XIX.—*On the Nature of the Curves whose Intersections give the Imaginary Roots of an Algebraic Equation.* By THOMAS BOND SPRAGUE, M.A., F.R.S.E.
(Plate XXIV.)

(Read 20th February 1882.)

“ If the n roots of the equation

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0$$

“ are all real, then all the roots of the equations

$$\cos\left(k\frac{d}{dx}\right)f(x) = f(x) - \frac{k^2}{2}f''(x) + \frac{k^4}{4!}f^{(4)}(x) - \dots = 0$$

“ and

$$\frac{1}{k}\sin\left(k\frac{d}{dx}\right)f(x) = f'(x) - \frac{k^2}{3!}f'''(x) + \frac{k^4}{5!}f^{(5)}(x) - \dots = 0$$

“ are also real, and one of the $n-1$ roots of the latter lies between each adjacent “ two roots of the former.”

In DE MORGAN'S *Trigonometry and Double Algebra*, and in TODHUNTER'S *Theory of Equations*, there is given a theorem of CAUCHY'S, which lays down a rule for determining the number of imaginary roots of the equation $f(x)=0$ that lie within any assigned limits. TODHUNTER says of it that it proposes to effect with respect to the roots in general what STURM'S theorem effects with respect to the *real* roots; but it seems to me that this is scarcely correct, and that the corresponding theorem as to the real roots, is that the number of such roots between a and b is the number of changes of sign of $f(x)$ when all values of x from a to b are substituted in $f(x)$.

If $h+ik$ is a root of $f(x)=0$ (where i is put for $\sqrt{-1}$), we have

$$f(h+ik) = e^{ik\frac{d}{dh}}f(h) = \left\{\cos\left(k\frac{d}{dh}\right) + i\sin\left(k\frac{d}{dh}\right)\right\}f(h) = 0,$$

whence

$$\cos\left(k\frac{d}{dh}\right)f(h) = 0, \text{ and } \sin\left(k\frac{d}{dh}\right)f(h) = 0.$$

It follows that h and k must satisfy these two equations; or, in other words, they are the co-ordinates of a point of intersection of the two curves represented by the equations

$$\cos\left(y\frac{d}{dx}\right)f(x) = 0, \quad \sin\left(y\frac{d}{dx}\right)f(x) = 0.$$

The number of points of intersection will be the number of roots (real or imaginary) of the equation $f(x)=0$, k being of course zero for the real roots. Writing $p=0$, $q=0$, for the last two equations, so that $f(x+iy)=p+iq$, and

assuming for the present that $f(x)=0$ has no equal roots, CAUCHY's theorem asserts that we can ascertain how many intersections of the curves lie within any assigned area, by observing how often $p:q$ passes through the value 0 and changes its sign, when we substitute for x and y the co-ordinates of a moving point that traverses the contour of the area in the positive direction. If $p:q$ so changes sign l times from $+$ to $-$, and m times from $-$ to $+$, then the number of points of intersection is $\frac{1}{2}(l-m)$. Calling the two curves P and Q, it is obvious that $p:q$ passes through the value 0 and changes sign, only when the moving point crosses the curve P. When the moving point crosses the curve Q, $p:q$ changes sign; but, as it passes through the value ∞ , this change of sign is not counted. Before proceeding further let us consider whether it is true of all curves that the number of intersections is determined by the above formula, or for what particular kinds of curves it holds good.

Suppose the point to move round the rectangle in any one of the figures (1), (2), (3); then in the first case it crosses the curve P twice, but on one occasion $p:q$ changes sign from $+$ to $-$, and on the other from $-$ to $+$. There is therefore, according to the rule, no point of intersection of P and Q within the rectangle; and this is the case. In fig. 2 the point also crosses P twice, and the same changes of sign occur, so that here again, according to the rule, there is no intersection within the rectangle; and this is the fact. In fig. 3 the point crosses P twice, and on each occasion there is a change of sign from $+$ to $-$, so that $l=2$, $m=0$, and the rule gives us one point of intersection, which is correct. If, however, we suppose the curves P and Q to intersect in two points within the rectangle, as in figs. 4 and 5, we see that the rule does not hold good. In fig. 4, the point crosses P twice, and we have $l=1$, $m=1$; in fig. 5, the point crosses P four times, and we have $l=2$, $m=2$; in both cases, therefore, the rule would give us no intersections, whereas there are two. This shows that the rule does not apply to all kinds of curves, and it is not difficult to determine the general characteristics of the curves to which it does apply.

Suppose a branch of P to cut two branches of Q, thus giving two intersections, then, if we draw a rectangle to include these intersections, as in fig. 6, we see that the rule does not hold good. If, however, a second branch of P lies between the two branches of Q, as in fig. 7, the proper sequence of signs will be obtained, and the rule will hold good. In this case we get $l=4$, $m=0$, $\frac{1}{2}(l-m)=2$, which is the actual number of intersections of P and Q. Proceeding in the same way we see that, if there are an odd number of branches of P between the two branches of Q, the rule holds good, but not if the number of branches is even; and we conclude that, when the rule applies, if two intersections of P and Q occur on a branch of P, an odd number of double points of P lie between them. Similar reasoning shows that if two intersections of P and Q occur on a branch of Q, the rule will not apply unless an odd number

of double points of Q lie between them. This explains why the rule does not apply, when the curves intersect as they do in figs. 4 and 5.

Looking again at fig. 7 we see that no two branches of P and Q can intersect again when prolonged; for, if they did, there would be two intersections of P and Q on one branch, either of P or Q , without any double point between. It is not at first sight obvious how far this conclusion would be modified by the existence of another branch of Q passing through the intersection, as the dotted curve in fig. 8. It is necessary therefore to consider multiple points of this kind, that is to say, points through which pass two or more branches of one curve and one or more of the other. First consider the case of two branches of P and two of Q passing through one point and arranged alternately, as shown in fig. 9. The simplest way in which such a point can arise is by two branches of P moving up towards a branch of Q , as if in fig. 10 the points A and B should move up and coincide with C . In that case two points of intersection of P and Q would coincide, and the multiple point in fig. 9 is therefore to be reckoned as two intersections of P and Q . The rule gives us $l=4$, $m=0$, $\frac{1}{2}(l-m)=2$, which is correct. If the branches are not arranged alternately, but occur as in fig. 11, the simplest way in which it can arise is by two branches of P joining to form a double point which is already a double point on Q . In this case it seems that the point should, notwithstanding its apparent complexity, not be counted as an intersection at all. This agrees with the rule, which gives $l-2$, $m=2$, $\frac{1}{2}(l-m)=0$. Similarly, if we have a point through which pass one branch of P and two branches of Q , this may be considered to have arisen from the union of two branches of Q which do not cut P , as the dotted curves in fig. 12, and the point will therefore not count as an intersection; this being again in agreement with the rule, which gives $l=1$, $m=1$, $\frac{1}{2}(l-m)=0$. In this way of reckoning intersections, the rule applies to such a curve as is shown in fig. 8. Here, if we take a contour to include the two points A , B , but not C , the rule gives $l=3$, $m=1$, $\frac{1}{2}(l-m)=1$; and this is correct, since, as we have seen, B does not count as an intersection. If the contour is drawn so as to include also C , we shall have $l=4$, $m=0$, $\frac{1}{2}(l-m)=2$, which also is correct.

Resuming consideration of figs. 7 and 8, we see that it follows from what we have proved that no branch of P which intersects Q can re-enter upon itself and form a closed curve, but it must always have two infinite branches; and the same is true of Q . We next observe that the rule will not hold unless the arrangement of the infinite branches follows a law similar to that we have seen to prevail among the intersections. If in figs. 13 and 14 we draw a contour to include the points of intersection A and B , we shall have in fig. 13, $l=2$, $m=2$, $\frac{1}{2}(l-m)=0$; and in fig. 14, $l=4$, $m=0$, $\frac{1}{2}(l-m)=2$; or the rule applies in the latter case, but not in the former. If, however, in fig. 13 we draw between A and B an infinite branch either of P or Q or an odd number of such branches

either of P or Q, the proper sequence of signs is obtained and the rule applies. We thus see that between any two infinite branches of P on which there is an intersection, there must lie an odd number of infinite branches, either of P or Q; and the same is true of any two such branches of Q.

Applying the principles we have established, it is easy to show how to draw very complicated curves to which the rule will apply; but it seems unnecessary to pursue this part of the subject further.

CAUCHY does not seem to have investigated the properties of the curves P and Q, but it will be found that an examination of them leads to a variety of interesting results, one of which I have stated at the outset of this paper.

As a first step in this direction we expand the equations

$$\cos\left(y\frac{d}{dx}\right)f(x)=0, \text{ and } \sin\left(y\frac{d}{dx}\right)f(x)=0,$$

and thus get

$$f(x) - \frac{y^2}{2}f''(x) + \frac{y^4}{4!}f^{iv}(x) - \dots = 0 \quad (P)$$

and

$$yf'(x) - \frac{y^3}{3!}f'''(x) + \frac{y^5}{5!}f^{v}(x) - \dots = 0.$$

The latter is satisfied by $y=0$, or the x -axis forms part of the locus of the equation: and putting $y=0$ in the former equation, we get $f(x)=0$. It is obvious that the points thus determined correspond to the real roots of the original equation, and that the imaginary roots correspond to the intersections of the curves represented by (P) and by

$$f'(x) - \frac{y^2}{3!}f'''(x) + \frac{y^4}{5!}f^{v}(x) - \dots = 0 \quad (Q).$$

The only case I propose to examine in any detail on the present occasion is when $f(x)=0$ has no imaginary roots. The two curves must then have no intersections corresponding to finite values of y , and we shall presently prove that this is the case. For this purpose we must trace the two curves, which for brevity we will call P and Q.

We first observe that since y enters the equations (P) and (Q) only in even powers, both curves are symmetrical with regard to the x -axis. Putting $y=0$, the points in which the curves cut that axis are determined by $f(x)=0, f'(x)=0$, respectively. By the theory of equations, since all the roots of the former are real, all the roots of the latter ($n-1$ in number) are also real, and one of them lies between each adjacent two roots of the former. We next observe that, if $f(x)=0$ has no equal roots, then $f'(x)=0$ has none, and each of the curves cuts the x -axis at right angles; that is to say, P has n branches and Q has $n-1$ branches, all of which cut the x -axis at right angles. In fact, if h is a value of

x that satisfies the equation $f(x)=0$, then $f(x)$ contains the factor $x-h$ once, and only once; and when x becomes equal to h and y vanishes, the limit of $y^2:(x-h)$ is finite, and the limit of $y:(x-h)$ is infinite. This is also an immediate result of the symmetry of each curve with regard to the x -axis.

If $f(x)=0$ has two equal roots, $f'(x)=0$ has one root equal to them. In this case P has a double point, through which passes a branch of Q cutting the x -axis at right angles, as shown in fig. 15, where the dotted curve belongs to Q . If $f(x)=0$ has three equal roots, two of the roots of $f'(x)=0$ and one of the roots of $f''(x)=0$, are equal to them. In this case P has a triple point, which is also a double point on Q , as indicated in fig. 16, where, as before, the dotted branches belong to Q and the others to P ; and the middle branch of P cuts the x -axis at right angles. In general, if m roots of $f(x)=0$ are equal, P has a multiple point of the order m on the x -axis, and this is also a multiple point of the order $m-1$ on Q . If m is odd, one branch of P cuts the x -axis at right angles, and the others are inclined to it, two and two, at equal angles; and one branch of Q lies between each adjacent two of P . If m is even, one branch of Q cuts the x -axis at right angles, and the other branches of Q and all the branches of P , are symmetrically arranged as just described.

We will henceforward suppose that the roots of $f(x)=0$ are all real and unequal, in which case, as already shown, there are n branches of P and $n-1$ branches of Q , all of which cut the x -axis at right angles. We have next to consider the nature of the infinite branches, and for this purpose we must develop the equations (P) and (Q). The former gives us

$$\begin{aligned} & x^n + p_1 x^{n-1} + p_2 x^{n-2} + p_3 x^{n-3} + \dots + p_n \\ & - \frac{y^2}{2} \left\{ n(n-1)x^{n-2} + (n-1)(n-2)p_1 x^{n-3} + \dots + 2p_{n-2} \right. \\ & \left. + \frac{y^2}{4!} \left\{ n(n-1)(n-2)(n-3)x^{n-4} + \dots \right\} + \dots \right\} = 0. \end{aligned}$$

Or, re-arranging the terms according to their dimensions,

$$\begin{aligned} & x^n - \frac{n \cdot n-1}{2} x^{n-2} y^2 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{4!} x^{n-4} y^4 - \dots \\ & + p_1 \left\{ x^{n-1} - \frac{n-1 \cdot n-2}{2} x^{n-3} y^2 + \frac{n-1 \cdot n-2 \cdot n-3 \cdot n-4}{4} x^{n-5} y^4 + \dots \right\} \\ & + p_2 \left\{ x^{n-2} - \frac{n-2 \cdot n-3}{2} x^{n-4} y^2 + \dots \right\} + \dots = 0. \end{aligned}$$

The directions of the infinite branches, n in number, are given by the terms of highest dimensions; and equating these to zero, we have

$$x^n - \frac{n \cdot n-1}{2} x^{n-2} y^2 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{4!} x^{n-4} y^4 - \dots = 0.$$

Now since

$$(x+iy)^n = x^n + nix^{n-1}y - \frac{n \cdot n-1}{2} x^{n-2}y^2 - \frac{n \cdot n-1 \cdot n-2}{3!} ix^{n-3}y^3 + \dots$$

$$(x-iy)^n = x^n - nix^{n-1}y - \frac{n \cdot n-1}{2} x^{n-2}y^2 + \dots$$

we see that the above equation is equivalent to $(x+iy)^n + (x-iy)^n = 0$.

If we put $x=r \cos \theta$ and $y=r \sin \theta$, this becomes

$$r^n \{(\cos \theta + i \sin \theta)^n + (\cos \theta - i \sin \theta)^n\} = 0,$$

or $2r^n \cos n\theta = 0$. Hence for the infinite branches we have $\cos n\theta = 0$, or

$$n\theta = \frac{\pi}{2}, \frac{3}{2}\pi, \frac{5}{2}\pi, \dots, \frac{2n-1}{2}\pi,$$

and

$$\theta = \frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \dots, \frac{2n-1}{2n}\pi.$$

This shows that there are n real asymptotes, the directions of which are arranged at equal intervals through the first two quadrants. Also since the equation

$$x^n - \frac{n \cdot n-1}{2} x^{n-2}y^2 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{4!} x^{n-4}y^4 - \dots = 0$$

is satisfied by $x=y \cot \frac{\pi}{2n}$, $y \cot \frac{3\pi}{2n}$, &c., the first member of it must be identical with the product

$$\left(x-y \cot \frac{\pi}{2n}\right) \left(x-y \cot \frac{3\pi}{2n}\right) \dots \left(x-y \cot \frac{2n-1}{2n}\pi\right).$$

Having thus determined the direction of the asymptotes, we have next to find where they cut the x -axis. Take the asymptote for which $\theta = (2r-1)\pi : 2n$; then, observing that the equation to P may be put in the form

$$\begin{aligned} \{(x+iy)^n + (x-iy)^n\} + p_1 \{(x+iy)^{n-1} + (x-iy)^{n-1}\} \\ + p_2 \{(x+iy)^{n-2} + (x-iy)^{n-2}\} + \dots = 0, \end{aligned}$$

and that $x-y \cot \frac{2r-1}{2n}\pi$ is a factor of the first term, we have

$$x-y \cot \frac{2r-1}{2n}\pi + p_1 \frac{x-y \cot \frac{2r-1}{2n}\pi}{(x+iy)^n + (x-iy)^n} \{(x+iy)^{n-1} + (x-iy)^{n-1}\}$$

$$\begin{aligned}
& + p_2 \frac{x-y \cot \frac{2r-1}{2n} \pi}{(x+iy)^n + (x-iy)^n} \{(x+iy)^{n-2} + (x-iy)^{n-2}\} \\
& + \dots \dots \dots = 0 \dots \dots \dots \quad (\text{A}).
\end{aligned}$$

If in the term multiplied by p_1 we substitute $y \cot \frac{2r-1}{2n} \pi$ for x , and neglect the following terms, which vanish when x and y are infinite, we shall obtain the equation to the asymptote. On making this substitution, the fraction,

$$\frac{x-y \cot \frac{2r-1}{2n} \pi}{(x+iy)^n + (x-iy)^n}, \text{ takes the form } \frac{0}{0}.$$

Differentiating both numerator and denominator with respect to x , the value of the fraction is the same as that of

$$\frac{1}{n(x+iy)^{n-1} + n(x-iy)^{n-1}}$$

and the term multiplied by p_1 therefore becomes $\frac{p_1}{n}$ and the equation to the asymptote is

$$x - y \cot \frac{2r-1}{2n} \pi + \frac{p_1}{n} = 0.$$

The intercept of the asymptote on the x -axis is therefore $-p_1 : n$, or is the same for all the asymptotes, which therefore all intersect the x -axis in the same point. This result might have been at once obtained by observing that, if we increase the roots of the original equation $f(x)=0$ by the same quantity $p_1 : n$, or transfer the origin of co-ordinates for that distance to the left, the term $p_1 x^{n-1}$ disappears from the equation.

We have next to determine the asymptotes of Q. Expanding the equation (Q), we get

$$\begin{aligned}
& nx^{n-1} + (n-1)p_1x^{n-2} + (n-2)p_2x^{n-3} + \dots \dots + p_{n-1} \\
& - \frac{y^2}{3!} \{ n(n-1)(n-2)x^{n-3} + (n-1)(n-2)(n-3)p_1x^{n-4} + \dots \} \\
& + \frac{y^4}{5!} \{ n(n-1)(n-2)(n-3)(n-4)x^{n-5} + \dots \} - \dots \dots = 0,
\end{aligned}$$

$$\begin{aligned}
\text{or} \quad & nx^{n-1} - \frac{n \cdot n-1 \cdot n-2}{3!} x^{n-3} y^2 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4}{5!} x^{n-5} y^4 - \dots \dots \\
& + p_1 \left\{ (n-1)x^{n-2} - \frac{n-1 \cdot n-2 \cdot n-3}{3!} x^{n-4} y^2 + \dots \dots \right\} \\
& + p_2 \{ (n-2)x^{n-3} - \dots \dots \} + \dots \dots = 0,
\end{aligned}$$

or

$$(x+iy)^n - (x-iy)^n + p_1 \{(x+iy)^{n-1} - (x-iy)^{n-1}\} \\ + p_2 \{(x+iy)^{n-2} - (x-iy)^{n-2}\} + \dots = 0.$$

The directions of the asymptotes are given by $(x+iy)^n - (x-iy)^n = 0$, and putting $x=r \cos \theta$, $y=r \sin \theta$, this leads to $r^n \sin n\theta = 0$,

whence

$$n\theta = 0, \pi, 2\pi, \dots, (n-1)\pi,$$

and

$$\theta = 0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \dots, \frac{n-1}{n}\pi.$$

The first of these values corresponds to the x -axis, which is not an asymptote, but is part of the locus of the equation $\sin \left(y \frac{d}{dx} \right) f(x) = 0$; and the other values show us that the $n-1$ asymptotes of Q are all real, and that their directions bisect the angles between the directions of the asymptotes of P . It may be proved, as in the case of P , that all the asymptotes pass through the same point on the x -axis at a distance $-p_1 : n$ from the origin, so that the n asymptotes of P and the $n-1$ asymptotes of Q all meet in a point, and midway between each adjacent two of the former lies one of the latter.

We will next examine whether the infinite branches lie above or below the asymptotes. For this purpose we resume the equation (A), (see p. 473), and expand x in a series proceeding by negative powers of y . It will, however, simplify the process very much if we transfer the origin to the point of intersection of the asymptotes, or, what comes to the same thing, put $p_1 = 0$, which may be done without any loss of generality. Putting also $\frac{2r-1}{2n}\pi = \alpha$, equation (A) becomes

$$x = y \cot \alpha - \frac{x-y \cot \alpha}{(x+iy)^n + (x-iy)^n} p_2 \{(x+iy)^{n-2} + (x-iy)^{n-2}\} + \dots$$

Our first approximation to the value of x is $x = y \cot \alpha$, and we shall get a second approximation by putting this value of x in the term involving p_2 . Making this substitution, we get

$$(x+iy)^{n-2} + (x-iy)^{n-2} = \frac{y^{n-2}}{\sin^{n-2} \alpha} \{ (\cos \alpha + i \sin \alpha)^{n-2} + (\cos \alpha - i \sin \alpha)^{n-2} \} \\ = \frac{2y^{n-2}}{\sin^{n-2} \alpha} \cos (n-2)\alpha.$$

Also we have seen that, when we make the same substitution,

$$\frac{x-y \cot \alpha}{(x+iy)^n + (x-iy)^n} \text{ becomes } = \frac{1}{n(x+iy)^{n-1} + n(x-iy)^{n-1}} = \frac{\sin_{n-1} \alpha}{2ny^{n-1} \cos (n-1)\alpha}$$

Hence we have approximately

$$x = y \cot \alpha - \frac{p_2 \sin \alpha \cos (n-2)\alpha}{ny \cos (n-1)\alpha}.$$

Now since the roots of $f(x)=0$ are all real and $p_1=0$, we know from the theory of equations that p_2 is negative. And since $\alpha = \frac{2r-1}{2n}\pi$, and is therefore $<\pi$, $\sin \alpha$ is positive. Again

$$\cos (n-2)\alpha = \cos \frac{(n-2)(2r-1)}{2n}\pi = \cos \left(r - \frac{1}{2} - \frac{2r}{n} + \frac{1}{n} \right) \pi = (-1)^{r+1} \sin (2r-1) \frac{\pi}{n};$$

$$\cos (n-1)\alpha = \cos \frac{(n-1)(2r-1)}{2n}\pi = \cos \left(r - \frac{1}{2} - \frac{r}{n} + \frac{1}{2n} \right) \pi = (-1)^{r+1} \sin (2r-1) \frac{\pi}{2n}.$$

Now r ranges from 1 to n , and $(2r-1) \frac{\pi}{2n}$ from $\frac{\pi}{2n}$ to $\frac{2n-1}{2n}\pi$ or $\pi - \frac{\pi}{2n}$; and $\sin \frac{2r-1}{2n}\pi$ is therefore always positive. Lastly, $(2r-1) \frac{\pi}{n}$ ranges from $\frac{\pi}{n}$ to $2\pi - \frac{\pi}{n}$, and $\sin (2r-1) \frac{\pi}{n}$ is positive when $(2r-1) \frac{\pi}{n} < \pi$, or $2r-1 < n$ and $r < \frac{n+1}{2}$.

The final result is that $x - y \cot \alpha$ is positive when (y being positive) the asymptote lies in the first quadrant, and negative when it lies in the second quadrant; or in both cases the infinite branch lies below the asymptote.

We have thus proved that the curve P has n branches, which cut the x -axis at right angles; and that it has n infinite branches, lying below the n asymptotes which are arranged at equal intervals through the first two quadrants.

For the infinite branches of Q, we have similarly, putting $\beta = r\pi : n$,

$$x = y \cot \beta - \frac{p_2 \sin \beta}{ny} \cdot \frac{\sin (n-2)\beta}{\sin (n-1)\beta}, \text{ approximately;}$$

and it may be proved, in the same way as for P, that (y being positive) the infinite branches lie below the asymptotes. We thus see that Q has $n-1$ branches, that cut the x -axis at right angles in points which lie singly between two corresponding points of P; also that it has $n-1$ infinite branches lying below the asymptotes which bisect the angles between the asymptotes of P. Again, since $f(x)=0$ has no imaginary roots, the curves P and Q cannot intersect; and it follows that from each point of intersection, either of P or Q, with the x -axis, a branch proceeds up to the corresponding asymptote, without cutting any other branch of either P or Q. This may be proved, as regards P and Q separately, directly from the equations.

Suppose the roots of $f(x)=0$, arranged in descending order of magnitude, to be $a, b, c, \dots l$, then

$$f(x) = (x-a)(x-b)(x-c) \dots (x-l).$$

Hence
$$f(x+iy) = (x-a+iy)(x-b+iy) \dots (x-l+iy)$$

Now let
$$\frac{y}{x-a} = \tan \alpha, \quad \frac{y}{x-b} = \tan \beta, \quad \dots \quad \frac{y}{x-l} = \tan \lambda;$$

and put
$$R = \sqrt{\{(x-a)^2 + y^2\}} \sqrt{\{(x-b)^2 + y^2\}} \dots \sqrt{\{(x-l)^2 + y^2\}}.$$

Then, by an obvious transformation,

$$\begin{aligned} f(x+iy) &= R (\cos \alpha + i \sin \alpha) (\cos \beta + i \sin \beta) \dots \\ &= R \{ \cos (\alpha + \beta + \dots + \lambda) + i \sin (\alpha + \beta + \dots + \lambda) \} \quad (B). \end{aligned}$$

But we have seen that

$$f(x+iy) = \cos \left(y \frac{d}{dx} \right) f(x) + i \sin \left(y \frac{d}{dx} \right) f(x) \quad (C).$$

Hence $\cos \left(y \frac{d}{dx} \right) f(x) = 0$ leads to $\cos (\alpha + \beta + \dots + \lambda) = 0$, and $\alpha + \beta + \dots + \lambda = \frac{\pi}{2} + m\pi$, where m is an integer; or to

$$\tan^{-1} \frac{y}{x-a} + \tan^{-1} \frac{y}{x-b} + \dots + \tan^{-1} \frac{y}{x-l} = \frac{\pi}{2} + m\pi.$$

This equation admits of an obvious geometrical interpretation. In fig. 17 let R be the point (x, y) and $A, B, C \dots$ be points on the x -axis at distances from the origin $a, b, c \dots$; then the equation expresses that the sum of the angles $RAX, RBX, RCX, \dots$ is equal to $\frac{\pi}{2}$, or $\frac{3}{2}\pi$, or $\frac{5}{2}\pi$, or $\dots$ $\frac{2n-1}{2}\pi$. When y is positive, each of the angles is $< \pi$ and their sum is there-

fore $< n\pi$; and the curve has n branches, corresponding to these n values $\frac{\pi}{2}, \frac{3}{2}\pi, \dots$, and passing through the points $A, B, C \dots$ respectively.

Considering first the branch passing through A , we see that when R moves up to A (see fig. 18), the angle RAX is equal to $\pi:2$ and each of the angles $RBX, RCX \dots$ vanishes, so that the sum is equal to $\pi:2$. Next considering the branch passing through B , when R approaches B , RAX becomes equal to π , RBX to $\pi:2$, and $RCX, \dots$ all vanish; so that the sum of the angles is $3\pi:2$. Similarly we see that for the branch passing through C , the sum of the angles when R approaches C , is $5\pi:2$; and so on, for all the n branches.

Consider next the infinite branches. We have seen that the asymptotes are inclined to the x -axis at angles $\frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \dots, \frac{2n-1}{2n}\pi$; and when the

point R moves off to an infinite distance, all the angles RAX, RBX become equal, and their sum will be $\pi : 2$ for the first asymptote, $3\pi : 2$ for the second, $5\pi : 2$ for the third, and so on, being equal to the sums of the angles when R approaches A, B, C respectively. We have thus proved that a branch proceeds from A to the first asymptote, another from B to the second asymptote, and so on; but it may be still more conclusively demonstrated as follows.

We have seen that the x -axis cuts the curve (P) in n real points, and that a line parallel to it at an infinite distance also cuts the curve in n real points, being coincident with the points in which it cuts the asymptotes; and I will now prove that any line parallel to the x -axis cuts the curve in n points. Suppose y to have the fixed value k , and x to receive all values from $+\infty$ to $-\infty$; then as x decreases from $+\infty$ to a , $\tan^{-1} \frac{k}{x-a}$ continually increases from 0 to $\frac{\pi}{2}$; and as x continues to decrease to $-\infty$, the angle similarly increases from $\pi : 2$ to π . A similar proposition is true of each of the angles: hence, as x decreases from $+\infty$ to $-\infty$, $\tan^{-1} \frac{k}{x-a} + \tan^{-1} \frac{k}{x-b} + \dots$ (which we may call σ) continually increases from 0 to $n\pi$. In other words, as x decreases, each value of σ occurs only once; therefore, if we assign any value to σ between 0 and $n\pi$, there will be one value of x corresponding to it, and only one. Hence, assigning to σ the values, $\frac{\pi}{2}, \frac{3\pi}{2}, \dots, \frac{2n-1}{2}\pi$, we get for each, one value of x ; or in other words, whatever value k we assign to y , there are n real values of x which satisfy the equation (P). It is easily seen that if, instead of supposing $\tan^{-1} \frac{k}{x-a}$ to increase from 0 to π , we suppose it to increase from π to 2π , or from 2π to 3π , &c., and if we make the like suppositions with regard to the other angles, we shall always get the same values of x if we take in each case the proper value of σ .

Similarly for the curve Q, the equations (B) and (C) (see p. 436) show that $\sin \left(y \frac{d}{dx} \right) f(x) = 0$, leads to $\sin (\alpha + \beta + \dots + \lambda) = 0$, or $\alpha + \beta + \dots = m\pi$, where m is an integer; or $\sigma = \pi, 2\pi, 3\pi \dots (n-1)\pi$. Now we know that Q cuts the x -axis in $(n-1)$ points, one of which lies between A and B, another between B and C, and so on; and reasoning as above we see that when R moves up to the first of these points, σ becomes equal to π ; for the second, 2π ; and so on. Also the asymptotes are inclined to the x -axis at angles $\frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n} \dots \frac{n-1}{n}\pi$; and when R moves to infinity along the first asymptote, each of the angles $\alpha, \beta, \gamma \dots$ becomes equal to $\pi : n$, and their sum is π ; for the

second asymptote, the sum is 2π ; and so on. Lastly, if we suppose y to have the fixed value k , and assign to σ the values, $\pi, 2\pi, 3\pi, \dots$, we get $(n-1)$ values of x ; and since the values of σ lie singly between two adjacent values of σ for the curve P, we see that the $n-1$ points in which the straight line $y=k$, cuts the curve Q, lie singly between the n points in which it cuts P. Thus the proposition enunciated at the outset is completely established.

When the roots of $f(x)=0$ are not all real, the equations (P) and (Q) still admit of a simple geometrical interpretation. Suppose there is a pair of imaginary roots $f+ig, f-ig$; then the corresponding factors in $f(x+iy)$ become

$$\begin{aligned}(x-f-ig+iy)(x-f+ig+iy) &= R_1 R_2 (\cos \mu + i \sin \mu)(\cos \nu + i \sin \nu) \\ &= R_1 R_2 \{ \cos (\mu + \nu) + i \sin (\mu + \nu) \}\end{aligned}$$

if $R_1^2 = (x-f)^2 + (y-g)^2$, $R_2^2 = (x-f)^2 + (y+g)^2$, $\tan \mu = \frac{y-g}{x-f}$, $\tan \nu = \frac{y+g}{x-f}$.

It is easy to see from this that $\cos \left(y \frac{d}{dx} \right) f(x) = 0$ leads to

$$\tan^{-1} \frac{y}{x-a} + \tan^{-1} \frac{y}{x-b} + \dots + \tan^{-1} \frac{y-g}{x-f} + \tan^{-1} \frac{y+g}{x-f} + \dots = \frac{\pi}{2} + m\pi.$$

In fig. 19, let R be the point (x, y) , $OA=a$, $OB=b$, $\dots$, $OF=f$, $RS=RT=g$; then the equation expresses that the sum of the angles RAX, RBX $\dots$ SFX, TFX, $\dots$ is equal to $\frac{\pi}{2}$, or $\frac{3}{2}\pi$, or $\frac{5}{2}\pi$, &c.; or the curve P is the locus of a point for which this is the case. Similarly the curve Q is the locus of a point for which the sum of the angles is $\pi, 2\pi, 3\pi, \dots$. Whether the roots of $f(x)=0$ are real or imaginary, the n asymptotes of P and $n-1$ asymptotes of Q, are all real; but when some or all of the roots are imaginary, our demonstration that the infinite branches lie below the asymptotes, no longer applies; for p_2 may then be either positive or negative.

In conclusion, it may be useful to give a few examples of the actual forms of the curves.

First, we will take a case where all the roots are real

$$x^4 - 25x^2 + 60x - 36 = (x-1)(x-2)(x-3)(x+6) = 0.$$

Then the equations to the two curves are

$$y^4 - 6x^2y^2 + x^4 + 25(y^2 - x^2) + 60x - 36 = 0 \quad \dots \quad (P)$$

$$2y^2x - 2x^3 + 25x - 30 = 0 \quad \dots \quad (Q)$$

and the curves assume the shape shown in fig. 20, where the thin branches belong to P and the thick belong to Q. Consistently with what has gone before, P and Q do not intersect, but a branch of Q lies between each adjacent two branches of P; and, the equation being of the fourth degree, P has four asymp-

totes and Q has three, which all pass through the origin, and are arranged at equal angles around it as already explained.

Next take an equation with all its roots imaginary, say $1 \pm \sqrt{-1}$, $-1 \pm \sqrt{-2}$, so that $f(x) = (x^2 - 2x + 2)(x^2 + 2x + 3)$ and the equation is $x^4 + x^2 - 2x + 6 = 0$.

Then the equations to the two curves are

$$y^4 - 6x^2y^2 + x^4 - y^2 + x^2 - 2x + 6 = 0 \quad . \quad . \quad . \quad . \quad (P)$$

$$2y^2x - 2x^3 - x + 1 = 0 \quad . \quad . \quad . \quad . \quad (Q)$$

and the curves intersect in four points, as shown in fig. 21.

When the four roots are all imaginary, the curves may be arranged in a widely different way from that here shown. Thus, if the roots are $1 \pm 4\sqrt{-1}$, $-1 \pm 8\sqrt{-1}$, so that

$$x^4 + 78x^2 - 96x + 1105 = 0$$

the equations to the two curves may be put in the form

$$y^2 = 3x^2 + 39 \pm \sqrt{8x^4 + 156x^2 + 96x + 416} \quad . \quad . \quad . \quad . \quad (P)$$

$$y^2 = x^2 + 39 - \frac{24}{x} \quad . \quad . \quad . \quad . \quad (Q)$$

and the curves will lie as shown in fig. 22. They still intersect in four points, but the branches of the P curve now touch the asymptotes (1, 4), (2, 3), instead of (1, 2), (3, 4), as in fig. 21. It follows that there will be a transition position, in which the branches will touch the asymptotes (1, 3), (2, 4), and will cross, so that the curve P will have two double points.

Lastly, take an equation with two real and two imaginary roots, say $1 \pm \sqrt{-1}$, $2, -4$, so that $x^4 - 10x^2 + 20x - 16 = 0$. Then the equations to the two curves become

$$y^2 = 3x^2 - 5 \pm \sqrt{8x^4 - 20x^2 - 20x + 41} \quad . \quad . \quad . \quad . \quad (P)$$

$$y^2 = x^2 - 5 + \frac{5}{x} \quad . \quad . \quad . \quad . \quad (Q).$$

In this case the curves intersect in two points, as shown in fig. 23. Here the real part of the imaginary roots lies between the two real roots; let us therefore take another instance in which this is not the case. Thus let the roots be $2 \pm \sqrt{-1}$, $-1, -3$, so that $x^4 - 8x^2 + 8x + 15 = 0$. Then the equations to the two curves are

$$y^2 = 3x^2 - 4 \pm \sqrt{8x^4 - 16x^2 - 8x + 1} \quad . \quad . \quad . \quad . \quad (P)$$

$$y^2 = x^2 - 4 + \frac{2}{x} \quad . \quad . \quad . \quad . \quad (Q)$$

and the curves intersect in two points, as shown in fig. 24. Comparing this with fig. 23, we see that, whereas the infinite branches of P then touched the asymptotes $(1, -4)$, $(2, 3)$, $(4, -1)$, $(-2, -3)$, they now touch the asymptotes $(1, 2)$, $(3, -2)$, $(4, -1)$, $(-3, -4)$. It follows that there must be a transition position in which the branches touch the asymptotes $(1, 3)$, $(2, -1)$, $(4, -1)$, $(2, -3)$, so that P has two double points.

If we now examine the relations of the P and Q curves in figs. 20-24, we see that they satisfy the conditions laid down at the outset, bearing in mind that the x -axis must in each case be considered, for this purpose, a part of the Q curve.

Since the foregoing was written I have met with some remarks of the late Professor DE MORGAN on the curves P and Q, contained in a paper of his read before the Cambridge Philosophical Society on 7th December 1857. He remarks that these curves "are such that two branches, one of each curve, "cannot inclose a space." This is a particular case of the properties investigated in the early part of this paper. He also remarks that the curves "always "intersect orthogonally," but he gives no proof of this. It may be proved as follows.

Let h, k , be the co-ordinates of a point of intersection of P and Q, and θ, θ' , the angles which the tangents to P and Q at the point, respectively make with the x -axis. If $\phi(xy)=0$ is the equation to a curve, then for any point (x, y) , $\frac{dy}{dx} = -\frac{d\phi}{dx} \div \frac{d\phi}{dy}$. Applying this formula to (P) and (Q) (p. 430) we have

$$\tan \theta = \frac{f'(h) - \frac{k^2}{2}f'''(h) + \frac{k^4}{4!}f^{(5)}(h) - \dots}{kf''(h) - \frac{k^3}{3!}f^{(4)}(h) + \dots}$$

$$\tan \theta' = \frac{f''(h) - \frac{k^2}{3!}f^{(4)}(h) + \frac{k^4}{5!}f^{(6)}(h) - \dots}{\frac{2k}{3!}f'''(h) - \frac{4k^3}{5!}f^{(5)}(h) + \dots}$$

Hence

$$\tan \theta \tan \theta' = \frac{f'(h) - \frac{k^2}{2}f'''(h) + \frac{k^4}{4!}f^{(5)}(h) - \dots}{\frac{2k^2}{3!}f'''(h) - \frac{4k^4}{5!}f^{(5)}(h) + \dots}$$

and

$$\tan \theta \tan \theta' + 1 = \frac{f'(h) - \frac{k^2}{3!}f'''(h) + \frac{k^4}{5!}f^{(5)}(h) - \dots}{\frac{2k^2}{3!}f'''(h) - \frac{4k^4}{5!}f^{(5)}(h) + \dots}$$

and the numerator of this fraction = 0, since h, k , satisfy the equation (Q). Hence $\tan \theta \tan \theta' = -1$, or the tangents are at right angles to each other.

Fig. 1

Fig. 2.

Fig. 3.

Fig. 4.

Fig. 5

Fig. 6.

Fig. 7

Fig. 8.

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 14

Fig. 15

Fig. 16.

Fig. 17

Fig. 18

Fig. 19

Fig. 20

Fig. 21

Fig. 22

Fig. 23.

Fig. 24