indicates the existence of one or more identical relations among the fundamental directrices of $S_{j,j+i+1}$; every such relation is a relation among the fundamental directrices of S^{i+1} , and so must be linearly dependent on the relations (65) and their derivative relations. Among those which involve $A_1^{t+i}, \dots, A_{r_{t-j}}^{t+i}$ the number of independent relations is $s_{1j} + \dots + s_{ij}$; hence

$$s_{1j} + \dots + s_{ij} \leq s_1 + \dots + s_i \quad (i = 1, \dots, u). \quad (69)$$

This theorem and its correlative may be stated as follows.

In ascending from the j th special enclosed spread S_{jj} of a given spread S to its i th tangent spread $S_{j,j+i}$ the resulting reduction in range cannot be greater than that which takes place in ascending from S to its i th tangent spread S^{i0} .

In descending from the i th special enclosing spread S^{ii} of a given spread S to its j th focal spread $S^{i,i+j}$ the resulting reduction in range cannot be greater than that which takes place in descending from S to its j th focal spread $S_{j,0}$.

It is to be observed that although $s_{1j} \leq s_1$, $s_{\varepsilon j}$ ($\varepsilon > 1$) is not necessarily $\leq s_\varepsilon$.

THE MINIMUM RANGE OF $S_{j,j+i}$ AND $S^{i,i+j}$.

73. Comparing (59), (67) and (69), we see that the range of $S_{j,j+i}$ cannot be less than $r_{t+i} + r_{t-j} - r (= r')$. If $r' > 0$, this is an actual restriction on the range of $S_{j,j+i}$; and the special case in which its range is equal to r' exists and is characterized by the equation $s_{1j} + \dots + s_{ij} = s_1 + \dots + s_i$. This case is obviously the one in which the relations (65) involve only the fundamental directrices of $S_{j,j+i+1}$, and in which, therefore, the equations $s_{\varepsilon j} = s_{\varepsilon}$ ($\varepsilon = 1, \dots, i$) all hold. Hence:

<p>If the total reduction in range is the same in ascending from S_{jj} to $S_{j,j+i}$ and from S to S', the partial reductions are the same for each step of the ascent.</p>	<p>If the total reduction in range is the same in descending from S'' to $S^{i,i+j}$ and from S to S_j, the partial reductions are the same for each step of the descent.</p>
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The following theorem has now been proved: *If a spread S , its i th tangent spread S^{i0} and its j th focal spread S_{j0} are of ranges r , r_{t+i} and r_{t-j} , respectively, where $r_{t+i} + r_{t-j} - r \geq 0$, then the minimum value of the range of $S_{j,j+i}$, and also of $S^{i,i+j}$, is $r_{t+i} + r_{t-j} - r$; and if $S_{j,j+i}$ has its minimum range, each of the spreads $S_{j,j+1}, \dots, S_{j,j+i-1}$ has also its minimum range.*

and if $S^{i,i+j}$ has its minimum range, each of the spreads $S^{i,i+1}, \dots, S^{i,i+j-1}$ has also its minimum range.

THE CHOICE OF THE FUNDAMENTAL RELATIONS.

74. We shall assume that the fundamental relations (65) have been selected as follows:

First choose a set of independent relations (if any exist), which, together with their derivative relations, fix the linear dependence of the fundamental directrices of $S_{t-1,t+i}$, and which, therefore, involve only the quantities in first r_1 columns of (24) and their derivatives; then choose a set of additional relations, independent of those already chosen and of each other, which, together with those already chosen and their derivative relations, fix the linear de-

pendence of the fundamental directrices of $S_{t-2, t+i-1}$; and continue this process until the complete set of relations is obtained. Number them in such a way that those which involve only the fundamental directrices of $S_{j, j+i+1}$ are obtained, for every value of j from 1 to $t-1$, by giving to μ , in (65), the values $1, 2, \dots, s_{\epsilon j}$ ($\epsilon = 1, \dots, i$).

THE TRANSFORMATION OF THE FUNDAMENTAL RELATIONS.

75. We now proceed to find the directrices of the successive focal spreads of $S^{i,0}$ and of $S_{j, j+i}$ ($j = 1, \dots, t-1$), in each case employing the corresponding part of the relations (65).

Following the method used in the special case (§§ 56-58), we first transform the equations (65) and write them in the form

$$\left. \begin{aligned} \sum_{\epsilon=0}^{t+\epsilon-\sigma} \left(\sum_{\lambda=1}^r l_{\rho\lambda\mu\epsilon\sigma} A_{\lambda}^{\sigma} \right)^{t+\epsilon-\sigma-\rho} + f_{\mu\epsilon\sigma}(A, C) = 0 \\ (\mu = 1, \dots, s_{\epsilon}; \epsilon = 1, \dots, i), \end{aligned} \right\} \quad (70)$$

where σ is a fixed non-negative integer to be chosen as small as the conditions of the particular case will allow, and $f_{\mu\epsilon\sigma}(A, C)$ denotes a linear homogeneous function of the quantities A_{λ}^{σ} and C_{λ} involving no derivatives of A_{λ} higher than the $(\sigma-1)$ -st. Evidently σ can always be chosen $\leq t-1$.

The transformation can always be accomplished and the new coefficients can be found by means of the equations

$$\left. \begin{aligned} l_{\rho\lambda\mu\epsilon\sigma} = \sum_{\beta=0}^{\rho} (-1)^{\rho-\beta} \binom{t+\epsilon-\sigma-\rho}{\rho-\beta} h_{\beta\lambda\mu\epsilon}^{\rho-\beta} \\ (\rho = 0, \dots, t+\epsilon-\sigma; \lambda = 1, \dots, r; \mu = 1, \dots, s_{\epsilon}; \epsilon = 1, \dots, i), \end{aligned} \right\} \quad (71)$$

The old coefficients are expressed in terms of the new by means of the equations

$$\left. \begin{aligned} h_{\beta\lambda\mu\epsilon}^{\rho-\beta} = \sum_{\rho=0}^{\beta} \binom{t+\epsilon-\sigma-\rho}{\beta-\rho} l_{\rho\lambda\mu\epsilon\sigma}^{\rho-\beta} \\ (\beta = 0, \dots, t+\epsilon-\sigma; \lambda = 1, \dots, r; \mu = 1, \dots, s_{\epsilon}; \epsilon = 1, \dots, i). \end{aligned} \right\} \quad (72)$$

In particular, putting $\rho=0$ in (71), we have

$$l_{0\lambda\mu\epsilon\sigma} = h_{0\lambda\mu\epsilon}. \quad (73)$$

77. Since the proof of these formulae is not essentially more difficult than that given in § 60 for the special case, we shall omit it. Only one point requires mention. The s_i directrices $L_{\alpha\mu\sigma}^{t+i-\sigma-1}$ of the spread $S^{i,1}$ are independent of one another and of the directrices of $S^{i-1,0}$, altho the r curves $A_1^{t+i-1}, \dots, A_r^{t+i-1}$, on which they are linearly dependent, are themselves not independent of the directrices of $S^{i-1,0}$.

This follows from (74) and the fact that the linear dependence of these curves is fixed by the first $i-1$ sets of relations (70), whereas the directrices under consideration depend on the i th set of relations. Similarly, in the general case of the spread $S^{i\nu}$ the directrices $L_{\beta\mu\sigma}^{t+i-\sigma-\nu}$ are all independent of one another and of the directrices of $S^{i-\nu,0}$ or $S_{\nu-i,0}$, as the case may be; and the dimensionality of $S^{i\nu}$ is equal to that of $S^{i-\nu,0}$ (or $S_{\nu-i,0}$) plus the number of the additional directrices.

78. If σ cannot be chosen $< t-1$, the formula (78) determines only one spread $S^{i,t+1}$. Since the descending series of spreads $S^{i\nu}$ is regular as far as S^i , inclusive, therefore $S^{i\nu}$ ($0 < \nu \leq i$) is of range r_{t+i} ($= r - \sum_{\delta=1}^i s_\delta$) and dimensionality $m_{t+i} - \nu r_{t+i}$.

When $\nu > i$ and $S^{i\nu}$ is determined by (78), its dimensionality is $m_{t-\nu+i} + \sum_{\delta=1}^i (\nu - i + \delta) s_\delta$ and its range (except when $\nu = t + i - \sigma$) is $r_{t-\nu+i} - \sum_{\delta=1}^i s_\delta = r_{t+i} + r_{t-\nu+i} - r$. This is the smallest value that the range of $S^{i\nu}$ can have, for given values of r , r_{t+i} and $r_{t-\nu+i}$.

When $\nu = t + i - \sigma$, the range of $S^{i\nu}$ can be found and the equations of its focal spread $S^{i,\nu+1}$, which are not given by (78), can be obtained by a method analogous to that of § 66.

THE SUCCESSIVE FOCAL SPREADS OF $S_{j,j+i}$.

79. In order to write the equations of the successive focal spreads of $S_{j,j+i}$, we employ only those particular relations (70), $s_{1j} + \dots + s_{ij}$ in number, which involve the directrices of $S_{j,j+i}$, and the corresponding auxiliary quantities (75); and we find that

$$S_{j,j+i,\nu} = [S_{j,j+i-\nu}; L_{\beta\mu\sigma}^{t+i-\sigma-\nu} (\beta=0, \dots, \nu-i+\varepsilon-1; \mu=1, \dots, s_{\varepsilon j}; \varepsilon=i, \dots, i-\nu+1)] \left. \vphantom{S_{j,j+i,\nu}} \right\} (79) \\ (\nu = 1, \dots, i),$$

$$S_{j,j+i,v} = [S_{j,j+i-v}; L_{\beta\mu\varepsilon\sigma}^{t+i-\sigma-v} (\beta=0,\dots, v-i+\varepsilon-1; \mu=1,\dots, s_{\varepsilon\gamma}; \varepsilon=i,\dots, 1)] \quad \left. \vphantom{S_{j,j+i,v}} \right\} \quad (80)$$

$$(v = i+1, \dots, i+j),$$

$$S_{j,j+i,v} = [S_{v-i}; L_{\beta\mu\varepsilon\sigma}^{t+i-\sigma-v} (\beta=0,\dots, v-i+\varepsilon-1; \mu=1,\dots, s_{\varepsilon\gamma}; \varepsilon=i,\dots, 1)] \quad \left. \vphantom{S_{j,j+i,v}} \right\} \quad (81)$$

$$(v = i+j+1, \dots, t+i-\sigma).$$

In these formulae σ can be chosen $\leq t-j-1$; if $\sigma = t-j-1$, the formula (81) determines only one spread $S_{j,j+i,j+i+1}$. The descending series of spreads $S_{j,j+i,v}$ is regular, of range

$$r_{t-j} = \sum_{\delta=1}^i s_{\delta j}, \quad (82)$$

as far as $S_{j,j+i,j+i}$; and since $S_{j,j+i}$ is evidently of dimensionality

$$m_{t-j} + (i+j) r_{t-j} = \sum_{\delta=1}^{i-1} (i-\delta) s_{\delta j}, \quad (83)$$

therefore, if $v \leq j+i+1$, $S_{j,j+i,v}$ is of dimensionality

$$m_{t-j} + (i+j-v) r_{t-j} + \sum_{\delta=1}^i (v-i+\delta) s_{\delta j}. \quad (84)$$

In particular, $S_{j,j+i,j+i}$ is of dimensionality

$$m_{t-j} + \sum_{\delta=1}^i (j+\delta) s_{\delta j}. \quad (85)$$

The descending series (81), like (78), can be continued one step further by the method of § 66. Comparing (79) and (80) with (77) and (78), we easily verify the fact that $S_{j,j+i,v}$ is enclosed in $S^{i'}$.

THREE SPECIAL CASES, §§ 80-89.

80. We shall now consider three special cases in which σ , in formulae (77) and (78), can be chosen $< t-1$.

First, if it happens that $r_t = r_{t-1} = \dots = r_{t-j}$, then σ can be chosen as small as $t-j-1$, and the equations of the spreads $S^{i',i+1}, \dots, S^{i',i+j}$ will be given by (78); hence the descending series of focal spreads of $S^{i',0}$ will con-

tinue regular, of range r_{i+i} , as far as $S^{i,i+j}$. This was to be expected, because the entire tree determined by S is in this case also determined by S_j , and $S^{i,i+j} = (S_j)^{i+j,i+j}$.

THE CASE IN WHICH $S_{j,j+i}$ IS OF MINIMUM RANGE.

81. The second special case, which is more interesting, is the one considered in § 73, in which $s_{\epsilon j} = s_{\epsilon}$ ($\epsilon = 1, \dots, i$), and in which, therefore, the identical relations that fix the linear dependence of the fundamental directrices of $S^{i+1,0}$ involve only the directrices of the first r_{i-j} columns of the array (24) and their derivatives, namely the fundamental directrices of $S_{j,j+i+1}$. An example illustrating this case was given in §§ 64 and 67.

We can again choose σ as small as $t - j - 1$; and the spreads $S^{i,i+1}, \dots, S^{i,i+j}$ will be given by the equations (78), but will no longer necessarily form a regular series.

82. In this case, putting $v = i + j$ in (78) and (80), we discover that $S^{i,i+j}$ coincides with $S_{j,j+i,j+i}$, and therefore that *the $(i + j)$ -th focal spread of $S_{j,j+i}$, which, as we saw in § 33, is always enclosed in $S^{i,i+j}$, in this case coincides with $S^{i,i+j}$* . In other words, $S_{j,j+i}$ is the $(i + j)$ -th special enclosed spread of $S^{i,0}$; and dually $S^{i,i+j}$ is the $(i + j)$ -th special enclosing spread of $S_{j,0}$.

CONNECTING SERIES.

83. Hence $S^{i,i+j}$ and $S_{j,j+i}$ are of the same range $r_{i+i} + r_{i-j} - r$ ($= r' \geq 0$), which, in view of §§ 72, 73, is a minimum for both; and *if $r' > 0$, they form the first and last terms of a regular connecting series of $i + j + 1$ spreads*.

The two following theorems, both converses of the one just given, are evidently true:

If the ranges r , r_{i+i} and r_{i-j} of S , $S^{i,0}$ and $S_{j,0}$, respectively, satisfy the inequality $r_{i+i} + r_{i-j} - r$ ($= r'$) ≥ 0 , and if the range of one of the two spreads $S^{i,i+j}$ and $S_{j,j+i}$ has its minimum value r' , the same is true of the other, and the first is the $(i + j)$ -th focal spread of the second.

If $S^{i,i+j}$ is the $(i+j)$ -th focal spread of $S_{j,j+i}$, then $r' \geq 0$ and both spreads are of minimum range r' .

84. Moreover, if $S^{i,i+j}$ is the $(i+j)$ -th focal spread of $S_{j,j+i}$, it is clear that $S^{\alpha,\alpha+\beta}$ ($\alpha \leq i$, $\beta \leq j$) is the $(\alpha+\beta)$ -th focal spread of $S_{\beta,\beta+\alpha}$, and therefore that they also are connected by a regular series. This connecting series will be merely a part of the one connecting $S^{i,i+j}$ and $S_{j,j+i}$, if $S^{i,0}$ and $S^{\alpha,0}$ are of the same range and if $S_{j,0}$ and $S_{\beta,0}$ are of the same range; otherwise it will be a distinct series. Hence if the number of reductions in range that occur in the ascent from S to $S^{i,0}$ is i' ($\leq i$) and the number of reductions in range that occur in the descent from S to $S_{j,0}$ is j' ($\leq j$), then there are *exactly* $i'j'$ distinct connecting series between the spreads $S^{\alpha,\alpha+\beta}$ ($\alpha = 1, \dots, i$; $\beta = 1, \dots, j$) on the one hand and the spreads $S_{\beta,\beta+\alpha}$ on the other.

$$S_{j,j+i} = S^{i,i+j} = \text{A FIXED FLAT.}$$

85. A subcase of some interest is the one in which $r' = 0$; $S^{i,i+j}$ is then of range zero and coincides with $S_{j,j+i}$.

That is, if $r_{i+i} + r_{i-j} - r = 0$, and if the $(i+j)$ -th focal spread of $S^{i,0}$ is a fixed flat F , then F is also the $(i+j)$ -th tangent spread of $S_{j,0}$. Every spread of the entire tree determined by $S^{i,0}$ encloses F , and every spread of the tree determined by $S_{j,0}$ is enclosed in F .

Moreover, if $S^{i-1,0}$ is of higher range than $S^{i,0}$, the former does not enclose F , and the spreads $S_{j,0}$, $S_{j,1}$, ..., $S_{j,j+i}$ ($= F$) are all distinct. Dually, if $S_{j-1,0}$ is of higher range than $S_{j,0}$, the former is not enclosed in F , and the spreads $S^{i,0}$, $S^{i,1}$, ..., $S^{i,i+j}$ ($= F$) are all distinct.

A CASE IN WHICH $S_{j,j+i}$ ENCLOSES $S^{i,i+j}$.

86. A third special case in which σ , in (77) and (78), can be chosen $< t - 1$, and which includes the second case but is somewhat more general, is that in which $s_{t,j-1} = s_t$ ($\varepsilon = 1, \dots, i$), and in which the $\sum_{\varepsilon=1}^i (s_\varepsilon - s_{t,j})$ ($= s'$) identical relations (70), that are independent of those fixing the linear depen-

dence of the fundamental directrices of $S_{j,j+i+1}$, involve merely the $r_{t-j+1} - r_{t-j}$ curves

$$A_{r_{t-j+1}}^{t-j}, \dots, A_{r_{t-j+1}}^{t-j}, \quad (86)$$

and not any higher derivatives of the same. Obviously $s' \leq r_{t-j+1} - r_{t-j}$. We can choose $\sigma = t - j$.

Putting $\sigma = t - j$ and $\rho = j + \varepsilon$ in those particular equations (70) that involve the quantities (86), we find that the coefficients of these quantities are $l_{j+\varepsilon, \lambda \mu \varepsilon, t-j}$. But these coefficients are not involved in the auxiliary quantities $L_{\lambda \mu \nu \varepsilon, t-j}$, defined by (75); for the highest value of ρ occurring there is $j + \varepsilon - 1$.

Hence, putting $\nu = i + j$ in (78), we see that the quantities (86) are not involved in the expression for the directrices of $S^{i,i+j}$, and that *the latter spread, therefore, is enclosed in $S_{j,j+i}$* . Moreover, it is easy to see that $r_{t+i} + r_{t-j} - r (= r') \geq 0$, and that $S^{i,i+j}$ and $S_{j,j+i}$ are of the same range $r' + s'$, although they do not necessarily belong to a connecting series.

87. To illustrate this case take the example considered in §§ 55-63, and let the relation (34) or (35), of § 56, involve A_3^2 but not A_3^3 , A_3^4 or A_3^5 ; that is, let $l_{03} = l_{13} = l_{23} = 0$ and $l_{33} \neq 0$. Take $i = 2$ and $j = 1$; then $r_{t+i} = 2$, $r_{t-j} = 2$, $r' = 1$, $s' = 1$; $S^{i,i+j} = S^{2,3} = S_6$, as given by (42); $S_{j,j+i} = S_{1,3} = S_9$, as given by (43); S_6 and S_9 are both of range $r' + s' = 2$. Equations (41) show that L_{02} , L_{12} and L_{22} do not involve A_3^2 or its derivatives, and therefore that S_6 is enclosed in S_9 . But S_6 is evidently not a focal spread of S_9 , as in § 64.

$$S^{i,i+j} \text{ COINCIDING WITH } S_{j,j+i}.$$

88. In the subcase in which $r' = 0$ the fundamental directrices of $S^{i,i+j}$, as given by (78), include exactly enough linear functions of the quantities A_λ^δ ($\lambda = 1, \dots, r_{t-j}$; $\delta = t - j, \dots, t + i - 1$) to determine all of the latter, and $S^{i,i+j}$ coincides with $S_{j,j+i}$.

Hence, if $r_{t+i} + r_{t-j} - r = 0$ and the conditions of § 86 are fulfilled, $S^{i,i+j}$ is of range s' and coincides with $S_{j,j+i}$. Consequently, $S^{i,i+j}$, its first $i + j$ focal spreads and its first $i + j$ tangent spreads together form a regular series of $2(i + j) + 1$ terms, all of range s' .

89. To illustrate this subcase we can again return to the example of §§ 55-63. This time let $i = j = 2$; then $r_{t+i} = 2$, $r_{t-j} = 1$, and $r' = 0$. Let (34)

involve A_i^1 but not any of the other elements in the 2nd or 3rd columns of (31). Accordingly, (34) can be written in the form (49), of § 67, and the condition

$$b_{\rho\lambda} = 0 \quad (\lambda = 2, 3; \rho = 0, 1, 2, 3), \quad b_{42} = 0, \quad (87)$$

is satisfied. $S_{j,j+i} = S_{2,4} = S_5$, as given by (45), and $S^{i,i+j} = S^{2,4} = S_5$, as given by (52). The quantities B_{01} , B_{11} , B_{21} , B_{31} , determined by (50), involve only A_1 and its derivatives. Hence $S^{2,4}$ coincides with $S_{2,4}$ and is of range 1. It determines a regular series of nine spreads, including its first four focal spreads and its first four tangent spreads.

If we alter the above condition (87) merely by putting $b_{42} = 0$, we get an illustration of the case of § 85. $S^{2,4}$ will still coincide with $S_{2,4}$, but will now be a fixed flat F . The spreads (42) will all enclose F , while the spreads (45) will all be enclosed in F .

THE GENERAL CASE.

90. Returning to the general case (§§ 68-79) and imposing the one condition that

$$r_{i+i} + r_{i-j} - r \leq 0, \quad (88)$$

we find, in view of (59), § 68, that $s_1 + \dots + s_i$ must be $\geq r_{i-j}$. This means that the number of independent relations involving A_λ^{i+i} ($\lambda = 1, \dots, r$) is large enough to make it possible for the r_{i-j} curves A_λ^{i+i} ($\lambda = 1, \dots, r_{i-j}$) to be directrices of $S^{i,0}$. If so, $S^{i,0}$, which always encloses $S_{j,j+i}$ will also enclose its tangent spread $S_{j,j+i+1}$. But this is obviously impossible, unless (88) is satisfied.

Hence, if $S^{i,0}$ encloses $S_{j,j+i+1}$, or, dually, if $S_{j,0}$ is enclosed in $S^{i,i+j+1}$, the condition (88) must be satisfied.

THE DIMENSIONALITIES OF $S_{j,j+i}$ AND $S^{i,i+j}$.

91. In all that follows let r_{ij} denote the integer $r_{i+i} + r_{i-j} - r$, where r , r_{i+i} and r_{i-j} are, as usual, the ranges of the spread S , its i th tangent spread $S^{i,0}$, and its j th focal spread $S_{j,0}$, respectively; also let d denote the

dimensionality of $S_{j,j+i}$, the $(j+i)$ -th tangent spread of $S_{j,0}$, and d' the dimensionality of $S^{i,i+j}$, the $(i+j)$ -th focal spread of $S^{i,0}$. We wish to prove the inequality

$$d - d' \geq (i+j) r_{ij}. \quad (89)$$

Proof: It follows from § 83 that if $r_{ij} \geq 0$ and if $S_{j,j+i}$ and $S^{i,i+j}$ are both of minimum range r_{ij} , then $d - d' = (i+j) r_{ij}$. If the ranges of these two spreads are increased above their minimum value, the dimensionality d of $S_{j,j+i}$ will not change, unless the range of $S_{j,j+i-1}$ is also increased, and if the value of d changes at all, it is evidently increased; similarly, the dimensionality d' of $S^{i,i+j}$ will not change, unless the range of $S^{i,i+j-1}$ is increased, and if the value of d' does change, it is decreased; for both reasons, therefore, the value of $d - d'$ is either increased or not changed. Hence, if $r_{ij} \geq 0$, (89) holds true. But if $r_{ij} < 0$, it obviously holds true a fortiori; and the theorem is proved.

In particular, if $r_{ij} > 0$, the dimensionality of $S_{j,j+i}$ exceeds that of $S^{i,i+j}$ by at least $(i+j) r_{ij}$ units; if $r_{ij} = 0$, the dimensionality of $S_{j,j+i}$ is at least as great as that of $S^{i,i+j}$; and if $S_{j,j+i}$ is of lower dimensionality than $S^{i,i+j}$, r_{ij} must be negative, and the difference between their dimensionalities cannot exceed $(i+j) \cdot |r_{ij}|$.

92. If $d - d' = (i+j) r_{ij}$, the range of $S_{j,j+i-1}$ must have its minimum value $r_{i-1,j} (\geq 0)$ which means (§ 73) that $s_{\epsilon,j} = s_{\epsilon}$ ($\epsilon = 1, \dots, i-1$); and the range of $S^{i,i+j-1}$ must have its minimum value $r_{i,j-1} (\geq 0)$, which means that $s_{\epsilon,j-1} = s_{\epsilon}$ ($\epsilon = 1, \dots, i$).

Hence, a necessary and sufficient condition that $d - d'$ is exactly equal to $(i+j) r_{ij}$ is that $r_{i-1,j} \geq 0$, $r_{i,j-1} \geq 0$, and that $S_{j,j+i-1}$ and $S^{i,i+j-1}$ are of minimum ranges $r_{i-1,j}$ and $r_{i,j-1}$, respectively. This means that

$$s_{\epsilon,j} = s_{\epsilon} \quad (\epsilon = 1, \dots, i-1) \quad \text{and} \quad s_{i,j-1} = s_i. \quad (90)$$

In other words, those particular relations of the set (65), or (70), which fix the linear dependence of the fundamental directrices of $S^{i,0}$ must involve only the fundamental directrices of $S_{j,j+i}$, and those additional relations, if any, which are independent of the former and their derivative relations and which help to fix the linear dependence of the fundamental directrices of $S^{i+1,0}$, must involve only the fundamental directrices of $S_{j-1,j+i}$.

In particular, if $r_{ij} = 0$, (90) is a necessary and sufficient condition that $S_{j,j+i}$ and $S^{i,i+j}$ are of the same dimensionality. It is also possible for r_{ij} to

be negative; in that case (90) is a necessary and sufficient condition that $d' - d = (i + j) \cdot |r_{ij}|$.

93. A little attention to formulae (78) and (80) will make the following theorems clear:

If $S_{j,j+i}$ encloses $S^{i,i+j}$ and does not coincide with it, r_{ij} is positive; if $S_{j,j+i}$ coincides with $S^{i,i+j}$, r_{ij} vanishes; and if $S_{j,j+i}$ is enclosed in $S^{i,i+j}$ and does not coincide with it, r_{ij} is negative.

<p>If $S_{j,j+i}$ encloses $S^{i,i+k}$ ($k < j$), $S_{k,0}$ and $S_{j,0}$ are of the same range, and $r_{ij} \geq 0$.</p>	<p>If $S^{i,i+j}$ is enclosed in $S_{j,j+k}$ ($k < i$), $S^{k,0}$ and $S^{i,0}$ are of the same range, and $r_{ij} \geq 0$.</p>
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<p>If $S_{j,j+i}$ is a fixed flat F, $S^{i,0}$ and all its focal spreads enclose F, and $r_{ij} \leq 0$.</p>	<p>If $S^{i,i+j}$ is a fixed flat F, $S_{j,0}$ and all its tangent spreads are enclosed in F, and $r_{ij} \leq 0$.</p>
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If $S_{j,j+i}$ and $S^{i,i+j}$ are both fixed flats and do not coincide, r_{ij} must be negative.

Cornell University, June, 1912.