



## LXIII. The differential equation of the most general substitution of one variable

Captain P.A. MacMahon R.A.

To cite this article: Captain P.A. MacMahon R.A. (1887) LXIII. The differential equation of the most general substitution of one variable , Philosophical Magazine Series 5, 23:145, 542-543, DOI: [10.1080/14786448708628049](https://doi.org/10.1080/14786448708628049)

To link to this article: <http://dx.doi.org/10.1080/14786448708628049>



Published online: 29 Apr 2009.



Submit your article to this journal [↗](#)



Article views: 2



View related articles [↗](#)

LXIII. *The Differential Equation of the most general Substitution of one Variable.* By Captain P. A. MACMAHON, R.A.\*

IN the Philosophical Magazine for February 1886, Dr. T. Muir considers the differential equations of the general conic and cubic curves by a perfectly general method.

The general linear substitution

$$y = \frac{(a, b)(x, 1)}{(a', b')(x, 1)}$$

leads, as is well known, to the differential equation

$$2 \frac{dy}{dx} \cdot \frac{d^3y}{dx^3} - 3 \left( \frac{d^2y}{dx^2} \right)^2 = 0,$$

wherein the expression on the left has been called the Schwarzian derivative: this is a reciprocant; but it is also an invariant, as may be seen by writing

$$\frac{dy}{dx} = 1!t, \quad \frac{d^2y}{dx^2} = 2!a, \quad \frac{d^3y}{dx^3} = 3!b,$$

when it assumes the form

$$12(tb - a^2).$$

In the case of the general substitution of order  $n$ , the resulting expression is no longer a reciprocant, but it is an invariant (catalecticant) of a certain binary quantic †.

For, writing

$$y = \frac{(a, b, c, \dots)(x, 1)^n}{(a', b', c', \dots)(x, 1)^n} = \frac{U_n}{V_n},$$

we have

$$yV_n = U_n.$$

Differentiating this equation  $n+1, n+2, n+3, \dots, 2n+1$  times successively by Leibnitz's theorem, and putting

$$\frac{d^p y}{dx^p} = y_p, \quad \frac{d^p V_n}{dx^p} = V_n^{(p)},$$

there results the set of equations:—

\* Communicated by the Author.

† The formation of the differential equation was recently set as a question in an examination for Fellowship at Trinity College, Dublin; but I am not aware that its connexion with the theory of Invariants has been before noticed.

$$\begin{aligned}
 y_{n+1}V_n + (n+1)y_nV_n^{(1)} + \frac{(n+1)!}{2!(n-1)!}y_{n-1}V_n^{(2)} + \dots + \frac{(n+1)!}{n!1!}y_1V_n^{(n)} &= 0, \\
 y_{n+2}V_n + (n+2)y_{n+1}V_n^{(1)} + \frac{(n+2)!}{2!n!}y_nV_n^{(2)} + \dots + \frac{(n+2)!}{n!2!}y_2V_n^{(n)} &= 0, \\
 y_{n+3}V_n + (n+3)y_{n+2}V_n^{(1)} + \frac{(n+3)!}{2!(n+1)!}y_{n+1}V_n^{(2)} + \dots + \frac{(n+3)!}{n!3!}y_3V_n^{(n)} &= 0, \\
 \vdots & \\
 y_{2n+1}V_n + (2n+1)y_{2n}V_n^{(1)} + \frac{(2n+1)!}{2!(2n-1)!}y_{2n-1}V_n^{(2)} + \dots + \frac{(2n+1)!}{n!(n+1)!}y_{n+1}V_n^{(n)} &= 0,
 \end{aligned}$$

or writing

$$y_1 = 1!t, \quad y_2 = 2!a_0, \quad y_3 = 3!a_1, \quad \dots \quad y_p = p!a_{p-2},$$

these are

$$\begin{aligned}
 t \frac{1}{n!}V_n^{(n)} + a_0 \frac{1}{(n-1)!}V_n^{(n-1)} + a_1 \frac{1}{(n-2)!}V_n^{(n-2)} + \dots + a_{n-1}V_n &= 0, \\
 a_0 \frac{1}{n!}V_n^{(n)} + a_1 \frac{1}{(n-1)!}V_n^{(n-1)} + a_2 \frac{1}{(n-2)!}V_n^{(n-2)} + \dots + a_nV_n &= 0, \\
 a_1 \frac{1}{n!}V_n^{(n)} + a_2 \frac{1}{(n-1)!}V_n^{(n-1)} + a_3 \frac{1}{(n-2)!}V_n^{(n-2)} + \dots + a_{n+1}V_n &= 0, \\
 \vdots & \\
 a_{n-1} \frac{1}{n!}V_n^{(n)} + a_n \frac{1}{(n-1)!}V_n^{(n-1)} + a_{n+1} \frac{1}{(n-2)!}V_n^{(n-2)} + \dots + a_{2n-1}V_n &= 0.
 \end{aligned}$$

Eliminating the  $n+1$  quantities,

$$\frac{1}{n!}V_n^{(n)}, \quad \frac{1}{(n-1)!}V_n^{(n-1)}, \quad \dots \quad V_n,$$

between these  $n+1$  equations, we find that the desired differential equation is

$$\begin{vmatrix}
 t & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\
 a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\
 a_1 & a_2 & a_3 & \dots & a_n & a_{n+1} \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 a_{n-1} & a_n & a_{n+1} & \dots & a_{2n-2} & a_{2n-1}
 \end{vmatrix} = 0.$$

This determinant is the catalecticant of the binary quantic

$$(t, a_0, a_1, \dots, a_{2n-1})(X, Y)^{2n};$$

and by counting the constants, we see that the general substitution is the complete primitive of the differential equation.

Royal Military Academy, Woolwich,

May 21st, 1887.