GRAPHICS OF THE THERMODYNAMIC FUNCTION.

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On reading Professor Thurston's article on the "Graphics of Thermodynamic Law,"* I thought that it might interest students of the subject to see that graphics can be applied even to the mystical function ϕ and to all theorems depending on it.

We start with equation (9) of Dr. Thurston's article:

$$dH = K_v dt + t \frac{dp}{dv} dv \quad (1)$$

From the "defining" equation of a gas, $p v = R t$, we readily obtain

$$p \left(= t \frac{dp}{dt} \right) = \frac{R t}{v} \quad (2)$$

and hence

$$dH = K_v dt + R t \frac{dv}{v} \quad (3)$$

and

$$\int_t'' dH = K_v \int_t'' dt + R \int_t'' t \frac{dv}{v} \quad (4)$$

which can be solved only when we know the law of variation between t and v , or when either t or v is constant.

Dividing equation (3) by t , we get

$$d\phi = \frac{dH}{t} = K_v \frac{dt}{t} + R \frac{dv}{v} \quad (5)$$

and

$$\begin{aligned} \phi &= \int_t \frac{dH}{t} = K_v \text{hyp. log. } t + R \text{hyp. log. } v + C \\ &= \text{hyp. log. } (t^{K_v} v^R) + C \\ &= \text{hyp. log. } (t v^{\gamma-1})^{K_v} + C \end{aligned} \quad (6)$$

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The quantity ϕ may also be expressed as a function of p and v , or of t and p :

$$\phi = \text{hyp. log.} \left(\frac{p v^\gamma}{R} \right)^{K_v} + C \quad (7)$$

$$\phi = \text{hyp. log.} \left(R^{\gamma-1} \frac{t^\gamma}{p^\gamma - 1} \right)^{K_v} + C \quad (8)$$

As is well known,

$$R = K_p - K_v = \left(\frac{K_p}{K_v} - 1 \right) K_v = (\gamma - 1) K_v = a p_0 v_0$$

By means of equations, 6, 7, 8, ϕ can be found if any two of the quantities p, v, t are known. If a curve be used to represent a series of changes in the pressure and volume of a given gas, a similar curve can be constructed referred to t and ϕ co-ordinates, representing varying values of t and ϕ corresponding to the different values of p and v , as shown by the curve.

The transforming equations present, however, some difficulty on account of the constant of integration C , which determines the position of the origin of co-ordinates. Apparently ϕ should reduce to zero when t, p or v is equal to zero. We then have

$$\phi = 0 = \text{hyp. log. } 0 + C = -\infty + C \text{ and } C = \infty$$

By placing the origin at a distance C from the true origin at infinity, we can and do disregard the constant. Besides, all calculations deal with differences and variations of the function ϕ , rather than with its absolute value.

Hence

$$\phi = 0 \text{ when } t_v^{\gamma-1} = \frac{p v^\gamma}{R} = t^\gamma p^{1-\gamma} = 1$$

Heat Absorbed during a Series of Changes.—From equation (5) we have

$$t d\phi = dH \text{ and } H = \int'' t d\phi \quad (9)$$

This can be integrated when the law of variation between t and ϕ is known, or when either t or ϕ is a constant.

Equation (9) can be represented by an area just as $\int p \, d v$ is represented. Let us assume the abscissas to correspond to the values of ϕ and the ordinates to represent values of t (Fig. 1).

In adiabatic expansion the value of H —the actual heat absorbed during expansion—reduces to zero, and hence

$$H = \int t \, d \phi = 0 \text{ or } \phi = \text{constant.}$$

In Fig. 1, the straight line $A B$ represents adiabatic (isentropic) changes between limits of temperature corresponding to A and B ; the straight line $C D$ represents isothermal changes between limits of ϕ corresponding to the points C and D ; while the line $E F$ may represent any known series of changes between the limits of ϕ and t corresponding to the co-ordinates of the points E and F .

Fig. 1.

The areas $A B A'$, $C D C' D'$ and $E F E' F'$ represent each the quantity of heat which must be supplied to produce the changes represented by the lines $A B$, $C D$ and $E F$, respectively, *i e.*, values of

$$\int_{\phi'}^{\phi''} t \, d \phi$$

If the series of changes take place in the opposite direction, from F to E , from an initial value of ϕ to a smaller final value,

then the area will be measured in the opposite direction and will be negative. It will then represent heat which must be abstracted from the changing gas.

Figs. 2 and 3 represent corresponding changes in p and v and t and φ ; they give all the particulars concerning the conditions of the gas. P (P') is supposed to be the initial condition, while $P I$ ($P' I'$), $P A$ ($P' A'$), $P D$ ($P' D'$), and $P F$ ($P' F'$) show respectively isothermal, adiabatic, isometric and isopiestic changes. Line $P C$ ($P' C'$) shows that heat is given out, although the gas is expanding and doing work; while line $P E$ ($P' E'$) is absorbing mechanical energy during compression and at the same time requires a supply of heat energy.

Fig. 2.

Fig. 3.

Heat Required during Cycle.—A cycle of operation may be represented by a closed figure (Fig. 4).

Draw two adiabatics φ' and φ'' and two isothermals t'' and t' tangent to the given area, thus circumscribing it, the points of tangency being the points $A B C$ and D . Let the changes follow one another in the direction indicated by the arrow in the diagram. The area $\varphi' A B C \varphi''$ will represent the quantity of heat which must be supplied during the changes from A through B to C . This quantity,

$$\int_{\varphi'}^{\varphi''} t d\varphi,$$

will be positive. In a similar manner the changes from C through D to A are accompanied by a production or evolution of sensible heat represented by the area $\phi'' C D A \phi'$. This area,

$$\int_{\phi''}^{\phi'} t d\phi,$$

will evidently be negative. The algebraic sum of these two areas gives the area enclosed by the irregular line $A B C D$ and represents the total amount of heat which disappears as heat, *i. e.*, is absorbed during the cycle and appears as mechanical energy or work.

Fig. 4

If, therefore, we chose our units in the two diagrams properly, this area referred to t and ϕ co-ordinates will be exactly equal to the area of the diagram of a cycle of changes referred to the ordinary co-ordinate axes v and p .

Mechanical Energy.—In order to get an area representing the mechanical energy developed during any changes, such as those represented by the line in Fig. 5, we proceed as follows:
Area

$$\phi' A B C \phi'' = \int t d\phi = H = K_v \int d t + R \int t \frac{dv}{v}$$

See eq. (4).

where

$$R \int t \frac{dv}{v} =$$

mechanical energy developed.

$$K_v \int dt = K_v (t'' - t')$$

where t' is the initial temperature at the point A' , and t'' is the temperature at the point C , the end of the changes.

Draw the line K_v , distant from the axis of t by an amount equal to K_v ; then will the shaded area $t' t'' D E$ represent $K_v (t'' - t')$.

Hence from equation (4)

$$\int t d\phi - K_v \int dt = R \int t \frac{dv}{v} = \text{work}$$

and area $\phi' A B C \phi''' + \text{area } t' t'' D E = \text{Work}$.

Here we assumed t' greater than t'' , and hence the area $t' t'' D E$ is negative, and the work performed is really the arithmetical sum of the two areas.

Passing back to A through any series of changes as along the line through F , the same area will represent $K_v (t' - t'')$, but will be positive since now the final temperature is higher than the initial.

Thus during a cycle represented by the successive changes along $A B C$ and then back along $C F A$, the work done is

represented by the area enclosed, $A B C F A$, since the two equal areas $E t'' t' D$ with opposite signs counteract each other and disappear in the result.

It is, therefore, evident that any area enclosed by lines representing a series of changes of values of φ and t is *equal* to the corresponding area enclosed by the lines representing varying values of p and v (Fig. 6).

Area $\varphi A B C \varphi''$ of Fig. 5 is equal to the area $\varphi' A B C \varphi''$ of Fig. 6, the curves ϕ' and ϕ'' being extended to infinity; and hence the latter represents the quantity of heat absorbed during the changes along the line $A B C$ (Theorem, p. 303, Rankine, Steam Engine, etc.).

Fig. 6.

The Second Law of Thermodynamics.—The second law of thermodynamics can now also be represented in a simple manner (Rankine, § 244, p. 308). In Fig. 7, $A B$ represents an isothermal expansion at temperature t ; and the area $\varphi' A B \varphi''$ represents both the heat absorbed during this expansion as well as the mechanical work performed.

This area may be divided into equal parts by simply dividing t into equal parts and drawing the isothermals $C D$, $E F$, etc., $A B C D = C D F E =$ etc. Hence every part of the temperature affects the work by the same amount.

Carnot's Cycle.—Fig. 8 represents the diagram of the elementary engine (Carnot's cycle). The area $(t' - t'') (\varphi' - \varphi'') = A B C D$ is evidently the work performed during the cycle,

while $\varphi'' A B \varphi' = t' (\varphi' - \varphi'')$ represents in the same units the quantity of heat absorbed during and hence required by the cycle. The efficiency of this engine is, therefore,

$$\frac{(t' - t'') (\varphi' - \varphi'')}{t' (\varphi' - \varphi'')} = \frac{t' - t''}{t'}$$

Maximum Efficiency.—Fig. 9 represents the diagram of any engine, and the circumscribing rectangle $A B C D$ represents the work done by an elementary heat engine acting between the same limits.

The efficiency of the former is the ratio: (Irreg. area, $E F G H I$, etc.) $\div \varphi'' E F G \varphi'$, while the efficiency of the

Fig. 7.

Fig. 8.

latter is the ratio: $A B C D / \varphi'' A B \varphi'$, where $A B C D = E F G H + E F A + F B G + G C H + H D E$ and $\varphi'' A B \varphi' = \varphi' E F G \varphi'' + E F A + F B G$.

Subtracting from both the numerator and the denominator of the fraction $A B C D / \varphi'' A B \varphi'$ the same quantity $E F A + F B G$, we get

$$\frac{E F G H + G C H + H D E}{\varphi'' E F G \varphi'}$$

which is less than the original ratio

$$\frac{A B C D}{\varphi'' A B \varphi'}$$

and hence

$$\frac{E F G H}{\varphi'' E F G \varphi'}$$

is still less than the same ratio

$$\frac{A B C D}{\varphi'' A B \varphi'}$$

(NOTE.—It is easily shown that if both numerator and denominator of a fraction whose value is less than unity be decreased by the same amount, the resulting fraction is reduced in value.)

We thus reach the conclusion given by Rankine, on pages 265, 266 and 267, that the highest possible efficiency between

given limits t' and t'' is the ratio $A B C D / \varphi'' A B \varphi' = (t' - t'') / t'$, and corresponds to the efficiency of an elementary heat engine.

Isodiabatic Lines—Regenerator.—Fig. 10 represents isodiabatic lines, which appear simply as lines parallel to each other, or rather as lines equidistant at all points, the distances being measured parallel to the axis of φ .

Points A and D have equal ordinates (temperature t'); similarly B and E correspond to the temperature t'' ; $A B$ and $D E$ are parallel. Therefore, the areas $\varphi' A B \varphi''$ and $\varphi''' D E \varphi''$ are equal, and the same quantity of heat is absorbed in chang-

ing from A to B as from D to E , i. e., between the same temperatures t' and t'' .

We can readily show that isometric as well as isopiestic lines are isodiabatic.

From equations (6) and (8), we have

$$\varphi = \text{hyp. log. } t^{Kv} + R \text{hyp. log. } v + C$$

and

$$\varphi = \text{hyp. log. } t^{Kp} + R \text{hyp. log. } R / p + C$$

If v is constant, we get the logarithmic curve

$$\varphi_v' = \text{hyp. log. } t^{Kv} + \text{constant } C'$$

depending on the value of v ($= v'$).

Fig. 10.

For any other value v ($= v''$), we get

$$\varphi_v'' = \text{hyp. log. } t^{Kv} + C''$$

depending on v'' .

At equal temperatures,

$$\varphi_v' - \varphi_v'' = C' - C'' = \text{constant.}$$

See distance AB , Fig. 11.

Similar reasoning applies to lines of equal pressure, isopiestic lines.

The area enclosed by two isothermals and two isodiabatics, whose abscissas differ by the constant $\varphi_1 - \varphi_2$, is equal to the

rectangle $(t' - t'')$ ($\varphi_1 - \varphi_2$). This is evident since the area is made up of infinitesimal areas $(\varphi_1 - \varphi_2) dt$, which, when added, give the total area $(\varphi_1 - \varphi_2) \int dt = (\varphi_1 - \varphi_2)(t' - t'')$.

Let *Fig. 12* represent such an area $A B C D$. Then the efficiency of such a cycle is represented by the ratio $A B C D / \varphi' A B C \varphi''$, or its equal $A' B C D' / \varphi' A B C \varphi''$, provided all the heat represented by the area $\varphi' A D C \varphi''$ is lost as available heat. If, however, the heat given out during the changes from C to D be taken up (absorbed) by a "regenerator" and then returned or re-supplied to the medium while passing from A to B , then the area $\varphi' D C \varphi''$ represents that heat. To find the efficiency in that case we must subtract this

area from the denominator of the above ratio. Since the lines $A B$ and $D C$ are isodiabatic, the area $\varphi' D C \varphi'' = \varphi' A B \varphi'''$, and hence the efficiency will be

$$\frac{A' B C D'}{\varphi' A B C \varphi'' - \varphi' D C \varphi''} = \frac{A' B C D'}{\varphi' A B C \varphi'' - \varphi' A B \varphi'''}$$

$$\frac{A' B C D'}{\varphi''' B C \varphi''} = \frac{t' - t''}{t'}$$

This value $(t' - t'') / t'$ we recall as the maximum possible between the given limits of temperature (vid. Rankine, § 273, etc.).

We notice that, theoretically, the successive portions of

the regenerator must assume the decreasing temperatures corresponding to the successive points of the line $C'D$. During the "regeneration" the temperature of the medium will change in the reverse order, and as the gas passes through the regenerator in the opposite direction, it will encounter temperatures corresponding to its own. Hence a transfer of heat is possible, if the "heat-capacity" (specific heat and mass) of the regenerator is high, and if its different portions are sufficiently "insulated" from each other.

If the lines AB and CD are not parallel, changes between equal temperatures will require different quantities of heat $\int t d \phi$, and hence a complete transfer will not be possible.

Fig. 12.

Applications.—The advantage of the $t\varphi$ diagram is the fact that it represents by finite areas both the work performed and the heat supplied during a cycle of operations, such as a revolution of the hot-air engine.

The discussion may be made general, but somewhat more difficult, by insisting on the value $d p / d t$ instead of p / t in the original equation, and introducing molecular internal action, as in the case of liquids and imperfect gases.

In that case we can still suppose the variation of ζ to be known, and we can still construct the diagram of a cycle.

given temperature and pressure. The adiabatic $C D$ gives the most efficient reduction of temperature, down to that of the feed water, t'' . $D A$ shows the isothermal recondensation of the steam. Here also the rectangle $A D \phi''' \phi'$ includes simultaneously the energy of compression and that of reintegration into water.

The efficiency of the steam engine is thus seen to correspond very closely to that of Carnot's cycle. Unfortunately, the line $C D$ is not attainable in practice, as the water in the expanding steam causes dissipation of energy by facilitating the radiation losses. The function φ is, therefore, diminished, and a curve similar to $C E$ ensues, yielding a much smaller enclosed area, and hence lower efficiency. The point C is the cut-off, and the point F may be used to represent the end of the stroke, unless the expansion is continued till the lower temperature t'' is reached. $F G$ may then show an isopiestic or other fall in the temperature, as met in reality.

$C H$ represents the logarithmic curve, showing superheating; $\varphi = K_v \text{ hyp. log. } t$, where K_v is the specific heat of the steam-gas. $H I$ then gives the adiabatic fall in temperature of the steam-gas. $\varphi''' C H \varphi''$ is the additional heat required, and $C H I E$, the additional work performed. The economy of superheating can be shown to depend on the fact that the fall in temperature can be more closely along the adiabatic line $H I$, than $C E$ can be made to approximate the adiabatic $C D$.

Lastly, the curve $C I$ corresponds to saturated steam at the various temperatures. It is seen that a jacket is required to supply the additional heat measured by the area $\varphi''' C I \varphi''$.

In a similar manner other topics, presenting greater or less difficulty, can be illustrated and simplified. Thus comparisons between different types of engines and engines using different fluids can be made diagrammatically, because the figures show both the work performed and the heat energy supplied, and hence also the efficiency. On the other hand, the $p v$ diagram gives only the mechanical energy developed.