determines a homogeneous strain in space; and, since it is proper, the three edges of the asymptote system remain unchanged. The point A (1, 0, 0) comes to U ; the plane containing P , γP , $\gamma^2 P$, which was parallel to the tangent plane at A , when transformed, contains the points νP , $\nu\gamma P$, $\nu\gamma^2 P$, and is parallel to the tangent plane at U , the new position of A . In the same way, using the automorph $\gamma\nu$, we find that the plane of the points $\gamma\nu P$, $\gamma\nu\gamma P$, $\gamma\nu\gamma^2 P$ is parallel to the tangent plane at γU .

It may be noted that the fundamental units must be represented by points on the negative sheets, unless indeed there are no points with integral coordinates upon these sheets. For, if (u, v, w) is on the positive sheet, the three conjugate units are positive. Hence every power has three positive conjugates, and cannot lie on a negative sheet because units on a negative sheet have two of the conjugates negative.

Examples illustrating Lord Rayleigh's Theory of the Stability or Instability of certain Fluid Motions. By A. E. H. LOVE.

Read January 9th, 1896.

In papers published in the *Proceedings*, Vols. XI. and XIX., Lord Rayleigh has discussed the oscillations possible in a stream of fluid flowing between two fixed planes, which arise from difference of spin (or molecular rotation) in different parts. He has especially attended to cases where the spin changes discontinuously at certain planes between the boundaries. A difficulty occurs in these solutions through the fact that the places where the stream-velocity is equal to the wave-velocity are singularities of the integrals of the differential equation on which the small varied motion depends. This difficulty Lord Rayleigh has sought to evade in a recent paper (*Proc.*, Nov., 1895). It was felt, however, that, in the case of continuously varying spin, the complete discussion of the problem for a particular law of velocity would be desirable. The present paper contains such a discussion as appears possible (without solving the differential equation) of a case where there are two separated singular places of the integral. The general conclusion seems to be that wave-

motions of Lord Rayleigh's type can only occur in some very special cases, and that his method does not avail for the determination of a criterion of stability when the disturbance is of a general character. Some further examples are given in which the exact analytical form of the disturbance can be calculated for a definite wave-velocity.

1. A steady motion of fluid in the plane (x, y) is possible in which the stream-lines are parallel to a fixed line, the axis x , and the velocity of the fluid, everywhere parallel to this axis, is independent of x , and is any differentiable function of y . Suppose U is this velocity. The fluid may be bounded by two lines parallel to the axis of x , and between these lines $-\frac{1}{2} \frac{\partial U}{\partial y}$ is everywhere finite. This quantity is the spin (or molecular rotation) of the fluid at any point, and it is variable only with y , and we shall suppose it to be a differentiable function* of y . We shall suppose that U and $\frac{\partial U}{\partial y}$ are given for all values of y between the two limiting lines which contain the fluid.

We seek now to determine a motion of the fluid which shall, at all times, and for all points between the given boundaries, differ very little from the steady motion just described. Suppose $U+u$ and v are the component velocities parallel to x and y in the varied motion, and suppose that u and v are at any one time each less than some given arbitrarily small quantity ϵ for all values of x , and for all values of y between the given boundaries,† and further, suppose that u and v do not tend to increase indefinitely with the time; then the quantities u and v are always small, of the order ϵ at most, and the steady motion is stable. For the investigation of such varied motions, the equations determining u and v are reduced to linear equations by rejecting all terms that involve their squares or products, or higher powers, and then it may be assumed that u and v are proportional to simple harmonic functions of the time, with a period to be determined. If the equation determining this period has only real roots, the motion is stable.

* The case in which $\partial U/\partial y$ is discontinuous has been sufficiently considered by Lord Rayleigh in the papers already cited.

† This limitation excludes *a priori* the solutions discussed by Lord Rayleigh for which $\partial v/\partial y$, and consequently u , becomes infinite. For the limitation itself, cf. Routh's *Treatise on the Stability of Motion*, p. 1.

2. In the problem before us the quantities u and v are to be determined from the equation of continuity, and the condition that the spin of any element is independent of the time. The former gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

and the latter gives

$$\frac{\partial}{\partial t}(Z + \zeta) + (U + u)\frac{\partial}{\partial x}(Z + \zeta) + v\frac{\partial}{\partial y}(Z + \zeta) = 0,$$

where $Z = -\frac{1}{2}\partial U/\partial y$, is the spin in steady motion, and $\zeta = \frac{1}{2}(\partial v/\partial x - \partial u/\partial y)$, is the added spin at any point. By the process already sketched this reduces to

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} + v\frac{\partial Z}{\partial y} = 0 \dots\dots\dots(1).$$

We shall now, following Lord Rayleigh, suppose that the varied motion is a wave-motion with wave-length $2\pi/\kappa$ and velocity V , so that

$$u = u_1 \cos \kappa(x - Vt + \alpha), \quad v = v_1 \sin \kappa(x - Vt + \alpha) \dots\dots\dots(2),$$

where u_1 and v_1 are functions of y only. The equation of continuity then becomes

$$-\kappa u_1 + \frac{\partial v_1}{\partial y} = 0 \dots\dots\dots(3),$$

and the equation (1) for the varied spin becomes

$$(U - V)\left(\frac{\partial^2 v_1}{\partial y^2} - \kappa^2 v_1\right) - \frac{\partial^2 U}{\partial y^2} v_1 = 0,$$

since
$$\zeta = \frac{1}{2} \cos \kappa(x - Vt + \alpha) \left(\kappa v_1 - \frac{\partial u_1}{\partial y} \right).$$

We may suppress the suffix attached to v_1 , and we see that v satisfies the equation

$$(U - V)\left(\frac{\partial^2 v}{\partial y^2} - \kappa^2 v\right) - \frac{\partial^2 U}{\partial y^2} v = 0 \dots\dots\dots(4).$$

The further conditions which have to be satisfied are that $v = 0$ at the two fixed boundaries, i.e., for two particular values of y .

3. In order that the disturbance may be propagated by waves in the manner supposed it must be possible to assign a real quantity V so that a function v may exist which (i.) satisfies the differential equation (4) for all values of y in a certain real interval, (ii.) vanishes at the limits of this interval, (iii.) is finite, and has a finite and continuous differential coefficient in this interval. Further, in order that the method may apply to an arbitrary initial disturbance, it is necessary that there should be a series of such quantities V_r , and, associated with each, a function v_r of such a character that an arbitrary function of y can be expanded in a series of the form

$$\sum A_r v_r,$$

which converges in the given interval. The quantities V_r are required to exist for all real values of κ .

Lord Rayleigh has proved* that it is impossible to satisfy the differential equation and the boundary conditions with a complex value of V , if $\frac{\partial^3 U}{\partial y^3}$ is one-signed between the boundaries; and he con-

cluded that, under this condition, the steady motion expressed by U must be stable. It appears however that this conclusion required additional justification, inasmuch as there is no evidence to show that every disturbance will be propagated by waves in the manner supposed. Lord Rayleigh has further remarked† that it is impossible to satisfy the differential equation and the boundary conditions with any value of V for which $U - V$ and $\frac{\partial^3 U}{\partial y^3}$ have the same sign everywhere between the boundaries.

We can construct an example for which $\frac{\partial^3 U}{\partial y^3}$ is one-signed, and for which $U - V$ has the opposite sign for values of y between the boundaries, by taking for U any quadratic function of y , and for V a real constant such that the roots of $U - V$ are two values of y between which the boundaries of the fluid lie.

4. Let
$$U = Ay^2 + 2By + C,$$

and
$$U - V = A(y - a)(y - b),$$

and let the boundaries of the fluid be $y = h_1$ and $y = h_2$; and to fix

* *Proc.*, Vol. xi., p. 69.

† *L.c.*, p. 70.

ideas take $b > h_1 > h_2 > a$; then the differential equation becomes

$$(y-a)(y-b) \left(\frac{\partial^2 v}{\partial y^2} - \kappa^2 v \right) = 2v \dots\dots\dots(5).$$

When κ is given we have only one parameter to dispose of, viz., V , and this is given by

$$V = \frac{1}{4}A(b-a)^2 + \frac{AC-B^2}{A} \dots\dots\dots(6),$$

so that the parameter which is in our power is $b-a$, the sum of the roots $(b+a)$ being given.

We transform the equation to one with fixed singularities by changing the independent variable from y to y' , where

$$\frac{y-a}{y-b} = \frac{y'}{y'-1} \dots\dots\dots(7),$$

i.e., by putting $y = (b-a)y' + a$. We then have

$$\frac{\partial^2 v}{\partial y'^2} - \kappa^2 (b-a)^2 v = -\frac{2v}{y'(1-y')},$$

and the interval within which it is to be solved is such that

$$1 > y' > 0,$$

but the interval does not extend so far as $y' = 0$ or $y' = 1$.

Writing now λ^2 for $\kappa^2(b-a)^2$, we see that we have to seek for real values of λ for which the differential equation

$$\frac{\partial^2 v}{\partial y'^2} = -\frac{2v}{y'(1-y')} + \lambda^2 v \dots\dots\dots(8)$$

can be satisfied by a value of v which vanishes for two values of y' lying in the interval $1 > y' > 0$. The limiting values, h'_1 and h'_2 , of y' are given by

$$h'_1 = (h_1 - a)/(b - a) = \frac{1}{2} \{ 1 - (a + b - 2h_1)/(b - a) \},$$

$$h'_2 = (h_2 - a)/(b - a) = \frac{1}{2} \{ 1 - (a + b - 2h_2)/(b - a) \},$$

so that they depend on h_1 , h_2 , and on the parameter $(b-a)$ in quite definite ways.

5. It is useless to attempt to solve the differential equation. Solutions in series can of course be obtained, but they afford no criterion for determining the roots of v . We can however use an intuitional method* by comparing the above equation (8) with

$$\frac{d^2x}{dt^2} = -\frac{2}{t(1-t)}x + \lambda^2x.$$

This equation may be regarded as the equation of motion of a particle, which moves in a straight line under the action of a repulsive force λ^2x proportional to its displacement x , and an attractive force $\frac{2}{t(1-t)}x$ also proportional to x , but variable with the time t . At a certain time t_0 ($=h'_1$), the particle is passing through the origin with a finite velocity, and we wish to discover whether the forces can be so adjusted that before $t=1$ it may have returned to the origin. We cannot determine the time that the particle would take to describe any distance x , or to return to the origin, but we may use the principle that any increase in the strength of the attractive force or decrease in the strength of the repulsive force would shorten the period that must elapse before it returns to the origin. We can now show that at least one of the limiting values must lie near to $t=0$ or $t=1$. Suppose, to take an example, that the instant of starting is $t=\frac{1}{2}$, we can see that the particle will not have returned before $t=\frac{2}{3}$; for during this interval the strength of the attractive force is constantly less than $\frac{2}{\frac{1}{2}(\frac{1}{2})}$, and the particle would not return in less time than $\frac{1}{2}\pi\sqrt{2}$ if the strength of the attractive force were constant and had its greatest value. More generally, if the particle starts at time $t_0(=\epsilon)$, and returns at time $t_1(>1-\epsilon)$, the time taken to return to the origin is not greater than $1-2\epsilon$. The maximum strength of the attractive force is $2/\{\epsilon(1-\epsilon)\}$. The period is greater than

$$\frac{\pi}{\sqrt{\left\{\frac{2}{\epsilon(1-\epsilon)} - \lambda^2\right\}}},$$

so that this quantity is certainly $<1-2\epsilon$. Further we have

$$\lambda = \kappa(b-a) = \kappa(a+b-2h_1)/(1-2\epsilon),$$

* Suggested by Klein's discussion of Lamé's equation in his lithographed *Vorlesungen über lineare Differentialgleichungen der zweiten Ordnung*, Göttingen, 1894.

and we hence find that

$$\epsilon < \frac{1}{2} \left[1 - \sqrt{\left\{ \frac{\pi^2 + \kappa^2 (a+b-2h_1)^2}{8 + \pi^2 + \kappa^2 (a+b-2h_1)^2} \right\}} \right].$$

In the case where κ is small, *i.e.*, for very long waves in the hydrodynamical problem, $\epsilon < \frac{1}{2}$ and for greater values of κ still smaller values of ϵ are required.

Again, let us suppose the particle starts at time t_0 ($\neq \epsilon$) and returns at time t_1 ($= 1 - \epsilon$); then, just as before, we find

$$\epsilon < \frac{1}{2} \left[1 - \sqrt{\left\{ \frac{\pi^2 + \kappa^2 (a+b-2h_2)^2}{8 + \pi^2 + \kappa^2 (a+b-2h_2)^2} \right\}} \right].$$

This discussion shows that equation (8) with the conditions $v = 0$ when $y' = h'_1$ and when $y' = h'_2$ cannot be satisfied unless either h'_1 is very near 0 or h'_2 is very near 1. In general we can only satisfy these conditions, either by taking a very near to h_1 , or by taking b very near to h_2 .

The results here obtained are by themselves sufficient to show that in many cases the problem can have no solution. We have found that either

$$\frac{a+b-2h_1}{b-a} > \text{a quantity} > \frac{3}{4},$$

or
$$\frac{a+b-2h_2}{b-a} > \text{a quantity} > \frac{3}{4}.$$

The former of these gives

$$b-a < \frac{4}{3} (a+b-2h_1),$$

but, since

$$b-a > h_2 - h_1,$$

we must certainly have

$$h_2 - h_1 < \frac{4}{3} (a+b-2h_1),$$

or

$$3h_2 + 5h_1 < 4(a+b).$$

In like manner the inequality

$$\frac{a+b-2h_2}{b-a} > \frac{3}{4}$$

leads to

$$11h_2 - 3h_1 < 4(a+b).$$

We conclude that there are some restrictive conditions on h_1 , h_2 in order that a solution may be possible, and the character of these would appear to be that the breadth of the stream must not exceed one or other of certain limits depending on the given quantity $(a+b)$.

6. But now suppose these conditions are satisfied, and consider the velocity-time curve of the motion of the particle. Let us suppose that λ is sufficiently small for the resultant force on the particle to be attractive throughout the interval $1 > t > 0$. Since the attractive force has minimum strength when $t = \frac{1}{3}$, this requires $\lambda^3 < 8$. Then the velocity constantly diminishes until the particle returns to the origin, and, as we shall see in § 8, the velocity becomes in general positively or negatively infinite at $t = 1$. The curve has therefore an asymptote $t = 1$. Further, we shall prove in § 8 that when the velocity is infinite the displacement remains finite and thus the area between the curve and the asymptote is finite. Now the area contained between the curve, the t axis, and two ordinates $t = t_0$ and $t = t_1$ is the space passed over between these times. If then the particle starts from the origin at time t_0 with a finite positive velocity, it may happen that there is a value t_1 (< 1) of t for which the positive area between t_0 and the point where the velocity vanishes is equal to the negative area between this point and t_1 . Since $\frac{d^2x}{dt^2} = 0$ when $x = 0$, the points t_0 and t_1 must be points on the velocity-time curve where the ordinate is a maximum or a minimum. Thus λ must be chosen so that the distance between a maximum and the next minimum ordinate may be $\kappa(h_2 - h_1)/\lambda$, and the area between the maximum ordinate and the point where the velocity vanishes must be equal to the area between this point and the minimum ordinate (see Fig. 1). It may be possible to secure the satisfaction of these conditions when h_1, h_2 are suitably chosen, but it may be impossible. We shall proceed on the supposition that one value of λ has been chosen which satisfies all the conditions.

FIG. 1.

7. It is next important to inquire whether the conditions can be satisfied by more than one value of λ . In the first place we show

that if the particle can return once to the origin with any value of λ it cannot return for any greater value of λ . For the period in which it is required to return, being $\kappa(h_2 - h_1)/\lambda$, is diminished by increasing λ ; but the period in which it does return is certainly increased by increasing λ , since λ^2 is the strength of the repulsive force. In the next place we observe that a diminution in the value of λ increases the time in which the particle is required to return to the origin, but diminishes the period in which the particle does return once to the origin. If then with one value of λ the particle can return once to the origin with any smaller λ for which the conditions can be satisfied, it must return more than once. Thus, if more than one value of λ can be chosen which satisfies all the conditions, with these different λ 's the particle returns to the origin once, twice, three times, and so on. Hence these values of λ form a discrete series of diminishing quantities. But, since $\lambda < \kappa(h_2 - h_1)$, there is only a finite number of possible values for λ .

Hence, going back to the hydrodynamical problem, it is proved that, though there may be a finite number of values of V for which the differential equation

$$(U - V) \left(\frac{\partial^2 v}{\partial y^2} - \kappa^2 v \right) = \frac{d^2 U}{dy^2} v$$

has a solution v , which vanishes when $y = h_1$ and when $y = h_2$, there cannot be an indefinite series of such values. It follows that, though there may be particular types of disturbance which can be propagated by wave-motion in the manner supposed, this cannot be true for a general disturbance.

Further, since, as we have seen, possible values of λ cannot exist for arbitrary values of h_1 and h_2 , there will not be such wave-motions unless some (unknown) relations subsist between the constants in U and the ordinates h_1, h_2 of the boundary lines.

We conclude that so long as the wave-velocity differs from the stream-velocity at all points between the boundaries there will be wave-motions of the kind supposed only for some special types of disturbance, and with the breadth of the stream subject to some restrictions.

We shall see presently that the application of the method is still more restricted when the wave-velocity is allowed to become equal to the stream-velocity.

8. Before passing on to examine how our results are modified if either of the roots of $U-V$ lies between the boundaries $y = h_1$ and $y = h_2$, it will be convenient to investigate the character of the solutions of the differential equation in the neighbourhood of the roots of $U-V$. Taking the equation in the form

$$\frac{\partial^2 v}{\partial y'^2} = \lambda^2 v - \frac{2v}{y'(1-y')},$$

we observe that in the neighbourhood of $y' = 0$ there is one particular integral, say v_1 , which can be expanded in powers of y' , the first term being the term of degree unity in y' . We may write

$$v_1 = g_1 y' + g_2 y'^2 + g_3 y'^3 + \dots + g_n y'^n + \dots;$$

then the differential equation gives us

$$(y' - 1) \left[\sum_2^n n(n-1) g_n y'^{n-1} - \lambda^2 \sum_2^n g_n y'^{n+1} \right] - 2 \sum_2^n g_n y'^n = 0,$$

from which

$$g_2 = -g_1,$$

$$6g_3 = \lambda^2 g_1,$$

$$n \left[(n-3) g_{n-1} - (n-1) g_n \right] + \lambda^2 (g_{n-2} - g_{n-3}) = 0, \quad n = 4, 5, \dots$$

We may assign an arbitrary value to g_1 and then all the other coefficients are determinate.

There is also a particular integral, say v_2 , which cannot be so expressed, but

$$v_2 = v_1 \log y' + v',$$

where v' is a series of the form

$$v' = h_0 + h_2 y'^2 + h_3 y'^3 + \dots,$$

in which h_0 is different from zero. The differential equation gives us

$$\frac{\partial^2 v'}{\partial y'^2} + \left(\frac{2}{y'(1-y')} - \lambda^2 \right) v' + \frac{2}{y'} \frac{dv_1}{dy'} - \frac{1}{y'^2} v_1 = 0,$$

$$\text{or} \quad (1-y') \left[\sum_2^n n(n-1) h_n y'^{n-1} - \lambda^2 (h_0 y' + \sum_2^n h_n y'^{n+1}) \right]$$

$$+ 2(h_0 + \sum_2^n h_n y'^n) + (1-y') \left[2 \sum_2^n n g_n y'^{n-1} - \sum_2^n g_n y'^{n-1} \right] = 0,$$

from which

$$2h_0 + g_1 = 0,$$

$$2h_2 - \lambda^2 h_0 + 3g_3 - g_1 = 0,$$

$$6h_3 + \lambda^2 h_0 + 5g_3 - 3g_2 = 0,$$

$$12h_4 - 4h_2 - \lambda^2 h_2 + 7g_4 - 5g_3 = 0,$$

$$n[(n-1)h_n - (n-3)h_{n-1}] - \lambda^2(h_{n-2} - h_{n-3}) + (2n-1)g_n - (2n-3)g_{n-1} = 0,$$

$$n = 5, 6, \dots$$

This work is the application to our particular equation of a known general theory.* It is further known that the series v_1 and v' converge when $1 > y' > -1$, the other singularity of the differential equation being at $y' = 1$, but they do not converge for the value $y' = 1$. There will in like manner be two particular integrals which in the neighbourhood of $y' = 1$ can be expressed in the forms

$$g_1(1-y') + g_2(1-y')^2 + \dots + g_n(1-y')^n + \dots,$$

$$\text{and } \log(1-y') [g_1(1-y') + g_2(1-y')^2 + \dots + g_n(1-y')^n + \dots]$$

$$+ h_0 + h_2(1-y')^2 + h_3(1-y')^3 + \dots + h_n(1-y')^n + \dots,$$

the coefficients being clearly the same as before, on account of the complete symmetry of the differential equation with respect to $y' = 0$ and $y' = 1$. The series here given converge when $1 > 1-y' > -1$, i.e., when $2 > y' > 0$.

We notice that neither solution becomes infinite at the singularities, but v_2 has an infinite differential coefficient at $y' = 0$. Thus, in general, $\partial v / \partial y$ is infinite at the singularities, but v is finite.

9. The results just found may be interpreted in accordance with the intuitional method in which y' is replaced by the time t , and v is regarded as the displacement of a particle which moves in a straight line under a repulsive force and an attractive force. The particle will generally have infinite velocity at the instants $t = 0$ and $t = 1$ (supposing the time of motion to include those instants), but it may have a finite velocity at one of these instants, and in that case it must be at the origin at that instant. Further, before the time $t = 0$, and after the time $t = 1$, the force is repulsive, so that if the particle

* See e.g. Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, Bd. I., Leipzig, 1895.

is moving through the origin at time $t = 0$ with a finite velocity it can never have been at the origin before, and if the particle is moving through the origin at time $t = 1$ with a finite velocity it can never return to the origin. Again, since the constant h_0 of the v_3 solution must be different from zero, we see that the particle can never have infinite velocity when at the origin.

Taking as before the supposition that the force never becomes repulsive, the velocity-time curves in the cases where the velocity is finite at $t = 0$ and $t = 1$ respectively are of the forms shown in Fig. 2 and Fig. 3, so that it appears probable that λ may be so chosen

FIG. 2.

FIG. 3.

that in the first the area between $t = 0$ and $t = t_1$ (< 1) shall vanish, and in the second that the area between $t = t_0$ (> 0) and $t = 1$ shall vanish.

10. Going back now to the hydrodynamical problem, if we take $a = h_1$, b will be fixed since $a + b$ is given. We shall suppose that $b > h_2$. Then the solution corresponding to v_1 with its differential coefficients remains finite when $y = h_1$ and throughout the range $h_2 > y > h_1$, and it appears probable that the value of V which makes $a = h_1$ will make v vanish for some definite values of $h_2 < b$. But it is not certain that this will be the case; and, if it does so happen,

the values of h_2 will be thereby fixed, and may not include the given value of h_2 . Thus there will be, with given $U (= Ay^2 + 2By + C)$, at most a finite number of definite breadths of stream for which a disturbance of wave-length κ can be propagated by waves in the manner supposed with a wave-velocity equal to the value of U on one of the boundary lines. Further, as in the preceding case, the disturbances which can be so propagated will not be arbitrary disturbances of wave-length κ but quite definite disturbances. Like results follow if we take $b = h_2$.

11. Any other supposition as to the relative situation of a, b, h_1, h_2 may be similarly disposed of. Without going through all the work we shall tabulate the results:—

(i.) $\left\{ \begin{array}{l} b > a > h_2 > h_1 \\ h_2 > h_1 > b > a \end{array} \right\}$ No disturbance possible. (This is the supposition disposed of by Lord Rayleigh in *Proc.*, Vol. XI., p. 70.)

(ii.) $b > h_2 > h_1 > a$. At most a finite number of values of V for any κ , so that special types of disturbance only are capable of being propagated by waves in the manner supposed.

(iii.) $b > h_2 > h_1, h_1 = a$. No value of V unless h_2 has one of certain definite values, and possibly no value of V for any h_2 . So that at most this would give a finite number of values of V when h_2 has one of certain definite values; i.e., at most there are a finite number of special types of disturbance, in streams of special breadths, which can be propagated by waves in the manner supposed, with $V = U(h_1)$.

(iv.) $b = h_2, h_2 > h_1 > a$. This case is exactly like the preceding.

(v.) $b < h_2, h_2 > h_1 > a$. In this case $\partial v / \partial y$ can only be finite, when $y = b$, if v vanishes there, and then v cannot vanish for any greater value of y , so that in particular it cannot vanish for h_2 . Hence there is no solution.

(vi.) $a > h_1, b > h_2 > h_1$. This case is like the preceding, and there is no solution.

(vii.) $a = h_1, b = h_2$. Since the value of v which makes $\partial v / \partial y$ finite at a vanishes at a , and is finite and different from zero at b , there is no solution.

12. The case where U is a linear function of y is very simple. Suppose as before that the boundaries are $y = h_1$ and $y = h_2$. The differential equation becomes

$$\frac{\partial^3 v}{\partial y^3} - \kappa^3 v = 0 \dots\dots\dots (9),$$

and a solution vanishing when $y = h_1$ is

$$v = A \sinh \kappa (y - h_1),$$

but we cannot make it vanish also when $y = h_2$. In this case it has been suggested that a possible wave-motion might be found by taking V equal to the value of U at one line, $y = a$ say, between h_1 and h_2 . Then the differential equation (9) need not be satisfied when $y = a$. We should then have to take

$$v = A \sinh \kappa (y - h_1), \quad a > y > h_1,$$

$$v = B \sinh \kappa (h_2 - y), \quad h_2 > y > a.$$

To make v and $\partial v / \partial y$ continuous at $y = a$, we should require

$$A \sinh \kappa (a - h_1) = B \sinh \kappa (h_2 - a),$$

$$A \cosh \kappa (a - h_1) = -B \cosh \kappa (h_2 - a),$$

and these cannot be satisfied when h_1 is different from h_2 . Thus there would be in this case no disturbance which could be propagated by waves in the manner supposed. Yet the example afforded by initial disturbances

$$u = C (2y - h_1 - h_2) \cos \kappa x,$$

$$v = \kappa C (y - h_1)(y - h_2) \sin \kappa x,$$

shows that some varied motion is possible which initially is periodic in x with given wave-length. Lord Rayleigh's method does not avail for the discovery of this motion, nor for determining whether the original steady motion is stable for this type of disturbance.

13. In the examples hitherto discussed there has either been no solution, or we have been able to show that the solution, if it exists, corresponds to a special type of disturbance, but we have not, in any case where there could be a solution, found the explicit form of the solution. In the examples now to be given the exact analytical form of v will be obtained when U has a definite form, and V has a

particular value. We observe that the differential equation depends only on $U - V$. Writing W for this, the equation is

$$\frac{\partial^2 v}{\partial y^2} - \kappa^2 v = \frac{\partial^2 W}{\partial y^2} \frac{v}{W} \dots\dots\dots(10).$$

If the form of W is assigned so that this equation can be solved, it is only necessary to determine whether the boundary conditions can be satisfied.

(a) Let $W = A \sin \mu (y - a) ;$

then $v = C \sin \lambda (y - h_1),$ where $\lambda^2 = \mu^2 - \kappa^2,$

and, if $\lambda (h_2 - h_1)/\pi$ is an integer, we have a solution. Thus there are in this case solutions corresponding to every integral value of s which is such that $\mu^2 - s^2 \pi^2 / (h_2 - h_1)^2$ is positive.

(b) Let $W = Ay \frac{d}{dy} \left(\frac{\sin \mu y}{y} \right),$

then $\frac{\partial^2 W}{\partial y^2} = W \left(-\mu^2 - \frac{2}{y^2} \right),$

and if $\mu^2 - \kappa^2$ is positive, and $= \lambda^2$ say, we have

$$v = y \frac{d}{dy} \left(\frac{C_1 \sin \lambda y + C_2 \cos \lambda y}{y} \right).$$

If λ is suitably chosen, v can vanish when $y = h_1,$ and when $y = h_2.$ In fact this requires

$$\tan \lambda (h_1 - h_2) = \lambda (h_1 - h_2) / (1 + \lambda^2 h_1 h_2).$$

This equation has an infinite number of real roots, but only such of them may be taken as make $\mu^2 - \lambda^2$ positive.

In this case the singular point $y = 0$ may be included in the interval between the boundaries if the latter are suitably chosen. We must then take $C_2 = 0,$ and the boundaries h_1, h_2 must be roots of the same equation of the form $\tan \lambda h = \lambda h.$ Thus if $h_2 = -h_1 = h$ any value of λ ($> \mu$) which satisfies this equation gives a solution. It is to be noted here that $W = 0$ at $y = 0,$ and v also $= 0$ at $y = 0;$ but the limits for y are not necessarily $y = 0$ and $y = h,$ in contrast with the case where U was a quadratic function, for which we found that, if a zero of $U - V$ is included in the region occupied by fluid, it must lie on the boundary.