ON SPHERICAL CURVES. PART II.

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1. In a previous paper* I discussed the properties of curves on a sphere which can be stereographically projected into plane algebraic curves. Every n -ic (curve of the n -th degree) on a sphere defined in the usual manner as a curve cut by any plane in n points is an algebraic curve, for its stereographic projection on a plane is evidently cut by any straight line in n points and is therefore algebraic.

2. In my previous paper I showed that, if a spherical n -ic has δ nodes and κ cusps, and is such that m tangent circles pass through any two points and such that τ bitangent circles and ι osculating circles cut any given circle orthogonally,

$$\begin{aligned} m &= \frac{1}{2}n^2 - 2\delta - 3\kappa, \dagger & n &= m(m-1) - 2\tau - 3\iota, \\ \iota &= \frac{3}{2}n(n-2) - 6\delta - 8\kappa, & \kappa &= 3m(m-2) - 6\tau - 8\iota, \end{aligned}$$

$$\text{deficiency} = \frac{1}{4}(n-2)^2 - \delta - \kappa \nless 0.$$

These results hold when a curve has ordinary singularities of any order, provided that a k -ple point‡ with superlinear branches of orders $r, s, t, \dots$ [$\sum r = k$] is considered equivalent to $\{\frac{1}{2}k(k-1) - \sum(r-1)\}$ nodes and $\sum(r-1)$ cusps. If $\delta + \kappa = \frac{1}{4}(n-2)^2$ and (ρ, θ) are the polar coordinates of any point of the curve, $\cos \rho$ and $\tan \theta$ can be expressed as rational algebraic functions of a single parameter.

* *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 267 (1904). To save space I shall assume that the reader has access to this paper.

† The number of circles through two given points normal to the curve = m ; the number of circles touching the curve, orthogonal to a given circle, and cutting another given circle at a given angle = $2m$; the number of circles touching the curve and cutting two given circles at given angles = $4m$.

‡ A multiple point of order k . Multiple points involving compound singularities (*e.g.*, cusps of the second species) are excluded.

3. Let A be any point on the sphere. Let one of the imaginary generators through A meet a spherical n -ic* in P and let the other meet it in Q . Let the other generators through P and Q meet in B . Then B will be called a *satellite of A with respect to the curve*: A is, of course, a satellite of B . Each point A has p^2 satellites ($n = 2p$), p of which are real, unless A is a k -ple point of the curve, when it has $(p-k)^2$ satellites, $(p-k)$ of which are real.

If the generators through A touch the curve, so that A is a *focus* (or if each generator through A passes through a double point of the curve), two of the real and two of the imaginary satellites of A coincide in one real 4-ple satellite of A . If the generators at A touch the curve at A , so that A is a *nodal focus*, A coincides with its multiple satellite. If the generators through A are inflexional tangents to the curve, so that A is an *inflexional focus* (or if each generator through A is a tangent at a double point of the curve), three of the real and six of the imaginary satellites of A coincide in one real 9-ple satellite of A .

4. It is easy to find the properties of the plane curve into which a spherical curve is projected from a point A on the sphere, when the stereographic projection of the spherical curve from some other point B is known: for we obtain the projection from A by inverting the projection from B with respect to a circle whose centre is the projection of A from B , and then reflecting it in a diameter of the sphere perpendicular to AB (*loc cit.*, § 5). Thus, when we project stereographically from A , the satellites of A project into the singular foci of the plane curve (the intersections of the tangents to the plane curve at the circles). If A is a nodal focus, the projection of the spherical curve is a plane $(n-2)$ -ic touching the line at infinity at each circle. If A is the 4-ple satellite of a focus B , B projects into a singular inflexional focus of the plane curve (the intersection of tangents of three-point contact at the circles). If A is a focus whose 4-ple satellite is B , B projects into the intersection of tangents at the circles to cusps of the plane curve. If A is an ordinary k -ple point of the spherical curve, the plane curve is of degree $(n-k)$ and has the projections of the k circles of curvature at A as asymptotes.† If A is a cusp of the first species, the plane curve touches the line at infinity. If A is a cusp of the second species, the plane curve has a cusp at infinity, &c., &c.

* I shall assume that the spherical curve and vertex of projection are both real unless the contrary is stated. The stereographic projection is then a real algebraic plane curve.

† It follows that $2k \geq n$.

5. The following theorems will illustrate properties of spherical curves :—

(1) Given four fixed points A, B, C, D on a sphere ; all spherical 4-ics through A, C, D having B for a focus whose satellite is A pass through a fifth fixed point and have four-point contact with the osculating circle at A .

(2) If A is a point on a spherical 4-ic, the generators through A and the osculating circle at A meet the curve at four concyclic points (including A).

(3) The osculating circle at a fixed point B of a spherical 4-ic meets the curve again in A ; and on any circle meeting the curve in A, H, B, K a point P is taken such that the pencil $A(BP, HK)$ is harmonic : then the locus of P is a circle through A .

(4) If two points A, B on a spherical 4-ic are satellites of each other, the points of contact of the two circles through A and B which touch the curve lie on two circles, one touching the 4-ic at A and the other at B .

(5) A spherical 4-ic has a cusp C and a focus A . If B is the 4-ple satellite of A , the circle ABC touches the 4-ic at C .

(6) Given in a cuspidal spherical 4-ic the osculating circle at a fixed point A and the satellite B of A , the locus of the cusp is a spherical 4-ic touching the given circle and having a focus at B and a node at A . Each branch of the locus at A cuts the given circle at an angle of 30° .

(7) If one of the four nodes A of a spherical 6-ic is the 4-ple satellite of a focus, the points of contact of the four osculating circles through A are concyclic.

(8) If the node A of a spherical 6-ic is the 4-ple satellite of a focus B , the twenty-two osculating circles through A all touch a curve with four real foci, B being one.

(9) A spherical 6-ic has a node, a nodal focus A , and two cusps H, K ; if the osculating circles at L and M pass through A , the circles AHK, ALM touch.

(10) If a spherical 6-ic has three nodes and a nodal focus A , the six points of contact of the three bitangent circles through A are concyclic.

The above theorems may be proved by projecting stereographically from A .

6. If P is any point on a spherical $2p$ -ic and A is any fixed point, then

$$\begin{aligned} & \sin^2 \tfrac{1}{2}PB_1 \sin^2 \tfrac{1}{2}PB_2 \dots \sin^2 \tfrac{1}{2}PB_p, \\ & \sin^2 \tfrac{1}{2}PC_1 \sin^2 \tfrac{1}{2}PC_2 \dots \sin^2 \tfrac{1}{2}PC_{p-1} \sin^2 \tfrac{1}{2}PA, \\ & \sin^2 \tfrac{1}{2}PD_1 \sin^2 \tfrac{1}{2}PD_2 \dots \sin^2 \tfrac{1}{2}PD_{p-2} \sin^4 \tfrac{1}{2}PA, \\ & \dots, \\ & \sin^{2p} \tfrac{1}{2}PA \end{aligned}$$

are connected by a homogeneous linear relation, where the B 's are the p real satellites of A and the C 's, D 's, ... are also fixed when A is fixed. Conversely, if the quantities are connected by a homogeneous linear relation when A and the B 's, C 's, D 's, ... are fixed, the locus of P is a spherical $2p$ -ic. The theorem leads to simple results in certain cases, *e.g.* :—

B_1, B_2 are the real satellites of A with respect to a spherical 4-ic ; A, C_1 are the real satellites of B_1 and A, C_2 of B_2 . If a circle orthogonal to the circles AB_1C_1, AB_2C_2 cuts the curve in two real points P_1 and P_2 , then

$$\sin \tfrac{1}{2}P_1B_1 \sin \tfrac{1}{2}P_1B_2 \operatorname{cosec}^2 \tfrac{1}{2}P_1A = \sin \tfrac{1}{2}P_2B_1 \sin \tfrac{1}{2}P_2B_2 \operatorname{cosec}^2 \tfrac{1}{2}P_2A.$$

If A is a focus of the 4-ic, B_1 and B_2 coincide (at B say). We have then

$$\sin \tfrac{1}{2}P_1B \operatorname{cosec} \tfrac{1}{2}P_1A = \sin \tfrac{1}{2}P_2B \operatorname{cosec} \tfrac{1}{2}P_2A,$$

where P_1 and P_2 are two real intersections of the curve with any circle orthogonal to all circles through A and B .

7. Two points P, Q on a sphere are inverse with respect to a circle j when the line PQ passes through the pole of the plane of j with respect to the sphere. The locus of Q is the inverse of the locus of P with respect to j . The values of $n, m, \delta, \kappa, \tau, \iota$ and the deficiency are the same for two inverse curves. The inverses of the satellites of a point are the satellites of the inverse point with respect to the inverse curve. The inverse of a focus is a focus of the inverse curve, &c.

8. Let O, O' be two diametrically opposite points on a sphere, and let P be any point of a spherical n -ic. The envelope of circles on OP as diameter is the first (positive) *pedal* of the curve with respect to O , and the first *negative pedal* with respect to O' . The second (positive) pedal is the first pedal of the first pedal, the third pedal is the first pedal of the second pedal, and so on ; similarly for the negative pedals. The p -th pedal of the q -th pedal is the $(p+q)$ -th pedal, p and q being any positive or negative integers (the 0-th pedal being the curve itself). The r -th pedal with respect to O is the r -th negative pedal with respect to O' . Properties

of the pedals may be proved by projecting stereographically from O or O' .

If $n_r, m_r, \dots$ are the quantities corresponding to $n, m, \dots$ for the r -th pedal with respect to O , when O and O' have no special relation towards the given curve, then

$$\begin{aligned} n_1 &= 2m, & n_2 &= 4m, & \dots, & n_r &= 2rm; \\ m_1 &= n+2m, & m_2 &= 5m, & \dots, & m_r &= (3r-1)m; \\ \kappa_1 &= \iota, & \kappa_2 &= \iota+n+m, & \dots, & \kappa_r &= \iota+n+(r-1)m; \\ \iota_1 &= \iota+3n, & \iota_2 &= \iota+n+4m, & \dots, & \iota_r &= \iota+n+4(r-1)m;^* \end{aligned}$$

the deficiency of all pedals is the same as the deficiency of the curve; O is an rm -ple point of the r -th pedal at which there are m superlinear branches of order r ; each of the generators through O' touches the first pedal at $\frac{1}{2}n$ places, has two-point contact with the second pedal at $(m-n)$ places and three-point contact at $\frac{1}{2}n$ places, has $(r-1)$ -point contact with the r -th pedal at $\frac{1}{2}n$ places, r -point contact at $(m-n)$ places, and $(r+1)$ -point contact at $\frac{1}{2}n$ places.[†]

If O or O' has a special relation towards the given curve, these results must be modified. Thus, if O is a focus of the curve, it is an $(m-2)$ -ple point of the first pedal, which is only of degree $(2m-2)$;[‡] if, in addition, O' is the 4-ple satellite of O ,§ the same is true of all the pedals. If O is an inflexional focus of the curve, it is an ordinary focus of the first pedal. If O is the 4-ple satellite of a focus of the given curve, O' is the 4-ple satellite of a focus with respect to the first pedal. If O' is a focus of the given curve, it is an inflexional focus of the first pedal.|| If O is a nodal focus of the curve, it is a nodal focus of the first pedal which has $(m-4)$ other branches through O and is of degree $(2m-2)$.

The properties of the negative pedals may be obtained by interchanging O and O' in the above.[¶]

* If a generator through O' cuts a circle j in P , the pair of generators through P counts as $(r-2)m$ of the ι circles osculating the r -th pedal and cutting j orthogonally.

† The first pedal has $\frac{1}{2}n^2$ foci besides O' , but no other positive pedal has a focus other than the multiple focus O' .

‡ If the generators through O each touch the curve at k places, O is an $(m-2k)$ -ple point of the first pedal which is of degree $2(m-k)$.

§ Or if O' is the focus and O its 4-ple satellite.

|| If the generators through O' have each r -point contact with the curve, they have $(r+1)$ -point contact with the first pedal.

¶ These theorems hold when we take O and O' not diametrically opposite, if, in the definition of a pedal, we substitute for "the circle on OP as diameter" "the circle through O and P cutting the circle $OP'O'$ orthogonally."

9. The *evolute* of a spherical curve with respect to a circle j is the locus of the limiting points* of j and the osculating circles of the curve, i.e., the envelope of circles orthogonal to the curve and to j .† It is anallagmatic (its own inverse) with respect to j . Let $n', m', \dots$ be the quantities for the evolute corresponding to $n, m, \dots$. Then $n' = 2i$; for any circle s orthogonal to j cuts the evolute only at the limiting points of j and the i osculating circles orthogonal to s ; the evolute meets j at the $2i$ points of contact of j and osculating circles. Again, $m' = 2m + 2i$; for, taking any two points H, H' inverse with respect to j , we see that the circles through H, H' touching the evolute are the m circles through H, H' normal to the given curve (each of which has *double* contact with the evolute) and the $2i$ intersections of the evolute with j .

Again, $i' = 2n + 6i$, for the only osculating circles of the evolute orthogonal to j are those whose points of contact are at the intersections of the curve or of the evolute with j ; for otherwise we should have two consecutive circles normal to the curve and j coinciding, which is not possible in general. Now, it may be readily shown that each intersection of the given curve with j is an inflexional focus of the evolute; and at each of the $2i$ points where the evolute cuts j we have three consecutive osculating circles of the evolute orthogonal to j (*loc. cit.*, p. 270, § 14).

Again, $\kappa' = 3n' - 3m' + i' = 2(3i - 3m + n)$, showing that the number of osculating circles of four-point contact of a spherical n -ic is $(3i - 3m + n)$ in general.

The above results must be modified if the given curve is anallagmatic with respect to j ; see § 13.

Every focus of a curve is a focus of its evolute; for, if a generating line touches the curve at P and meets j at Q , the generator-pair through Q is a circle cutting j orthogonally at Q and the curve orthogonally at P .

10. Let V be the vertex of a given cone and j the circle in which the polar plane of V meets the sphere. The envelope of the polar planes with respect to the sphere of the points in which the given cone meets the plane of j is a cone with vertex V which is the *reciprocal of the given cone with respect to the sphere*. The reciprocal cones cut the plane of j in two curves which are polar reciprocals of each other with respect to j , and cut the sphere in two curves both anallagmatic with respect to j which are

* The limiting points of two circles are the two points through which pass all circles cutting the given circles orthogonally; i.e., the points in which the sphere meets the line conjugate to the intersection of the planes of the circles.

† If the circle j is not mentioned, it is assumed to be the circle at infinity.

‡ Not m , as erroneously stated, *loc. cit.* p. 272, § 21.

called *reciprocal spherical curves*. The reciprocal of a circle whose plane passes through V is a pair of points collinear with V , and *vice versa*. The reciprocal of a pair of nodes on a curve is a circle orthogonal to j and touching the reciprocal curve in four places; the reciprocal of a pair of cusps is a circle orthogonal to j and osculating the reciprocal curve in two places. If N, M, Δ, K, T, I correspond in the reciprocal curve to $n, m, \delta, \kappa, \tau, \iota$ in the given curve, then

$$N = (m - n), \quad M = m, \quad \Delta = \delta + \frac{1}{4}(m - 2n)(m - 18),$$

$$K = \kappa + 8(m - 2n), \quad T = \tau - \frac{1}{2}(m - 2n), \quad I = \iota,$$

$$\frac{1}{2}(N - 2)^2 - (\Delta + K) = \frac{1}{4}(n - 2)^2 - (\delta + \kappa) + \frac{1}{2}(m - 2n).^*$$

If we are given a property of a certain type of anallagmatic spherical curve (the type being defined by the quantities $n, m, \dots$), we can deduce by reciprocation a property of another type of anallagmatic spherical curve. If $m = 2n$, we have

$$n = N, \quad m = M, \quad \delta = \Delta, \quad \kappa = K, \quad \tau = T, \quad \iota = I;$$

so that all properties of an anallagmatic spherical curve for which $m = 2n$ may be duplicated.†

11. The $\frac{1}{2}n(n-1)$ circles each of which touches a spherical n -ic anallagmatic with respect to j at two of the n points where it meets j are called the *asymptotic arcs* of the curve. The pairs of points reciprocal to the asymptotic arcs are $n(n-1) = (M-N)^2 - (M-N)$ foci of the reciprocal curve, the remaining $(M-N)$ foci being the n intersections of j with the given curve. For let $P_1, P_2, \dots, P_n$ be these intersections; then the tangent planes at $P_1, P_2, \dots, P_n$ touch the cone reciprocal to the cone on which the given curve lies, so that $P_1, P_2, \dots, P_n$ are foci of the reciprocal curve (*loc. cit.*, p. 275, line 12). Again, the generators through P_1 and P_2 meet again in a pair of foci of the reciprocal curve, the pair being the reciprocal of a circle cutting j orthogonally at P_1 and P_2 . Therefore the generators through $P_1, P_2, \dots, P_n$ meet again in $n(n-1)$ foci of the reciprocal curve which are reciprocal in pairs to the asymptotic arcs of the given curve.‡

* From these equations we can, of course, deduce

$$n = (M - N), \quad \delta = \Delta + \frac{1}{4}(M - 2N)(M - 18), \quad \dots$$

† In the same way that Pascal's theorem for a conic is duplicated into Pascal's and Brianchon's theorems. The condition for a plane curve corresponding to $m = 2n$ is class = degree.

‡ It should be noted that the foci of a curve anallagmatic with respect to j are completely determined when we know the foci lying on j . In general the foci of any curve with r^2 foci are completely determined when we know r foci no two of which lie on the same generator.

12. We may prove properties of a spherical curve anallagmatic with respect to a circle j and lying on a cone whose vertex is V by projecting it stereographically into the intersection of a sphere with a cylinder or a cone whose vertex is the centre of the sphere (*loc. cit.*, §§ 30–35). A better method, however, is to project the curve from V on to the polar plane of V with respect to the sphere (the plane of j). A circle on the sphere (not orthogonal to j) projects into a conic having double contact with j . The generator-pair through a point P on j projects into the tangent at P to j .^{*} A circle orthogonal to j and the reciprocal pair of points project into a straight line and its pole with respect to j . Two circles orthogonal to j and each other project into two lines conjugate with respect to j .

For example:—The envelope of a chord of a conic h whose extremities are conjugate with respect to a fixed circle j is a conic touching the tangents to h and j at their points of intersection; and, reciprocally, the locus of the intersection of two tangents to h which are conjugate with respect to j is a conic passing through the points of contact with h and j of their common tangents. Hence:—If a spherical 4-ic h is anallagmatic with respect to a circle j , the envelope of a circle orthogonal to j cutting h in four points subtending a harmonic pencil at any point of the circle is a 4-ic anallagmatic with respect to j having the intersections of h and j for foci; and, reciprocally, the locus of the intersection of two circles touching h and orthogonal to j and each other is a 4-ic anallagmatic with respect to j which meets j at the four foci of h lying on j . If V lies within the sphere, we may by stereographic projection reduce these two theorems to the well known theorems:—The envelope of a quadrant chord of a sphero-conic is a sphero-conic confocal with the reciprocal of the given sphero-conic; and, reciprocally, the locus of the intersection of two orthogonal tangents to a sphero-conic is a sphero-conic whose reciprocal is confocal with the given sphero-conic.

Again:—If a plane cuspidal 3-ic touches three given straight lines at three given collinear points, the cusp lies on a fixed conic touching the three fixed lines; and, reciprocally, if the 3-ic touches three given concurrent lines at three given points, its inflexional tangent touches a fixed conic through the given points. Hence:—A spherical 6-ic has two fixed nodes A, B lying on a circle j with respect to which it is anallagmatic and

^{*} This gives a simple proof of the theorem “Four generators of a conicoid of the same system cut all generators of the opposite system in a constant cross-ratio,” and of the allied theorems.

has two cusps : if it has a fixed guiding arc,* the cusps lie on a fixed 4-ic anallagmatic with respect to j which touches the given guiding arc at two points and has A, B for foci ; if it touches at a fixed point P a fixed circle with respect to which A and B are inverse points, the circle orthogonal to j and osculating the curve in two places touches a fixed 4-ic anallagmatic with respect to j passing through A, B , and P .

Again :—A spherical 6-ic with two nodes is anallagmatic with respect to a circle j , has fixed asymptotic arcs which meet at A, B , and C , and passes through a pair of fixed points P, P' on the asymptotic arc BC . The locus of the nodes is a spherical 6-ic through A anallagmatic with respect to j whose asymptotic arcs are AB, AC , and a circle whose plane contains the line conjugate to PP' . The tangents at A to the curve and the circles through A orthogonal to j and passing through P, B , and C form a harmonic pencil.

This may be proved as before ; or, if j is imaginary, we may reduce the curve to a sphero-3-ic lying on the cone

$$(x^2 + y^2 + z^2)(\lambda y + \mu z) + z(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0,$$

where the expression in the last pair of brackets breaks up into two linear factors ; and it is readily seen (by considering the intersection of the cone with the plane $z = 1$) that the nodal line of this cone lies on the cone

$$(x^2 + y^2 + z^2)(ax + hy + gz) = x(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy).$$

Again :—If a fixed circle is touched by a circle cutting orthogonally in P and Q a fixed circle j through two fixed points A and B , the circles cutting j orthogonally at A and P, B and Q meet again on a fixed 4-ic anallagmatic with respect to j .

Again :—Any spherical n -ic anallagmatic with respect to j meets j at $P_1, P_2, \dots, P_n$. If the asymptotic arcs through P_1 and P_2, P_2 and $P_3, \dots, P_{n-1}$ and P_n pass through fixed coplanar points, then the asymptotic arc through P_n and P_1 passes through two fixed points or touches two fixed circles according as n is even or odd.

13. Let a spherical curve be anallagmatic with respect to a circle j and lie on a cone whose vertex is V , and let the quantities corresponding to $n, m, \dots$ for its evolute with respect to j be denoted by $n', m', \dots$; then

* The guiding arc is the circle through the six points not lying on j in which the asymptotic arcs meet the curve. If the curve passes through the intersection of two asymptotic arcs (or if, as in the present case, two of the asymptotic arcs are replaced by a circle through A and B orthogonal to j), the guiding arc touches the curve (*loc. cit.*, p. 280, § 35).

$n', m', \dots$ have no longer the values obtained in § 9. The projections from V on the plane of j of circles normal to the curve and orthogonal to j are straight lines joining points on the projection of the curve to the corresponding points on the reciprocal polar of the projection with respect to j . The projection of the evolute is the envelope of these straight lines. Hence we obtain

$$n' = \iota, \quad m' = \iota + m, \quad \iota' = 3\iota$$

(for the projection of the evolute from V on the plane of j has no inflexions). Therefore

$$\kappa' = 3n' - 3m' + \iota' = 3(\iota - m);$$

and we see, as before, that the number of osculating circles* of the given curve which have four-point contact is $(3\iota - 3m + n)$. Two reciprocal curves have the same evolute with respect to j .

14. In the same way that we deduce properties of spherical curves from those of plane curves either by stereographic projection or by projection from any point on to the polar plane of the point, so we may deduce properties of plane curves from those of spherical curves. We may add to the examples given in the previous paper:—

The number of conics through two fixed points having double contact with a fixed conic j and touching a plane curve whose Plücker's numbers are $n, m, \delta, \kappa, \tau, \iota$ is $2(m + n)$; the number of conics through one fixed point having double contact with j and osculating the curve is

$$(6n + 2\iota) = (6m + 2\kappa);$$

and the number of conics having double contact with j and four-point contact with the curve is

$$3(2n - m + \iota) = 3(2m - n + \kappa).^\dagger$$

These theorems are proved by projecting j into a circle, passing a sphere through the circle, and then projecting on to the sphere from the pole of the plane of the circle.

15. As before pointed out (*loc. cit.*, §§ 36, 37), to many of the theorems proved for the sphere correspond theorems for a conicoid. The definitions of "focus," "satellite," &c., still hold good, though, if the conicoid has

* n of these circles touch the curve at its intersections with j .

† If the fixed conic is a pair of points, the number of conics is $(m + 2n)$, $(3n + \iota)$, $(6m - 4n + 3\kappa)$ respectively; if the fixed conic is a pair of straight lines, the number of conics is $4(m + n)$, $(6n + 2\iota)$, $(6n - 4m + 3\iota)$ respectively.

real generators, we can no longer draw conclusions as to how many foci or satellites are real. Every *real* algebraic spherical curve (which is projected stereographically into a *real* algebraic plane curve) is necessarily of even degree, but this is not true of a real algebraic curve on a conicoid with real generators. As examples of properties of curves of a conicoid we may take:—

The intersection of the osculating planes of a conicoidal 3-ic at P and Q meets the conicoid at A . If the plane APQ meets the curve again at R , the osculating plane at R also passes through A .

One of the four nodes of a conicoidal 6-ic is the 4-ple satellite of a focus; if the osculating planes at four points of the curve pass through that node, the four points are coplanar.

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