XXIII.—*On a Problem in Combinations.* By The Rev. PHILIP KELLAND, M.A.,
Professor of Mathematics in the University of Edinburgh.

(Read 3d December 1855.)

Several years ago, when discussing the question of the distribution of the stars, a problem occurred to Professor FORBES, which, simple as it is, appears to have escaped notice prior to that time. Having been consulted as to its solution, I communicated my results to Professor FORBES, who has inserted one of them in his paper printed in the Philosophical Magazine for 1850, vol. xxxvii., p. 425. But for the very ingenious application which Professor FORBES has there made of it, the problem might probably not be worth recurring to. As it is, I have thought it would not be altogether uninteresting to give the complete solution.

The Problem is as follows:—There are n dice, each of which has p faces, p being not less than n ; it is required to find the number of arrangements which can be formed with them, 1°, So that no two show the same face; 2°, That no three show the same face; 3°, That no four do so, and so on.

1. The number of arrangements in which no two show the same face is easily seen to be the same as the number of permutations of the p faces, taken n together; and is therefore $p(p-1)(p-2)\dots(p-n+1)$.

2. Remove the dice A and B, and cover the face 1 of the remainder. The number of arrangements which these can now form, omitting the covered face, and no two showing the same face, will be—

$$\begin{aligned} & (p-1)(p-2)\dots(p-1-\overline{n-2}+1) \\ & = (p-1)(p-2)\dots(p-n+2). \end{aligned}$$

Place with each of these the dice A and B, showing face 1, and you have the arrangements in which the dice A and B, and these alone, show face 1. The same applies to each of the other faces. Consequently there are $p(p-1)\dots(p-n+2)$ arrangements in which the dice A and B, and these alone, show the same face. The same is true of every other pair of dice. Hence the number of arrangements in which two, and two only, show the same face, is—

$$n \frac{(n-1)}{1 \cdot 2} p(p-1)\dots(p-n+2).$$

3. Remove the dice A, B, C, D, and cover the faces 1 and 2 of the others. The number of arrangements which can now be formed in which no two show the same face is—

$$\begin{aligned} & (p-2)(p-3)\dots(p-2-\overline{n-4}+1) \\ & = (p-2)(p-3)\dots(p-n+3). \end{aligned}$$

By placing A and B showing face 1, and C and D showing face 2, with each of these, you have the arrangements in which two particular duplications of the dice A, B, C, D, and those only, occur. Now the number of arrangements of faces 1 and 2, on dice A, B, C, D, is the number of permutations, all together, of four things, two of one kind and two of another, or, $\frac{4 \cdot 3 \cdot 2 \cdot 1}{1^2 \cdot 2^2}$.

Hence $\frac{4 \cdot 3 \cdot 2 \cdot 1}{2^2} (p-2) (p-3) \dots (p-n+3)$

is the number of arrangements in which two duplications occur of faces 1 and 2, and on dice A, B, C, D. The same is true of any other pair of faces; consequently the number of arrangements in which two duplications are found on the dice A, B, C, D, but on no others, nor any repetition of the faces shown on these four dice, is—

$$\frac{p(p-1)}{1 \cdot 2} \cdot \frac{4 \cdot 3 \cdot 2 \cdot 1}{2^2} (p-2) (p-3) \dots (p-n+3).$$

In like manner, any other four dice form the same number of arrangements; and hence the total number of arrangements in which two duplications and no more occur, is—

$$\frac{4 \cdot 3 \cdot 2 \cdot 1}{2^2} \cdot \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{1}{1 \cdot 2} p(p-1) \dots (p-n+3).$$

4. Similarly, the number of arrangements in which three duplications, and three only, occur without any other repetition, is,—

$$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2^3} \cdot \frac{n(n-1) \dots (n-5)}{1 \cdot 2 \dots 6} \cdot \frac{1}{1 \cdot 2 \cdot 3} p(p-1) \dots (p-n+4),$$

and the law of formation is evident.

5. We may now write the number of arrangements in which no triplication occurs, in the following form:—

$$\begin{aligned} & p(p-1) \dots (p-n+1) + \frac{n(n-1)}{1 \cdot 2} p(p-1) \dots (p-n+2) \\ & + \frac{4 \cdot 3 \cdot 2 \cdot 1}{2^2} \cdot \frac{n(n-1) \dots (n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{1}{1 \cdot 2} p(p-1) \dots (p-n+3) \\ & + \frac{6 \cdot 5 \dots 1}{2^3} \cdot \frac{n(n-1) \dots (n-5)}{1 \cdot 2 \dots 6} \cdot \frac{1}{1 \cdot 2 \cdot 3} p(p-1) \dots (p-n+4) + \&c. \\ & = p(p-1) \dots (p-n+1) \left\{ 1 + \frac{n(n-1)}{p-n+1} \cdot \frac{1}{2} + \right. \\ & \left. \frac{n(n-1) \dots (n-3)}{(p-n+1)(p-n+2)} \cdot \frac{1}{1 \cdot 2} \cdot \frac{1}{2^2} + \frac{n(n-1) \dots (n-5)}{(p-n+1)(p-n+2)(p-n+3)} \cdot \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2^3} + \&c. \right\} \end{aligned}$$

6. This series may be exhibited as the solution of a differential equation, but it is doubtful whether, with our present knowledge, we can simplify its form. We obtain the differential equation thus:—

$$\text{Let } u = 1 + \frac{n(n-1)}{p-n+1} x + \frac{n(n-1) \dots (n-3)}{(p-n+1)(p-n+2)} \cdot \frac{x^2}{1 \cdot 2} + \&c.$$

$$\begin{aligned} \text{Then } \frac{d u}{d x} &= \frac{n(n-1)}{p-n+1} + \frac{n(n-1) \dots (n-3)}{(p-n+1)(p-n+2)} x \\ &\quad + \frac{n(n-1) \dots (n-5)}{(p-n+1) \dots (p-n+3)} \frac{1}{1.2} x^2 + \&c. \\ \frac{d}{d x} \left(x^{p-n+1} \frac{d u}{d x} \right) &= n(n-1) x^{p-n} + \frac{n \dots (n-3)}{(p-n+1)} x^{p-n+1} \\ &\quad + \frac{n(n-1) \dots (n-5)}{(p-n+1)(p-n+2)} \frac{x^{p-n+2}}{1.2} + \&c. \\ &= n(n-1) z^{2n-2p} + \frac{n \dots (n-3)}{p-n+1} z^{2n-2p-2} + \&c. \end{aligned}$$

$$\text{when } x = \frac{1}{z^2}.$$

$$\text{Hence } z^{2p-n-2} \frac{d}{d x} \left(x^{p-n+1} \frac{d u}{d x} \right) = \frac{d^2}{d z^2} (z^n u);$$

$$\begin{aligned} \text{or } z^{2p-n-2} \left\{ z^{-2p+2n-2} \frac{d^2 u}{d x^2} + (p-n+1) z^{-2p+2n} \frac{d u}{d x} \right\} \\ = 4 z^{n-6} \frac{d^2 u}{d x^2} - (4n-6) z^{n-4} \frac{d u}{d x} + n(n-1) z^{n-2} u; \end{aligned}$$

$$\text{whence } (x-4x^2) \frac{d^2 u}{d x^2} + \left\{ p-n+1 + (4n-6)x \right\} \frac{d u}{d x} - n(n-1)u = 0.$$

7. To find the number of arrangements in which no quadruplications occur, let us write the result of Art. 5 under the form C_p^n , being the total number of arrangements of n dice, each having p faces, in which no triplications occur. Remove the dice A, B, C, and cover the face 1 of the others. Then C_{p-1}^{n-3} is the number of arrangements of those dice in which no triplications occur; and, consequently, the number in which A, B, C, and those only, show face 1. Hence $p C_{p-1}^{n-3}$ is the number of arrangements in which a triplication is found on the first three dice, and on those alone. Consequently the number of arrangements in which one triplet only occurs is—

$$\frac{n(n-1)(n-2)}{1.2.3} p C_{p-1}^{n-3}.$$

8. Remove the dice A, B, C, D, E, F, and cover faces 1 and 2 of the others; C_{p-2}^{n-6} is the number of arrangements in which A, B, C show face 1, and D, E, F face 2.

$$\text{Hence } \frac{6.5.4.3.2.1}{1.2.3.1.2.3} C_{p-2}^{n-6}$$

is the number of arrangements in which A, B, C, D, E, F have faces 1 and 2 tripled; consequently

$$\frac{6.5.4.3.2.1}{(1.2.3)^2} \cdot \frac{p(p-1)}{1.2} C_{p-2}^{n-6}$$

is the number in which A, B, C, D, E, F form two triplets. Hence the number of arrangements in which there are two triplets, and no more, is—

$$\frac{n(n-1) \dots (n-5)}{1 \cdot 2 \dots 6} \cdot \frac{6 \cdot 5 \dots 1}{(1 \cdot 2 \cdot 3)^2} \frac{p(p-1)}{1 \cdot 2} C_{p-2}^{n-6}$$

$$= \frac{n(n-1) \dots (n-5)}{(1 \cdot 2 \cdot 3)^2} \frac{p(p-1)}{1 \cdot 2} C_{p-2}^{n-6}.$$

9. The whole number of arrangements in which no quadruplication occurs is—

$$C_p^n + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} p C_{p-1}^{n-3} + \frac{n(n-1) \dots (n-5)}{(1 \cdot 2 \cdot 3)^2} \frac{p(p-1)}{1 \cdot 2} C_{p-2}^{n-6}$$

$$+ \frac{n(n-1) \dots (n-8)}{(1 \cdot 2 \cdot 3)^3} \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} C_{p-3}^{n-9} + \&c.$$

10. In the same manner it may be shown, that, if the above series be represented by D_p^n , the whole number of arrangements in which no quintuplications occur is—

$$D_p^n + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} p D_{p-1}^{n-4} + \frac{n(n-1) \dots (n-7)}{(1 \cdot 2 \cdot 3 \cdot 4)^2} \cdot \frac{p(p-1)}{1 \cdot 2} D_{p-2}^{n-8} + \&c.$$

11. It is evident that the total number of arrangements of the faces is p^n . Hence the probability that no two show the same face is—

$$\frac{p(p-1) \dots (p-n+1)}{p^n}.$$