The Theory of Continuants in the Historical Order of its
Development up to 1870. By Thomas Muir, LL.D.

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The more or less disguised use of continued fractions has been traced back to the publication of Bombelli's *Algebra* in 1572, eighty-four years, that is to say, before the publication of Wallis' *Arithmetica Infinitorum*, in which Brouncker's discovery was announced and the fractions explicitly expressed.* The study of the numerators and denominators of the convergents viewed as functions of the partial denominators was first seriously undertaken by Euler in his *Specimen Algorithmi Singularis* of the year 1764, in which denoting by

$$(a), \quad \frac{(a, b)}{(b)}, \quad \frac{(a, b, c)}{(b, c)}, \dots$$

the convergents to

$$a + \frac{1}{b} + \frac{1}{c} + \dots$$

he established a long series of identities, such as

$$\begin{aligned} (a, b, c, d, \dots) &= a(b, c, d, \dots) + (c, d, \dots) \\ (a, b, c, \dots, l) &= (l, \dots, c, b, a), \\ (a, b)(b, c) - (b)(a, b, c) &= 1, \\ (a, b, c)(d, e, f) - (a, b, c, d, e, f) &= -(a, b)(e, f), \\ &\dots \end{aligned}$$

The study was pursued by Hindenburg and his followers during the last twenty years of the eighteenth century, but not with any great profit; and, although in the first half of the nineteenth century considerable attention was given to the theory of continued fractions as a whole, little advance was made in elucidating

* For the early history see Favaro's *Notizie storiche sulle frazioni continue dal secolo decimoterzo al decimosettimo* published in vol. vii. of Boncompagni's *Bollettino*: and as regards Bombelli see a paper by G. Wertheim in the *Abhandl. zur Gesch. d. Math.*, viii. pp. 147-160.

the properties of the functions referred to.* Their connection with determinants, after the awakening of interest in the latter about 1841, was sure sooner or later to be detected: there is no evidence, however, of the discovery having been made before the year 1853.

SYLVESTER, J. J. (1853, May 13).

[On a remarkable modification of Sturm's theorem. *Philos. Mag.* (4), v. pp. 446-457.]

The mention of Sturm's theorem in the title of a paper renders not improbable the occurrence therein of matter connected with continued fractions. Especially likely is this in the case of a writer like Sylvester when in a characteristic mood; and, assuredly, the present communication is in structure, style, and originality redolent of its author. It must have been written in the white heat of discovery. The main part of it consists of six pages: this is followed by a "Remark" a page and a quarter long; then comes a "Postscript" of three and a half pages; and finally a small-page footnote as long as the "Remark."

It is the postscript which particularly concerns us. It begins thus:—

"Suppose that we have any series of terms, $u_1, u_2, u_3, \dots, u_n$, where

$$u_1 = A_1, \quad u_2 = A_1 A_2 - 1, \quad u_3 = A_1 A_2 A_3 - A_1 - A_3, \quad \dots$$

and in general

$$u_i = A_i u_{i-1} - u_{i-2},$$

then $u_1, u_2, u_3, \dots, u_n$ will be the successive principal coaxial determinants of a symmetrical matrix. Thus suppose $n=5$; if we write down the matrix

$$\begin{array}{ccccc} A_1 & 1 & 0 & 0 & 0 \\ 1 & A_2 & 1 & 0 & 0 \\ 0 & 1 & A_3 & 1 & 0 \\ 0 & 0 & 1 & A_4 & 1 \\ 0 & 0 & 0 & 1 & A_5 \end{array}$$

* The state of the theory in 1833 can best be gathered from Stern's monograph published in vol. x. of *Crelle's Journal*.

(the mode of formation of which is self-apparent), these successive coaxal determinants will be

$$1, A_1, \begin{vmatrix} A_1 & 1 \\ 1 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & 1 & 0 \\ 1 & A_2 & 1 \\ 0 & 1 & A_3 \end{vmatrix}, \begin{vmatrix} A_1 & 1 & 0 & 0 \\ 1 & A_2 & 1 & 0 \\ 0 & 1 & A_3 & 1 \\ 0 & 0 & 1 & A_4 \end{vmatrix}, \text{ etc.,}$$

i.e.

$$\begin{aligned} &1, A_1, A_1A_2 - 1, A_1A_2A_3 - A_1 - A_3, \\ &A_1A_2A_3A_4 - A_1A_2 - A_1A_4 - A_3A_4 + 1, \\ &A_1A_2A_3A_4A_5 - A_1A_2A_5 - A_1A_4A_5 - A_3A_4A_5 - A_1A_2A_3 \\ &\quad + A_5 + A_3 + A_1. \end{aligned}$$

It is proper to introduce the unit because it is, in fact, the value of a determinant of zero places, as I have observed elsewhere."

After using this as an aid to prove his proposition regarding Sturm's theorem, he returns to his new determinant in the following words:—

"I may conclude with noticing that the determinative [determinantal?] form of exhibiting the successive convergents to an improper continued fraction affords an instantaneous demonstration of the equation which connects any two consecutive such convergents as

$$\frac{N_{i-1}}{D_{i-1}} \quad \text{and} \quad \frac{N_i}{D_i} \quad \text{viz.} \quad N_i D_{i-1} - N_{i-1} D_i = 1.$$

For if we construct the matrix which for greater simplicity I limit to five lines and columns,

$$\begin{vmatrix} A & 1 & 0 & 0 & 0 \\ 1 & B & 1 & 0 & 0 \\ 0 & 1 & C & 1 & 0 \\ 0 & 0 & 1 & D & 1 \\ 0 & 0 & 0 & 1 & E \end{vmatrix}$$

and represent umbrally as

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5; \end{matrix}$$

and if, by way of example, we take the fourth and fifth convergents, these will be in the umbral notation represented by

$$\frac{\begin{smallmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{smallmatrix}}{\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{smallmatrix}} \quad \text{and} \quad \frac{\begin{smallmatrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \end{smallmatrix}}{\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{smallmatrix}}$$

respectively. Hence

$$\begin{aligned} & N_5 D_4 - N_4 D_5 \\ = & \frac{a_2}{b_2} \frac{a_3}{b_3} \frac{a_4}{b_4} \frac{a_5}{b_5} \times \frac{a_2}{b_2} \frac{a_3}{b_3} \frac{a_4}{b_4} \frac{a_1}{b_1} - \frac{a_2}{b_2} \frac{a_3}{b_3} \frac{a_4}{b_4} \times \frac{a_2}{b_2} \frac{a_3}{b_3} \frac{a_4}{b_4} \frac{a_5}{b_5} \frac{a_1}{b_1}, \\ = & \frac{a_2}{b_2} \frac{a_3}{b_3} \frac{a_4}{b_4} \frac{a_5}{b_5} \times \frac{a_2}{b_2} \frac{a_3}{b_3} \frac{a_4}{b_4} \frac{a_1}{b_1} - \frac{a_2}{b_2} \frac{a_3}{b_3} \frac{a_4}{b_4} \frac{a_5}{b_5} \frac{a_2}{b_2} \frac{a_3}{b_3} \frac{a_4}{b_4} \frac{a_1}{b_1} \\ = & \frac{a_2}{b_2} \frac{a_3}{b_3} \frac{a_4}{b_4} \frac{a_5}{b_5} \times \frac{a_2}{b_2} \frac{a_3}{b_3} \frac{a_4}{b_4} \frac{a_1}{b_1}, \\ = & \frac{a_2}{b_1} \frac{a_3}{b_2} \frac{a_4}{b_3} \frac{a_5}{b_4} \times \frac{a_1}{b_2} \frac{a_2}{b_3} \frac{a_3}{b_4} \frac{a_4}{b_5}, \\ = & \begin{matrix} 1 & B & 1 & 0 \\ 0 & 1 & C & 1 \\ 0 & 0 & 1 & D \\ 0 & 0 & 0 & 1 \end{matrix} \times \begin{matrix} 1 & 0 & 0 & 0 \\ B & 1 & 0 & 0 \\ 1 & C & 1 & 0 \\ 0 & 1 & D & 1 \end{matrix}, \\ = & 1 \times 1 = 1, \end{aligned}$$

as was to be proved. And the demonstration is evidently general in its nature."

In regard to this there has to be noted, first, the use of

$$\frac{\begin{smallmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{smallmatrix}}$$

when it would have been equally effective to use

$$\begin{matrix} 2 & 3 & 4 \\ 2 & 3 & 4; \end{matrix}$$

and, second, the use of a theorem for expressing the product of a five-line determinant and one of its secondary minors as an aggregate of products of pairs of four-line determinants.

Following on this comes the assertion that

"We may treat a proper continued fraction [*i.e.* with positive unit numerators] in precisely the same manner, substituting throughout $\sqrt{-1}$ in place of 1 in the generating matrix,

and we shall thus, by the same process as has been applied to improper continued fractions, obtain

$$\begin{aligned} N_{i+1}D_i - N_iD_{i+1} &= (\sqrt{-1})^i \times (\sqrt{-1})^i \\ &= (-1)^i. \end{aligned}$$

This would seem to imply that as yet Sylvester had not observed that an alternative mode of representation was obtainable by merely changing the sign of the units on one side of the diagonal.

The footnote contains two additional observations, the first being to the effect that the new mode of representation

“gives an immediate and visible proof of the simple and elegant rule for forming any such numerators or denominators by means of the principal terms [term ?] in each ; the rule, I mean, according to which the i^{th} denominator may be formed from

$$q_1q_2q_3q_4 \dots q_i$$

($q_1, q_2, \dots, q_i$ being the successive quotients) and the i^{th} numerator from

$$q_2q_3q_4 \dots q_i$$

by leaving out from the above products respectively any pair or any number of pairs of consecutive quotients as q_pq_{p+1} . For instance, from $q_1q_2q_3q_4q_5$ by leaving out q_1q_2, q_2q_3, q_3q_4 and q_4q_5 we obtain

$$q_3q_4q_5 + q_1q_4q_5 + q_1q_2q_5 + q_1q_2q_3;$$

and by leaving out $q_1q_2, q_3q_4, q_1q_2, q_4q_5, q_2q_3, q_4q_5$ we obtain

$$q_5 + q_3 + q_1;$$

so that the total denominator becomes

$$q_1q_2q_3q_4q_5 + q_3q_4q_5 + q_1q_4q_5 + q_1q_2q_5 + q_1q_2q_3 + q_5 + q_3 + q_1;$$

and in like manner the numerator of the same convergent is

$$q_2q_3q_4q_5 \left\{ 1 + \frac{1}{q_2q_3} + \frac{1}{q_3q_4} + \frac{1}{q_4q_5} + \frac{1}{q_2q_3q_4q_5} \right\}$$

i.e.

$$q_2q_3q_4q_5 + q_4q_5 + q_2q_5 + q_2q_3 + 1."$$

The "rule" here spoken of is that enunciated for the more general case of

$$a_1 + \frac{b_1}{a_2} + \frac{b_2}{a_3} +$$

in Stern's *Theorie der Kettenbrüche*, the fourth section of which is given up to the consideration of such rules (*Crelle's Journ.*, x. pp. 4-7).

The other observation is to the effect that

"every progression of terms constructed in conformity with the equation

$$u_n = a_n u_{n-1} - b_n u_{n-2} + c_n u_{n-3} \pm \dots$$

may be represented as an ascending series of principal coaxal determinants to a common matrix. Thus if each term in such progression is to be made a linear function of the three preceding terms, it will be representable by means of the matrix

$$\begin{array}{ccccc} A & B & C'' & 0 & 0 \\ 1 & A' & B'' & C''' & 0 \\ 0 & 1 & A'' & B''' & C'''' \\ 0 & 0 & 1 & A''' & B'''' \\ 0 & 0 & 0 & 1 & A'''' \end{array}$$

indefinitely continued, which gives the terms

$$1, A, AA' - B, AA'A'' - BA'' - AB'' + C'', \dots$$

This exhausts the paper so far as determinants are concerned: the results announced in it, one can readily own, were such as fairly to entitle the enthusiastic author to express his belief that "the introduction of the method of determinants into the algorithm of continued fractions cannot fail to have an important bearing upon the future treatment and development of the theory of numbers."

SPOTTISWOODE, W. (1853, August).*

[Elementary theorems relating to determinants. Second edition, rewritten and much enlarged by the author. *Crelle's Journ.*, li. (1856) pp. 209-271, 328-381.]

Save the utilisation of the fact that the denominator of any convergent of the continued fraction

$$a_1 + \frac{b_1}{a_2 + \frac{b_2}{a_3 + \dots}}$$

* This is the author's date at the end of the paper (p. 381). The first two parts of the volume, however, are dated 1855, and the remaining two 1856.

is the differential-quotient of the numerator, Spottiswoode did nothing but report the fundamental result reached by Sylvester. The full passage (p. 374) is as follows:—

“The improper continued fraction

$$\frac{1}{A - \frac{1}{B - \frac{1}{C - \dots}}} = \frac{d}{dA} \log_e \nabla$$

where

$$\nabla = \begin{vmatrix} A & 1 & 0 & \dots & 0 & 0 \\ 1 & B & 1 & \dots & 0 & 0 \\ 0 & 1 & C & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & M & 1 \\ 0 & 0 & 0 & \dots & 1 & N \end{vmatrix}$$

in which any number of rows may be taken at pleasure, and the formula will give the corresponding convergent fraction.

The same holds good for the continued fraction

$$\frac{1}{A} + \frac{1}{B} + \dots$$

if we write

$$\nabla = \begin{vmatrix} A & 1 & 0 & \dots \\ 1 & B & 1 & \dots \\ 0 & 1 & C & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

SYLVESTER, J. J. (1853, Sept.).

[On a fundamental rule in the algorithm of continued fractions. *Philos. Mag.* (4), vi. pp. 297-299.]

Without any reference to his previous paper on the subject Sylvester here announces that if

$$(a_1, a_2, \dots, a_i)$$

be the denominator of the i^{th} convergent to

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots$$

then

$$(a_1, \dots, a_m, a_{m+1}, \dots, a_{m+n}) = (a_1, \dots, a_m)(a_{m+1}, \dots, a_{m+n}) \\ + (a_1, \dots, a_{m-1})(a_{m+2}, \dots, a_{m+n}),$$

—a possibly new result which he considers “the fundamental theorem in the theory of continued fractions.” This, he says, is an immediate consequence of the fact that $(a_1, \dots, a_{m+n})$ can be expressed as a determinant, all that is further necessary being the application of the “well-known simple rule for the decomposition of determinants.” Thus, *e.g.*, the determinant

$$\begin{array}{cccc} a & 1 & & \\ -1 & b & 1 & \\ & -1 & c & 1 \\ & & -1 & d & 1 \\ & & & -1 & e & 1 \\ & & & & -1 & f \end{array}$$

is obviously decomposable into

$$\begin{array}{ccc} a & 1 & \\ -1 & b & 1 \\ & -1 & c \end{array} \times \begin{array}{ccc} d & 1 & \\ -1 & e & 1 \\ & -1 & f \end{array} + \begin{array}{ccc} a & 1 & \\ -1 & b & \\ & & \end{array} \times \begin{array}{ccc} e & 1 & \\ -1 & f & \end{array},$$

or into

$$\begin{array}{ccc} a & 1 & \\ -1 & b & \end{array} \times \begin{array}{ccc} c & 1 & \\ -1 & d & 1 \\ & -1 & e & 1 \\ & & -1 & f \end{array} + \begin{array}{ccc} a & & \\ & & d & 1 \\ & & -1 & e & 1 \\ & & & -1 & f \end{array},$$

or into

$$\begin{array}{ccc} a & & \\ & & b & 1 \\ & & -1 & c & 1 \\ & & & -1 & d & 1 \\ & & & & -1 & e & 1 \\ & & & & & -1 & f \end{array} + \begin{array}{ccc} c & 1 & \\ -1 & d & 1 \\ & -1 & e & 1 \\ & & -1 & f \end{array}.$$

Following this is what is called “Corollary I.” viz.,

$$(a_1, a_2, \dots, a_m) \cdot (a_2, a_3, \dots, a_{m+i}) - (a_2, a_3, \dots, a_m) \cdot (a_1, a_2, \dots, a_{m+i}) \\ = (-)^m (a_{m+i} a_{m+i-1} \dots \text{to } i-1 \text{ terms}),$$

its connection with the expression for the difference of two convergents being illustrated by the instances $i=1, 2, 3, 4, \dots$

The next "corollary," viz.,

$$\begin{aligned} & (a_1, \dots, a_p, a_{p+1}, \dots, a_{p+f})(a_1, \dots, a_p, a_{p+1}, \dots, a_{p+k}) \\ & - (a_1, \dots, a_p, a_{p+1}, \dots, a_{p+g})(a_1, \dots, a_p, a_{p+1}, \dots, a_{p+h}) \\ & = (-)^p \{ (a_{p+1}, \dots, a_{p+f})(a_{p+1}, \dots, a_{p+k}) - (a_{p+1}, \dots, a_{p+g})(a_{p+1}, \dots, a_{p+h}) \} \end{aligned}$$

is clearly incorrect, it being impossible for the value of the left-hand side to be independent of the elements $a_1, a_2, \dots, a_p$. Further, as the author gives no accompanying word of comment, the difficulty of suggesting the true theorem is increased. A "sub-corollary" is appended dealing with the case where all the a 's are equal, and leading up, not without some misprints or inaccuracies, to a theorem of Euler's quoted from the *Nouvelles Annales de Math.*, v. (Sept. 1851) pp. 357-358, to the effect that if $T_{n+1} = aT_n - bT_{n-1}$ be the generating equation of a recurrent series, then

$$\frac{T_{n+1}^2 - aT_nT_{n+1} + bT_n^2}{b^n}$$

is a constant with respect to n . Of course the more natural form of this expression is

$$\frac{T_{n+1}^2 - T_nT_{n+2}}{b^n},$$

the numerator of which being

$$\begin{vmatrix} T_{n+1} & T_{n+2} \\ T_n & T_{n+1} \end{vmatrix}$$

is successively transformable by means of the recursion-formula into

$$b \begin{vmatrix} T_n & T_{n+1} \\ T_{n-1} & T_n \end{vmatrix}, \quad b^2 \begin{vmatrix} T_{n-1} & T_n \\ T_{n-2} & T_{n-1} \end{vmatrix}, \quad b^3 \begin{vmatrix} T_{n-2} & T_{n-1} \\ T_{n-3} & T_{n-2} \end{vmatrix}, \dots$$

so that the constant in question is

$$\begin{vmatrix} T_1 & T_2 \\ T_0 & T_1 \end{vmatrix}.$$

This, however, Sylvester does not show.*

* An interesting extension of this is given by Brioschi in the *Nouvelles Annales de Math.*, xiv. (Jan. 1854) p. 20.

Finally, and to more purpose, it is noted that if we pass from $(a_1, a_2, \dots, a_i)$ to the readily-suggested extension

$$\begin{array}{ccccccc} m_1 & l_1 & & & & & \\ n_1 & m_2 & l_2 & & & & \\ & n_2 & m_3 & l_3 & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & n_{i-1} & m_i & l_i & \\ & & & & n_i & m_{i+1} & \end{array}$$

the corresponding fundamental theorem is

$$\begin{pmatrix} l_1 \dots l_{i+j} \\ m_1, m_2, \dots, m_{i+j+1} \\ n_1 \dots n_{i+j} \end{pmatrix} = \begin{pmatrix} l_1 \dots l_{i-1} \\ m_1, m_2, \dots, m_i \\ n_1 \dots n_{i-1} \end{pmatrix} \begin{pmatrix} l_{i+1} \dots l_{i+j} \\ m_{i+1}, m_{i+2}, \dots, m_{i+j+1} \\ n_{i+1} \dots n_{i+j} \end{pmatrix} \\ - l_i n_i \begin{pmatrix} l_1 \dots l_{i-2} \\ m_1, m_2, \dots, m_{i-1} \\ n_1 \dots n_{i-2} \end{pmatrix} \begin{pmatrix} l_{i+2} \dots l_{i+j} \\ m_{i+2}, m_{i+3}, \dots, m_{i+j+1} \\ n_{i+2} \dots n_{i+j} \end{pmatrix}.$$

SYLVESTER, J. J. (1853, Oct., Nov.).

[On a theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraical common measure. *Phil. Trans. Roy. Soc. London*, cxliii. pp. 407-548.]

Although this lengthy memoir in its original form bears date "16th June 1853," it is the equally lengthy "supplements" added later while passing through the press that claim attention in the present connection. In the first of these (§ i., p. 474) the denominator of the fraction

$$\frac{1}{q_1} - \frac{1}{q_2} - \dots - \frac{1}{q_n}$$

is denoted by $[q_1, q_2, \dots, q_n]$, and termed a "cumulant," and throughout the later portion of the paper this name constantly recurs. It is not, however, until we come to the second

“supplement” that anything apparently new in substance is met with. There in § a (p. 497) the following lemma occurs:—

“The roots of the cumulant $[q_1, q_2, \dots, q_i]$ in which each element is a linear function of x , and wherein the coefficient of x for each element has the like sign, are all real: and between every two of such roots is contained a root of the cumulant $[q_1, q_2, \dots, q_{i-1}]$ and *ex converso* a root of the cumulant $[q_2, q_3, \dots, q_i]$: and (as an evident corollary) for all values of ζ and ζ' intermediate between 1 and i the greatest root of $[q_1, q_2, \dots, q_i]$ will be greater, and the least root of the same will be less than the greatest and least roots respectively of $[q_\rho, q_{\rho+1}, \dots, q_{\rho'-1}, q_{\rho'}]$.”

Even this, however, may be placed under the well-known theorem regarding the roots of the equation

$$\begin{vmatrix} a_{11} - x & a_{12} & a_{13} & \dots \\ a_{12} & a_{22} - x & a_{23} & \dots \\ a_{13} & a_{23} & a_{33} - x & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0$$

which had been enunciated by Cauchy in 1829.*

The next noteworthy result occupies § i. (p. 502). As a preparation for it the theorem

$$[a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n] = [a_1, a_2, \dots, a_m][b_1, b_2, \dots, b_n] \\ - [a_1, a_2, \dots, a_{m-1}][b_2, b_3, \dots, b_n]$$

may be recalled, the group of elements on the left being now viewed as consisting of two sub-groups. This theorem Sylvester writes in the form

$$[\Omega_1 \Omega_2] = [\Omega_1][\Omega_2] - [\Omega'_1][\Omega'_2]$$

and he succeeds in including in it a general theorem, not explicitly formulated, in which the number of groups is i , the next two cases being

$$[\Omega_1 \Omega_2 \Omega_3] = [\Omega_1][\Omega_2][\Omega_3] \\ - [\Omega'_1][\Omega'_2][\Omega_3] - [\Omega_1][\Omega'_2][\Omega'_3] + [\Omega'_1][\Omega'_2][\Omega'_3],$$

* v. The theory of orthogonants . . . in *Proc. Roy. Soc. Edinburgh*, xxiv. p. 261.

and

$$\begin{aligned} [\Omega_1 \Omega_2 \Omega_3 \Omega_4] = & [\Omega_1][\Omega_2][\Omega_3][\Omega_4] \\ & - [\Omega'_1][\Omega_2][\Omega_3][\Omega_4] - [\Omega_1][\Omega'_2][\Omega_3][\Omega_4] - [\Omega_1][\Omega_2][\Omega'_3][\Omega_4] \\ & + [\Omega'_1][\Omega'_2][\Omega_3][\Omega_4] + [\Omega'_1][\Omega_2][\Omega'_3][\Omega_4] + [\Omega_1][\Omega'_2][\Omega'_3][\Omega_4] \\ & - [\Omega'_1][\Omega'_2][\Omega'_3][\Omega_4]. \end{aligned}$$

The general theorem is described as giving an expression for $[\Omega_1 \Omega_2 \dots \Omega_i]$ in terms of

$$\begin{aligned} & [\Omega_1], [\Omega_2], \dots, [\Omega_{i-1}], [\Omega_i] \\ & [\Omega'_1], [\Omega'_2], \dots, [\Omega'_{i-1}] \\ & [\Omega_2], \dots, [\Omega_{i-1}], [\Omega_i] \\ & [\Omega'_2], \dots, [\Omega'_{i-1}]; \end{aligned}$$

that is to say, in terms of all the unaltered Ω 's, all the curtailed Ω 's except the last, all the beheaded Ω 's except the first, and all the "doubly-apocopated" Ω 's except the first and the last; and it is pointed out that the number of products (or terms) in the expansion is 2^{i-1} "separable into i alternately positive and negative groups containing respectively 1, $(i-1)$, $\frac{1}{2}(i-1)(i-2)$, $\dots$, $i-1$, 1 products." Further, it is noted that "in every one of the above groups forming a product the accents enter in pairs and between contiguous factors, it being a condition that if any Ω have an accent on the right the next Ω must have one on the left, and if it have one on the left the preceding Ω must have an accent on the right, and the number of pairs of accents goes on increasing in each group from 0 to $i-1$."

In a footnote the case where each Ω has only one element, and where, therefore, each singly-accented Ω becomes 1, and each doubly-accented Ω vanishes, is stated to be identical with the "rule"

$$\begin{aligned} [a_1, a_2, \dots, a_i] = & a_1 a_2 \dots a_i - \sum \frac{1}{a_e a_{e+1}} \cdot a_1 a_2 \dots a_i \\ & + \sum \frac{1}{a_e a_{e+1} a_f a_{f+1}} \cdot a_1 a_2 \dots a_i - \dots \end{aligned}$$

SYLVESTER, J. J. (1854, August).

[Théorème sur les déterminants de M. Sylvester. *Nouv. Annales de Math.*, xiii. p. 305.]

This communication in its entirety is as follows:—

“Soient les déterminants

$$\begin{array}{ccccccc} \lambda, & \lambda & 1 & & \lambda & 1 & 0 & & \lambda & 1 & 0 & 0 \\ & 1 & \lambda, & & 2 & \lambda & 2 & & 3 & \lambda & 2 & 0 \\ & & & & 0 & 1 & \lambda, & & 0 & 2 & \lambda & 3 \\ & & & & & & & & 0 & 0 & 1 & \lambda, \\ & & & & \lambda & 1 & 0 & 0 & 0 & & & \\ & & & & 4 & \lambda & 2 & 0 & 0 & & & \\ & & & & 0 & 3 & \lambda & 3 & 0 & & & \\ & & & & 0 & 0 & 2 & \lambda & 4 & & & \\ & & & & 0 & 0 & 0 & 1 & \lambda, & \dots & \dots & \end{array}$$

la loi de formation est évidente; effectuant, on trouve

$$\lambda, \lambda^2 - 1, \lambda(\lambda^2 - 2^2), (\lambda^2 - 1^2)(\lambda^2 - 3^2), \lambda(\lambda^2 - 2^2)(\lambda^2 - 4^2), \\ (\lambda^2 - 1^2)(\lambda^2 - 3^2)(\lambda^2 - 5^2), \lambda(\lambda^2 - 2^2)(\lambda^2 - 4^2)(\lambda^2 - 6^2),$$

et ainsi de suite.”

That Sylvester was the author of the implied theorem may be considered proved by an entry in the index of the volume (v. p. 478), and by a statement of Cayley's in the *Quarterly Journal of Mathematics*, ii. p. 163. Probably the title of the communication was prefixed by the editors, who, knowing of Sylvester's papers in the *Philosophical Magazine*, felt themselves justified in applying the name “Sylvester's determinants.”

SCHLÄFLI, L. (Nov. 1855).

[Réduction d'une intégrale multiple qui comprend l'arc de cercle et l'aire du triangle sphérique comme cas particuliers. *Journ. de Liouville*, xx. pp. 359-394.]

Here there appears the equation

$$\frac{\Delta(a, \beta, \dots, \zeta, \eta)}{\Delta(\beta, \dots, \zeta, \eta)} = 1 - \frac{\cos^2 a}{1} - \frac{\cos^2 \beta}{1} - \dots - \frac{\cos^2 \zeta}{1 - \cos^2 \eta}$$

$$a + \frac{b}{a} + \frac{b}{a} + \dots (n \text{ } b's) = \frac{1}{\sqrt{a^2 + 4b}} \left\{ \left(\frac{a + \sqrt{a^2 + 4b}}{2} \right)^{n+2} - \left(\frac{a - \sqrt{a^2 + 4b}}{2} \right)^{n+2} \right\},$$

$$= a^{n+1} + C_{n,1} a^{n-1} b + C_{n-1,2} a^{n-3} b^2 + \dots$$

which by putting $a=1=b$ give the number of terms in Y_n ,—a number also obtained in the form

$$\frac{1}{2^{n+1}} \left\{ C_{n+2,1} + C_{n+2,3} \cdot 5 + C_{n+2,5} \cdot 5^2 + \dots \right\}.$$

Anything else is of small moment.

CAYLEY, A. (1857, April).

[On the determination of the value of a certain determinant.
Quart. Journ. of Math., ii. pp. 163–166 ; or *Collected Math. Papers*, iii. pp. 120–123.]

The determinant in question is rather more general than Sylvester's of the year 1854 (*Nouv. Annales de Math.*, xiii. p. 305), being

$$\begin{vmatrix} \theta & 1 & . & . & \dots & . & . \\ x & \theta & 2 & . & \dots & . & . \\ . & x-1 & \theta & 3 & \dots & . & . \\ . & . & x-2 & \theta & \dots & . & . \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ . & . & . & . & \dots & \theta & n-1 \\ . & . & . & . & \dots & x-n+2 & \theta \end{vmatrix},$$

while the other is obtained from this by putting $x=n-1$. Denoting his own form by U_n , Cayley, with Sylvester's results before him, found

$$U_2 = (\theta^2 - 1) - (x-1),$$

$$U_3 = \theta(\theta^2 - 4) - 3(x-2)\theta,$$

$$U_4 = (\theta^2 - 1)(\theta^2 - 9) - 6(x-3)(\theta^2 - 1) + 3(x-3)(x-1);$$

so that, if he put H_n for the value of U_n in Sylvester's case (viz., when $x=n-1$), he could write

$$U_2 = H_2 - (x-1)H_0,$$

$$U_3 = H_3 - 3(x-2)H_1,$$

$$U_4 = H_4 - 6(x-3)H_2 + 3(x-3)(x-1)H_0,$$

$$\dots$$

and thence, doubtless, divined the generalisation

$$U_n = H_n - B_{n,1} \cdot (x-n+1) \cdot H_{n-2} + B_{n,2} \cdot (x-n+1)(x-n+3) \cdot H_{n-4} - \dots$$

where

$$H_n = (\theta + n - 1)(\theta + n - 3)(\theta + n - 5) \dots \text{to } n \text{ factors}$$

and

$$B_{n,s} = \frac{n(n-1)(n-2) \dots (n-2s+1)}{2^s \cdot 1 \cdot 2 \cdot 3 \dots s}.$$

The establishment of the truth of this is all that the paper is occupied with, the procedure being to expand U_n in terms of the elements of its last row and their complementary minors, thus obtaining

$$U_n = \theta U_{n-1} - (n-1)(x-n+2)U_{n-2}$$

and thence

$$U_n + \left\{ (n-1)(x-n+2) + (n-2)(x-n+3) - \theta^2 \right\} U_{n-2} \\ + (n-2)(n-3)(x-n+3)(x-n+4)U_{n-4} = 0,$$

and showing that the above conjectural expression for U_n satisfies the latter equation. The process of verification is troublesome, and was not viewed with satisfaction by Cayley himself.

As a preliminary the coefficients of the H 's in the value of U_n are for shortness' sake denoted by $A_{n,0}, -A_{n,1}, \dots$, and for the same and an additional reason the coefficient of U_{n-2} in the difference-equation is denoted by

$$M_{n,s} = \left\{ \theta^2 - (n-2s-1)^2 \right\},$$

which is equivalent to putting

$$M_{n,s} \equiv (n-1)(x-n+2) + (n-2)(x-n+3) - (n-2s-1)^2.$$

The operation to be performed being thus the substitution of

$$A_{n,0}H_n - A_{n,1}H_{n-2} + \dots + (-)^s A_{n,s}H_{n-2s} + \dots$$

for U_n in the expression

$$+ \left[M_{n,s} - \left\{ \theta^2 - (n-2s-1)^2 \right\} \right] U_{n-2} + (n-2)(n-3)(x-n+3)(x-n+4)U_{n-4},$$

it is readily seen that the result will be an aggregate of expressions like

$$A_{n,s}H_{n-2s} + \left[M_{n,s} - \left\{ \theta^2 - (n-2s-1)^2 \right\} \right] A_{n-2,s}H_{n-2-2s} \\ + (n-2)(n-3)(x-n+3)(x-n+4)A_{n-4,s}H_{n-4-2s}.$$

Now if we bear in mind that by definition

$$\left\{ \theta^2 - (n-2s-1)^2 \right\} H_{n-2-2s} = H_{n-2s},$$

the second of the three terms of this

$$= M_{n,s}A_{n-2,s}H_{n-2-2s} - A_{n-2,s}H_{n-2s},$$

or, if we write $s-1$ for s in one case,

$$= -M_{n,s-1}A_{n-2,s-1}H_{n-2s} - A_{n-2,s}H_{n-2s} \\ = -H_{n-2s} \left\{ M_{n,s-1}A_{n-2,s-1} + A_{n-2,s} \right\};$$

and the third, by writing $s-2$ for s ,

$$= (n-2)(n-3)(x-n+3)(x-n+4)A_{n-4,s-2}H_{n-2s}.$$

Consequently the sum of the three will vanish if

$$A_{n,s} - (M_{n,s-1}A_{n-2,s-1} + A_{n-2,s}) + (n-2)(n-3)(x-n+3)(x-n+4)A_{n-4,s-2} = 0,$$

and therefore if

$$B_{n,s}(x-n+1) - B_{n-2,s}(x-n+2s+1) \\ - B_{n-2,s-1}M_{n,s-1} + B_{n-4,s-2}(n-2)(n-3)(x-n+4) = 0,$$

that is, if

$$(x-n) \left[B_{n,s} - B_{n-2,s} - (2n-3)B_{n-2,s-1} + (n-2)(n-3)B_{n-4,s-2} \right] \\ + \left[B_{n,s} - (2s+1)B_{n-2,s} - \left\{ 5n-8-(n-2s+1)^2 \right\} B_{n-2,s-1} + 4(n-2)(n-3)B_{n-4,s-2} \right] = 0.$$

But this is the case; for, as Cayley shows, both the cofactor of $x-n$ and the other similar expression following it vanish identically. The verification aimed at is thus attained.

PAINVIN (1858, February).

[Sur un certain système d'équations linéaires. *Journ. de Liouville* (2), iii. pp. 41-46.]

The system of equations referred to in the title of Painvin's paper had presented themselves to Liouville in the course of the research which led to his "Mémoire sur les transcendentes elliptiques . . ." (*Journ. de Liouville* (1), v. pp. 441-464). Painvin's reason for taking up the subject was his belief that one of Liouville's results could be more simply arrived at by the use of determinants; and in a few lines of introduction he succeeds in showing that the result in question can be viewed as merely the resolution of the determinant

$$\begin{vmatrix} r & a & . & . & \dots & . & . \\ n(a-1) & r-1 & 2a & . & \dots & . & . \\ . & (n-1)(a-1) & r-2 & 3a & \dots & . & . \\ . & . & (n-2)(a-1) & r-3 & \dots & . & . \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ . & . & . & . & \dots & r-n+1 & na \\ . & . & . & . & \dots & a-1 & r-n \end{vmatrix}$$

into factors.

In explanation of the process followed the case of the fourth order

$$\begin{vmatrix} r & a & . & . \\ 3(a-1) & r-1 & 2a & . \\ . & 2(a-1) & r-2 & 3a \\ . & . & a-1 & r-3 \end{vmatrix}$$

will suffice. Increasing each element of the first row by the corresponding elements of the other rows,—an operation which we may for the nonce symbolise by

$$\text{row}_1 + \text{row}_2 + \text{row}_3 + \dots \dots \dots,$$

—he removes the factor $r + 3a - 3$ and finds left the cofactor

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 3(a-1) & r-1 & 2a & . \\ . & 2(a-1) & r-2 & 3a \\ . & . & a-1 & r-3 \end{vmatrix}.$$

On this are performed the operations

$$\text{col}_1 - \text{col}_2, \quad \text{col}_2 - \text{col}_3, \quad \text{col}_3 - \text{col}_4, \quad \dots$$

the result being a determinant of the next lower order

$$- \begin{vmatrix} 3a - r - 2 & r - 2a - 1 & 2a \\ 2 - 2a & 2a - r & r - 3a - 2 \\ . & 1 - a & a - r + 2 \end{vmatrix}.$$

Finally, after changing the signs of all the elements here, the operations

$\text{row}_1 + \text{row}_2 + \text{row}_3 + \dots$, $\text{row}_2 + \text{row}_3 + \dots$, $\text{row}_3 + \dots$, ... are performed, the result

$$\begin{vmatrix} r - a & a & . \\ 2(a - 1) & r - a - 1 & 2a \\ . & a - 1 & r - a - 2 \end{vmatrix}$$

being a determinant exactly similar in form to the original but with $r - a$ instead of r . This, therefore, in turn may be transformed into

$$(r + a - 2) \begin{vmatrix} r - 2a & a \\ a - 1 & r - 2a - 1 \end{vmatrix},$$

and so on.

The value thus obtained for the above-written determinant of the $(n+1)^{\text{th}}$ order is

$$(r + na - n)(r + na - n - 2a + 1)(r + na - n - 4a + 2) \dots (r - na)$$

each factor being less than the preceding by $2a - 1$, and the whole a function of $a(a - 1)$.

The special case is noted where $a = \frac{1}{2}$, and where therefore all the $n + 1$ resulting factors are alike. This Painvin writes in the form

$$\begin{vmatrix} r & \frac{1}{2} & . & . & . & . & . \\ -\frac{n}{2} & r - 1 & \frac{2}{2} & . & . & . & . \\ . & -\frac{n-1}{2} & r - 2 & \frac{3}{2} & . & . & . \\ . & -\frac{n-2}{2} & r - 3 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & r - n + 1 & \frac{n}{2} \\ . & . & . & . & . & -\frac{1}{2} & r - n \end{vmatrix} = \left(r - \frac{n}{2}\right)^{n+1};$$

but a preferable is, evidently,

$$\begin{vmatrix} \rho & 1 & . & . & . & . & . \\ -n & \rho-2 & 2 & . & . & . & . \\ . & -n+1 & \rho-4 & 3 & . & . & . \\ . & . & -n+2 & \rho-6 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & \rho-2n+2 & n \\ . & . & . & . & . & -1 & \rho-2n \end{vmatrix} = (\rho-n)^{n+1}.$$

HEINE, E. (1858, Sept.).

[Auszug eines Schreibens über die *Laméschen* Functionen an den Herausgeber. Einige Eigenschaften der *Laméschen* Functionen. *Crelle's Journ.*, lvi. pp. 79-86, 87-99.]

In the case of Heine the functions afterwards known as "continuants" made their appearance under totally different circumstances, viz., while he was engaged in transforming a special homogeneous function of the second degree by means of an orthogonal transformation. It will be remembered that if the quadric

$$\begin{aligned} a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \dots \\ + a_{22}x_2^2 + 2a_{23}x_2x_3 + \dots \\ + a_{33}x_3^2 + \dots \end{aligned}$$

be transformed by an orthogonal transformation into

$$A_{11}\xi_1^2 + A_{22}\xi_2^2 + A_{33}\xi_3^2 + \dots$$

the coefficients of the latter expression are the roots of the equation

$$\begin{vmatrix} a_{11}-A & a_{12} & a_{13} & \dots \\ a_{12} & a_{22}-A & a_{23} & \dots \\ a_{13} & a_{23} & a_{33}-A & \dots \\ . & . & . & . \end{vmatrix} = 0.$$

Now Heine's peculiar quadric was

$$\begin{aligned} c_0^2x_0^2 - 2\kappa c_0c_1x_0x_1 \\ + (c_1^2 + c_2^2)x_1^2 - 2\kappa c_2c_3x_1x_2 \\ + (c_3^2 + c_4^2)x_2^2 - \dots \\ + (c_{2\sigma}^2 - 1 + c_{2\sigma}^2)x_{\sigma}^2 \end{aligned}$$

where in every case the coefficient of the product of two x 's vanishes if their suffixes differ by more than 1, and where

$$\begin{aligned} c_0^2 &= \frac{1}{2}(n)(n+1), \\ c_1^2 &= \frac{1}{4}(n-1)(n+2), \\ &\dots \dots \dots \\ c_r^2 &= \frac{1}{4}(n-r)(n+r+1), \quad (r > 0) \\ &\dots \dots \dots \\ c_{n-1}^2 &= \frac{1}{2}n, \\ \text{and } \kappa &= \frac{c^2 - b^2}{c^2 + b^2}. \end{aligned}$$

He was thus naturally led to the equation in z

$$\begin{array}{cccccccc} z - c_0^2 & \kappa c_0 c_1 & . & . & . & . & . & . \\ \kappa c_0 c_1 & z - c_1^2 - c_2^2 & \kappa c_2 c_3 & . & . & . & . & . \\ . & \kappa c_2 c_3 & z - c_3^2 - c_4^2 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & \kappa c_{2\sigma-2} c_{2\sigma-1} & . \\ . & . & . & . & . & . & z - c_{2\sigma-1}^2 - c_{2\sigma}^2 & . \end{array} \Bigg| = 0,$$

where either $c_{2\sigma}^2$ is c_{n-1}^2 , or $c_{2\sigma-1}^2$ is c_n^2 and, if the latter, $c_{2\sigma}^2 = 0$. From a knowledge of Painvin's paper he recognised the left-hand side of the equation as being the numerator of the continued fraction

$$z - c_0^2 - \frac{\kappa c_0^2 c_1^2}{z - c_1^2 - c_2^2} - \frac{\kappa c_2^2 c_3^2}{z - c_3^2 - c_4^2} - \dots$$

but he ventured nothing in elucidation of it. Even the special case where $b=0$ and where therefore $\kappa=1$ appears to have proved at the time too troublesome, although he knew otherwise that in this case the continued fraction

$$= \frac{z(z-2^2)(z-4^2) \dots (z-n^2)}{(z-1^2)(z-3^2) \dots (z-n-1^2)} \text{ if } n \text{ be even}$$

and

$$= \frac{(z-1^2)(z-3^2)(z-5^2) \dots (z-n^2)}{(z-2^2)(z-4^2) \dots (z-n-1^2)} \text{ if } n \text{ be odd};$$

for his words are—"Einen directen Beweis für diese Summirung des Kettenbruchs habe ich noch nicht aufgefunden."

SCHLÄFLI, L. (1858).

[On the multiple integral $\int^n dx dy \dots dz$ whose limits are

$p_1 = a_1x + b_1y + \dots + h_1z > 0, p_2 > 0, \dots, p_n > 0,$
 and $x^2 + y^2 + \dots + z^2 > 1$. *Quart. Journ. of Math.*, ii.
 pp. 269-301, iii. pp. 54-68, 97-108.]

The determinant which makes its appearance in the course of Schläfli's research is

$$\begin{vmatrix} 1 & -\cos \alpha & . & . & . & . & . & . & . \\ -\cos \alpha & 1 & -\cos \beta & . & . & . & . & . & . \\ . & -\cos \beta & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & 1 & -\cos \eta & . & . \\ . & . & . & . & -\cos \eta & 1 & -\cos \theta & . & . \\ . & . & . & . & . & -\cos \theta & 1 & . & . \end{vmatrix}$$

which for shortness' sake he denotes by

$$\Delta(\alpha, \beta, \gamma, \dots, \eta, \theta)$$

and whose connection with continued fractions he therefore specifies by the equation

$$\frac{\Delta(\alpha, \beta, \gamma, \dots, \eta, \theta)}{\Delta(\beta, \gamma, \dots, \eta, \theta)} = 1 - \frac{\cos^2 \alpha}{1} - \frac{\cos^2 \beta}{1} - \dots - \frac{\cos^2 \eta}{1 - \cos^2 \theta}$$

The first property noticed is, naturally,

$$\Delta(\alpha, \beta, \gamma, \dots, \theta) = \Delta(\beta, \gamma, \dots, \theta) - \cos^2 \alpha \cdot \Delta(\gamma, \dots, \theta).$$

Later there is given what may be viewed as an extension of this, viz.,

$$\Delta(\alpha, \dots, \delta, \epsilon, \zeta, \eta, \theta, \dots, \lambda) = \Delta(\alpha, \dots, \delta, \epsilon) \cdot \Delta(\eta, \theta, \dots, \lambda) \\ - \cos^2 \zeta \cdot \Delta(\alpha, \dots, \delta) \cdot \Delta(\theta, \dots, \lambda),$$

the proof being said to present no difficulty. The third is a little more complicated, and is logically led up to by taking four instances of the first property, viz.,

$$\begin{aligned} \Delta(\alpha, \beta, \gamma, \dots, \zeta) &= \Delta(\beta, \gamma, \dots, \zeta) - \cos^2 \alpha \cdot \Delta(\gamma, \delta, \dots, \zeta), \\ \Delta(\beta, \gamma, \delta, \dots, \zeta, \eta) &= \Delta(\gamma, \delta, \dots, \eta) - \cos^2 \beta \cdot \Delta(\delta, \dots, \zeta, \eta), \\ \Delta(\gamma, \delta, \dots, \zeta, \eta, \theta) &= \Delta(\gamma, \delta, \dots, \eta) - \cos^2 \theta \cdot \Delta(\gamma, \delta, \dots, \zeta), \\ \Delta(\delta, \dots, \zeta, \eta, \theta, \alpha) &= \Delta(\delta, \dots, \zeta, \eta, \theta) - \cos^2 \alpha \cdot \Delta(\delta, \dots, \zeta, \eta), \end{aligned}$$

using in connection with these the multipliers

$$\Delta(\delta, \dots, \zeta, \eta), \quad -\Delta(\delta, \dots, \zeta), \quad \Delta(\delta, \dots, \zeta), \quad -\Delta(\gamma, \delta, \dots, \zeta),$$

respectively, performing addition, and then showing that the right-hand sum vanishes, the result thus being

$$\begin{aligned} \Delta(\alpha, \beta, \gamma, \delta, \dots, \zeta) \cdot \Delta(\delta, \dots, \zeta, \eta) &- \Delta(\delta, \dots, \zeta, \eta, \theta, \alpha) \cdot \Delta(\gamma, \delta, \dots, \zeta) \\ &= \{ \Delta(\beta, \gamma, \delta, \dots, \zeta, \eta) - \Delta(\gamma, \delta, \dots, \zeta, \eta, \theta) \} \cdot \Delta(\delta, \dots, \zeta). \end{aligned}$$

The fourth property concerns the determinant

$$\begin{vmatrix} \Delta(\beta, \gamma, \dots, \eta, \theta) & \Delta(\alpha, \beta, \gamma, \dots, \eta, \theta) \\ \Delta(\beta, \gamma, \dots, \eta) & \Delta(\alpha, \beta, \gamma, \dots, \eta) \end{vmatrix}$$

which by reason of the first property can be shown equal to

$$\begin{vmatrix} \Delta(\beta, \gamma, \dots, \eta, \theta) & -\Delta(\gamma, \dots, \eta, \theta) \\ \Delta(\beta, \gamma, \dots, \eta) & -\Delta(\gamma, \dots, \eta) \end{vmatrix} \cos^2 \alpha,$$

or

$$\begin{vmatrix} \Delta(\gamma, \dots, \eta, \theta) & \Delta(\beta, \gamma, \dots, \eta, \theta) \\ \Delta(\gamma, \dots, \eta) & \Delta(\beta, \gamma, \dots, \eta) \end{vmatrix} \cos^2 \alpha,$$

and ultimately, "by repeating this sort of transformation," equal to

$$\cos^2 \alpha \cos^2 \beta \cos^2 \gamma \dots \cos^2 \theta.$$

If we use for a moment the present-day notation for continuants, viz., where

$$a_1 + \frac{b_1}{a_2} + \frac{b_2}{a_3} + \dots = \frac{K \begin{pmatrix} b_1 & b_2 \\ a_1 & a_2 & a_3 \dots \end{pmatrix}}{K \begin{pmatrix} b_2 \\ a_2 & a_3 \dots \end{pmatrix}}$$

Schläfli's results are seen to be

$$\begin{aligned} K \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ 1 & 1 & 1 \dots \end{pmatrix} &= K \begin{pmatrix} \beta_2 & \beta_3 \\ 1 & 1 \dots \end{pmatrix} + \beta_1 K \begin{pmatrix} \beta_3 & \beta_4 \\ 1 & 1 \dots \end{pmatrix}, \\ K \begin{pmatrix} \beta_1 & \beta_2 & \beta_\kappa & \beta_n \\ 1 & 1 & 1 \dots 1 \end{pmatrix} &= K \begin{pmatrix} \beta_1 & \beta_2 & \beta_{\kappa-1} \\ 1 & 1 & 1 \dots \end{pmatrix} \cdot K \begin{pmatrix} \beta_{\kappa+1} & \beta_n \\ 1 & 1 \dots 1 \end{pmatrix} \\ &\quad - \beta_\kappa K \begin{pmatrix} \beta_1 & \beta_{\kappa-2} \\ 1 & 1 \dots 1 \end{pmatrix} \cdot K \begin{pmatrix} \beta_{\kappa+2} & \beta_n \\ 1 & 1 \dots 1 \end{pmatrix}, \end{aligned}$$

$$\left. \begin{aligned} &K\left(\begin{smallmatrix} \beta_1 & \beta_{n-2} \\ 1 & 1 \end{smallmatrix} \dots \begin{smallmatrix} \beta_{n-1} \\ 1 \end{smallmatrix}\right) \cdot K\left(\begin{smallmatrix} \beta_4 & \beta_{n+1} \\ 1 & 1 \end{smallmatrix} \dots \begin{smallmatrix} \beta_{n+1} \\ 1 \end{smallmatrix}\right) \\ &K\left(\begin{smallmatrix} \beta_4 & \beta_n & \beta_1 \\ 1 & 1 & 1 \end{smallmatrix} \dots \begin{smallmatrix} \beta_{n-2} \\ 1 \end{smallmatrix}\right) \cdot K\left(\begin{smallmatrix} \beta_3 & \beta_{n-2} \\ 1 & 1 \end{smallmatrix} \dots \begin{smallmatrix} \beta_{n-2} \\ 1 \end{smallmatrix}\right) \end{aligned} \right\} = \left\{ \begin{aligned} &K\left(\begin{smallmatrix} \beta_2 & \beta_{n-1} \\ 1 & 1 \end{smallmatrix} \dots \begin{smallmatrix} \beta_{n-1} \\ 1 \end{smallmatrix}\right) \\ &-K\left(\begin{smallmatrix} \beta_3 & \beta_n \\ 1 & 1 \end{smallmatrix} \dots \begin{smallmatrix} \beta_n \\ 1 \end{smallmatrix}\right) \end{aligned} \right\} K\left(\begin{smallmatrix} \beta_4 & \beta_{n-2} \\ 1 & 1 \end{smallmatrix} \dots \begin{smallmatrix} \beta_{n-2} \\ 1 \end{smallmatrix}\right)$$

$$\left| \begin{aligned} &K\left(\begin{smallmatrix} \beta_2 & \beta_n \\ 1 & 1 \end{smallmatrix} \dots \begin{smallmatrix} \beta_n \\ 1 \end{smallmatrix}\right) \cdot K\left(\begin{smallmatrix} \beta_1 & \beta_n \\ 1 & 1 \end{smallmatrix} \dots \begin{smallmatrix} \beta_n \\ 1 \end{smallmatrix}\right) \\ &K\left(\begin{smallmatrix} \beta_2 & \beta_{n-1} \\ 1 & 1 \end{smallmatrix} \dots \begin{smallmatrix} \beta_{n-1} \\ 1 \end{smallmatrix}\right) \cdot K\left(\begin{smallmatrix} \beta_1 & \beta_{n-1} \\ 1 & 1 \end{smallmatrix} \dots \begin{smallmatrix} \beta_{n-1} \\ 1 \end{smallmatrix}\right) \end{aligned} \right| = (-1)^n \beta_1 \beta_2 \beta_3 \dots \beta_n,$$

the only change being the writing of $\beta_1, \beta_2, \dots$ for $-\cos^2 \alpha$,
 $-\cos^2 \beta, \dots$

WORPITZKY (1865, April).

[Untersuchungen über die Entwicklung der monodromen
 und monogenen Functionen durch Kettenbrüche. (Sch.
 Progr.) 39 pp., Berlin.]

Of the six sections into which the paper giving the results of
 Worpitzky's painstaking investigation is divided it is only the
 first headed "Fundamentalrelationen" which concerns us, these
 relations being nothing else than what we should now call "pro-
 perties of continuants."

He takes his continued fraction in the same form as Schläfli,
 viz.,

$$1 + \frac{a_1}{1} + \frac{a_2}{1} + \dots + \frac{a_n}{1},$$

showing of course that it equals

$$\frac{N_{1,n}}{N_{2,n}},$$

where

$$N_{\kappa,n} = \begin{vmatrix} 1 & 1 & . & . & . & . & . & . \\ -a_{\kappa} & 1 & 1 & . & . & . & . & . \\ . & -a_{\kappa-1} & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & -a_{n-1} & 1 & 1 \\ . & . & . & . & . & . & . & -a_n & 1 \end{vmatrix}.$$

The first matter of interest is the expansion of $N_{\kappa,n}$ as a sum of products of $a_{\kappa}, a_{\kappa-1}, \dots, a_n$, *e.g.*,

$$N_{1,3} = 1 + (a_1 + a_2 + a_3) + a_1 a_3.$$

This is written in the form

$$1 + a_{\kappa,n}^1 + a_{\kappa,n}^2 = \dots$$

where, he says, " $a_{\kappa,n}^r$ die Summe aller möglichen (als Producte aufgefassten) Combinationscomplexionen ohne Wiederholung bedeutet, welche sich aus $a_{\kappa}, a_{\kappa+1}, \dots, a_n$ so zu je r Elementen bilden lassen, dass nicht zwei neben einander stehende Elemente a_s, a_{s+1} dieser Reihe in den einzelnen Producten zugleich vorkommen." By way of proof it is pointed out (1) that the term independent of all the a 's is

$$\begin{vmatrix} 1 & 1 & & & \\ 0 & 1 & 1 & & \\ & \cdot & \cdot & \cdot & \cdot \\ & & 0 & 1 & 1 \\ & & & 0 & 1 \end{vmatrix} \quad i.e. \quad +1;$$

(2) that the cofactor* of $(-a_r)(-a_s)(-a_t) \dots$ when two of the a 's are consecutive is

$$\begin{vmatrix} 1 & 1 & & & & \\ 0 & 1 & 1 & & & \\ & \cdot & \cdot & \cdot & \cdot & \\ & & 0 & 1 & 1 & \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 & 0 \\ & & & & & 0 & 1 & 1 \\ & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & 0 & 1 & 1 \\ & & & & & & & 0 & 1 \end{vmatrix} \quad i.e. \quad 0;$$

* To obtain the cofactor of the product of a number of a set of elements in a determinant Worpitzky puts a 1 in the determinant in place of each element occurring in the said product, 0's in all the other places of the rows to which these elements belong, and 0's for all the other elements of the set.

and (3) that the cofactor of $(-a_r)(-a_s)(-a_t) \dots$ when no two of the a 's are consecutive and their number is p , is

$$\begin{vmatrix} 1 & 1 & & & & \\ 0 & 1 & 1 & & & \\ & & & & & \\ & & 0 & 1 & 1 & \\ & & & 1 & 0 & 0 \\ & & & & 0 & 1 & 1 \\ & & & & & & \\ & & & & 0 & 1 & 1 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 0 & 1 & 1 \\ & & & & & & & & \\ & & & & & & & 0 & 1 & 1 \\ & & & & & & & & 0 & 1 \end{vmatrix}$$

$$\text{i.e. } \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^p \quad \text{i.e. } (-1)^p,$$

and that, therefore, the cofactor of $a_r a_s a_t \dots$ in this case is $+1$.

In exactly similar fashion by partitioning $N_{k,n}$ into terms which contain $-a_s$ and terms which do not, he finds

$$N_{k,n} = D_0 - a_s D_s,$$

where

$$D_0 = \begin{vmatrix} 1 & 1 & & & & \\ -a_k & 1 & 1 & & & \\ & & & & & \\ & & & & & \\ & & -a_{s-1} & 1 & 1 & \\ & & & 0 & 1 & 1 \\ & & & & -a_{s+1} & 1 & 1 \\ & & & & & & \\ & & & & & & \\ & & & & & -a_{n-1} & 1 & 1 \\ & & & & & & -a_n & 1 \end{vmatrix},$$

$$= \begin{vmatrix} 1 & 1 & & & \\ -a_k & 1 & 1 & & \\ & & & & \\ & & & & \\ & & -a_{s-2} & 1 & 1 \\ & & & -a_{s-1} & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & & & \\ -a_{s+1} & 1 & 1 & & \\ & & & & \\ & & & & \\ & & -a_{n-1} & 1 & 1 \\ & & & -a_n & 1 \end{vmatrix} = N_{k,s-1} \cdot N_{s+1,n},$$

and

$$D_s = \begin{vmatrix} 1 & 1 & & & & & & & & \\ -a_k & 1 & 1 & & & & & & & \\ & & \dots & & & & & & & \\ & & & -a_{s-2} & 1 & 1 & & & & \\ & & & & -a_{s-2} & 1 & 0 & & & \\ & & & & & -a_{s-1} & 0 & 1 & & \\ & & & & & & 1 & 0 & 0 & \\ & & & & & & & -a_{s+1} & 1 & 1 \\ & & & & & & & & \dots & \\ & & & & & & & & & -a_{n-1} & 1 & 1 \\ & & & & & & & & & & -a_n & 1 \end{vmatrix},$$

$$= \begin{vmatrix} 1 & 1 & & & & & & & & \\ -a_k & 1 & 1 & & & & & & & \\ & & \dots & & & & & & & \\ & & & -a_{s-3} & 1 & 1 & & & & \\ & & & & -a_{s-2} & 1 & & & & \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & & & & & & & & \\ -a_{s+2} & 1 & 1 & & & & & & & \\ & & \dots & & & & & & & \\ & & & -a_{n-1} & 1 & 1 & & & & \\ & & & & -a_n & 1 & & & & \end{vmatrix},$$

$$= -N_{k,s-2}N_{s+2,n},$$

and thus reaches the result

$$N_{k,n} = N_{k,s-1}N_{s+1,n} + a_s N_{k,s-2}N_{s+2,n}$$

already obtained in a different way by Schläfli.

Lastly, taking a determinant of the same form as $N_{k,n}$, but having

$$-a_s, -a_{s-1}, \dots, -a_{k+1}, -a_k, -a_k, -a_{k+1}, \dots, -a_{n-1}, -a_n$$

for its minor diagonal of a 's, he obtains for it by isolating the first a_k the expression

$$N_{s,k+1}N_{k,n} + a_k N_{s,k+2}N_{k+1,n},$$

and by isolating the second a_k

$$N_{s,k}N_{k+1,n} + a_k N_{s,k+1}N_{k+2,n};$$

and thus deduces

$$N_{k,n}N_{k+1,s} - N_{k,s}N_{k+1,n} = -a_k(N_{k+1,n}N_{k+2,s} - N_{k+1,s}N_{k+2,n}).$$

It is then noted that the bracketed expression on the right differs

from the expression on the left merely in having $k+1$ in place of k ; so that there results

$$\begin{aligned} N_{k,n}N_{k+1,s} - N_{k,s}N_{k+1,n} &= (-1)^2 a_k a_{k+1} (N_{k+2,n}N_{k+3,s} - N_{k+2,s}N_{k+3,n}) \\ &= \dots\dots\dots \\ &= (-1)^{s-k+1} a_k a_{k+1} \dots a_{s+1} N_{s+3,n} . \end{aligned}$$

This also, it will be seen, is connected with a result of Schläfli's; for putting $s = n - 1$ we have*

$$\begin{vmatrix} N_{k+1,n-1} & N_{k,n-1} \\ N_{k+1,n} & N_{k,n} \end{vmatrix} = (-1)^{n-k} a_k a_{k+1} \dots a_n,$$

which becomes identical with Schläfli's last proposition on transposing the two rows of the determinant and (what is equally immaterial) putting $k = 1$.

THIELE, T. N. (1869, 1870).

[Bemærkninger om Kjædebrøker. *Tidsskrift for Math.* (2),
v. pp. 144-146.

Den endelige Kjædebrøksfunktions Theori. *Tidsskri forjtt*
Math. (2), vi. pp. 145-170.]

The first of the two notes comprising Thiele's first paper contains only one result, viz.,

$$a_1 + \frac{b_1}{a_2} + \frac{b_2}{a_3} + \dots + \frac{b_n}{a_n} = \frac{(a_1, a_2, \dots, a_n)}{(a_2, \dots, a_n)},$$

where $(a_1, a_2, \dots, a_n)$ is used to stand for

$$\begin{vmatrix} a_1 & b_1 & . & \dots & . & . \\ -1 & a_2 & b_2 & \dots & . & . \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ . & . & . & \dots & a_{n-1} & b_{n-1} \\ . & . & . & \dots & -1 & a_n \end{vmatrix}$$

* In giving to $N_{s+1,s}$, $N_{s+2,s}$, $N_{s+3,s}$ the values 1, 1, 0 which are necessitated by assuming the generality of the recursion-formula

$$N_{k,n} = N_{k+1,n} + a_k N_{k+2,n}$$

Worpitzky forgets to note that in these cases the proposition $N_{k,n} = N_{n,k}$, used by him in the demonstration, does not hold.

There is nothing to indicate that this is not viewed as a fresh discovery, notwithstanding the fact that Ramus' paper of 1856 containing virtually the same identity was published in the same city.

The other paper may be described as a careful study of finite continued fractions with the help of determinants. Instead of $b_1, b_2, \dots$ are used $a_{12}, a_{23}, \dots$; and

$$\begin{vmatrix} a_p & a_{p,p+1} & . & . & . & . & . \\ 1 & a_{p+1} & a_{p+1,p+2} & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & a_{q-1} & a_{q-1,q} \\ . & . & . & . & . & 1 & a_q \end{vmatrix}$$

is denoted by

$$K(p, q).$$

Further, this determinant is spoken of as a "Kjædebrøksdeterminant," or, shortly, a "K-Determinant"; and a section (§ 3, pp. 149–152) is devoted to a statement of its properties.

There is no need to rehearse all of these, the last portion (D) of the section being alone that which contains fresh matter. Opening with the double use of a previous property, viz.,

$$\begin{aligned} K(h, m) &= K(h, k-1) \cdot K(k, m) - a_{k-1,k} K(h, k-2) \cdot K(k+1, m), \\ K(h, n) &= K(h, k-1) \cdot K(k, n) - a_{k-1,k} K(h, k-2) \cdot K(k+1, n), \end{aligned}$$

where h, k, m, n are in ascending order of magnitude, the author eliminates $K(h, k-1)$ and obtains

$$\begin{vmatrix} K(h, m) & K(k, m) \\ K(h, n) & K(k, n) \end{vmatrix} = a_{k-1,k} K(h, k-2) \cdot \begin{vmatrix} K(k, m) & K(k+1, m) \\ K(k, n) & K(k+1, n) \end{vmatrix}. \quad (a)$$

Then by taking the particular case of this where k appears in place of h and $k+1$ in place of k there results

$$\begin{vmatrix} K(k, m) & K(k+1, m) \\ K(k, n) & K(k+1, n) \end{vmatrix} = a_{k,k+1} \begin{vmatrix} K(k+1, m) & K(k+2, m) \\ K(k+1, n) & K(k+2, n) \end{vmatrix},$$

which when applied to one of the determinants occurring in itself gives

$$\begin{vmatrix} K(k, m) & K(k+1, m) \\ K(k, n) & K(k+1, n) \end{vmatrix} = a_{k,k+1} a_{k+1,k+2} \begin{vmatrix} K(k+2, m) & K(k+3, m) \\ K(k+2, n) & K(k+3, n) \end{vmatrix},$$

and finally

$$= a_{k,k+1} a_{k+1,k+2} \dots a_{m,m+1} \cdot \begin{vmatrix} K(m+1, m) & K(m+2, m) \\ K(m+1, n) & K(m+2, n) \end{vmatrix},$$

$$= a_{k,k+1} a_{k+1,k+2} \dots a_{m,m+1} \cdot K(m+2, n). \quad (\beta)$$

Further, by using this to make a substitution in the previous result (a) there is obtained

$$\begin{vmatrix} K(h, m) & K(k, m) \\ K(h, n) & K(k, n) \end{vmatrix} = a_{k-1,k} a_{k,k+1} \dots a_{m,m+1} \cdot K(h, k-2) K(m+2, n), \quad (\gamma)$$

which on putting $k=h+1$ and $m=n-1$ becomes

$$\begin{vmatrix} K(h, n-1) & K(h+1, n-1) \\ K(h, n) & K(h+1, n) \end{vmatrix} = a_{h,h+1} a_{h+1,h+2} \dots a_{n-1,n},$$

—a result which may be compared with one of Schläfli's and Worpitzky's, but which is more general in that the main diagonal of each "K-Determinant" does not consist of units.

LIST OF AUTHORS

whose writings are herein dealt with.

	PAGE		PAGE
1853. SYLVESTER	130	1857. CAYLEY	144
1853. SPOTTISWOODE . . .	134	1858. PAINVIN	147
1853. SYLVESTER	135	1858. HEINE	149
1853. SYLVESTER	138	1858. SCHLÄFLI	151
1854. SYLVESTER	141	1865. WORPITZKY	153
1855. SCHLÄFLI	141	1869. THIELE	157
1856. RAMUS	142	1870. THIELE	157

(Issued separately, February 26, 1904.)