Moreover, corresponding to Lemmata II., III., and IV., in § 1, we get

$$\frac{1}{(\lambda \delta_1)} = \frac{1}{\left(-\frac{\lambda}{\lambda_1}\right)} \left(-\frac{\lambda}{\lambda_1 \eta_1}\right) \dots\dots\dots (2),$$

$$(-\lambda q \delta_1)(\lambda_1 - \lambda) \psi(\lambda_1) = \lambda_1 (-\lambda \delta_1) \psi(\lambda_1) \dots\dots\dots (3),$$

$$\text{and } (-\lambda q \delta_1)(-\lambda_1 \mu q) \psi(\lambda_1) = (-\lambda_1 \mu q) \left(-\frac{\lambda \mu q}{\eta_1}\right) (-\lambda q \delta_1) \psi(\lambda_1) \dots (4).$$

By the help of these results we deduce that, if $B_r(\theta)$ be defined by the equation

$$P(-\lambda q) \equiv 1 + \frac{B_1(\theta)}{1-q} \lambda + \frac{B_2(\theta)}{(1-q)(1-q^2)} \lambda^2 + \dots,$$

then $P(-xq, \theta + \phi) P(-xq, \theta - \phi) / (x^2 q)$

$$= 1 + \frac{B_1(\theta) B_1(\phi) q^{-1}}{1-q} + \frac{B_2(\theta) B_2(\phi)}{(1-q)(1-q^2)} q^{-2} + \dots,$$

and that $\frac{P(-\lambda_1 q) P(-\lambda_2 q) P(-\lambda_3 q)}{(\lambda_2 \lambda_3 q)(\lambda_3 \lambda_1 q)(\lambda_1 \lambda_2 q)}$

$$= 1 + \Sigma \Pi_r(\lambda_1 q, \lambda_2 q, \lambda_3 q / \lambda_1 \lambda_2 \lambda_3 q) q^{-ir(r+1)} B_r(\theta),$$

corresponding to § 5.]

Note on some Properties of Gauche Cubics. By T. R. LEE.

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1. If a system of quadrics pass through seven points taken arbitrarily in space, they have also one other point common (Salmon's *Geometry of Three Dimensions*, p. 91, first edition, pp. 97, 98, third edition).

This eighth point might be called the eighth point homologous to the seven given points, while the system of eight points might be called eight homologous points.

It is the object of this note to prove, by purely geometrical methods,

(1) the existence of the eighth homologous point, in a certain class of cases,

(2) certain properties belonging to such a system of eight points, and

(3) two properties of gauche cubics—one of them analogous to Desargues' theorem, that "a chord of a conic is cut in involution by the opposite sides of an inscribed quadrilateral"; the other property being the converse of the first, and affording us a test by which we may determine whether a given line is a chord of a given cubic or not.

It will be found that the theory of eight homologous points is closely connected with the theory of gauche cubics.

2. As the demonstrations in this paper are to be purely geometrical, we are not at liberty to assume the theorem stated above relating to a system of quadrics passing through seven points. We are obliged, therefore, to put the definition of an eighth homologous point into a hypothetical form. We shall use the following definition:—

DEF. 1.—If three quadrics pass through seven given points in space, and if they have one, and only one, other point common, this eighth point is called the eighth homologous point with respect to the seven given points and the three given surfaces.

It is convenient, also, to use the following definitions:—

DEF. 2.—When a system of six points in space is given, if we draw a plane through any three of them, and then draw a second plane through the three remaining points, these two planes are called a pair of opposite planes of the system.

Thus, if the points be 1, 2, 3, 4, 5, 6; then 235 and 461 are opposite planes, and 345, 612 are opposite planes, &c.

DEF. 3.—Given a system of points, the line joining any two of them is called a line of the system.

DEF. 4.—The plane determined by two successive sides of a gauche polygon is called a face of the polygon.

The polygon, therefore, has as many faces as it has sides or corners.

DEF. 5.—A diagonal plane of a polyhedron is any plane determined by three of the corners which are not in the same face.

3. Thus, in a gauche hexagon 123456, the faces are the planes 123, 234, 345, 456, 561, 612. The opposite faces are 123, 456; 234, 561; 345, 612.

There are two triplets of alternate faces: 123, 345, 561, which intersect in p ; and 234, 456, 612, which intersect in p' . The whole figure forms a hexahedron, whose opposite faces are the same as the opposite faces of the hexagon. The points p , p' might be called the poles of the hexagon.

If we take the pair of opposite planes 135, 246 (Def. 2), determined by the two sets of alternate corners of the hexagon, we have

a system of eight planes forming an octahedron; the four pairs of opposite faces being 123, 456; 234, 561; 345, 612; and 135, 246. A plane such as 124, which is not a face of the octahedron, but which passes through three of its corners, is one of its diagonal planes; and the opposite diagonal plane is 356.

It is easy to see that there are altogether ten pairs of opposite planes determined by a system of six points, and therefore an octahedron has four pairs of opposite faces, and six pairs of opposite diagonal planes.

4. Whenever the term "hyperboloid" is used in this paper, what is meant is "hyperboloid of one sheet." This will include a "hyperbolic paraboloid," which is nothing but a hyperboloid of one sheet having one generator at infinity.

A hyperboloid is the locus of lines which intersect three fixed lines. From this definition is easily deduced the anharmonic property that "A hyperboloid is generated by the intersections of two homographic systems of planes, which turn round two fixed lines" (Chasles, *Géométrie Supérieure*, Chap. xx., No. 411, 1^{re} édition; No. 420, 2^{me} édition).

If the two fixed lines intersect each other, the hyperboloid becomes a cone; and a cone is, therefore, generated by the intersections of

two homographic systems of planes, which turn round two fixed intersecting lines. This property of a cone of the second degree may be regarded as a consequence of the anharmonic property of a conic, namely, that "A conic is the locus of the intersection of two homographic systems of lines, which turn in one plane round two fixed points." Now the anharmonic property of a conic can be expressed in this way: "The anharmonic ratio of the lines joining a point on a conic to four fixed points on the curve, is constant." And, in the same way, the anharmonic property of a cone of the second degree may be expressed thus: "The anharmonic ratios of the planes determined by a generator of a cone of the second degree and four fixed generators of the surface, is constant."

5. If 1, 2, 3, 4 are four points in a line, then (1234) denotes their anharmonic ratio. If 1, 2, 3, 4 are any four points in space, and L is any line, then L (1234) denotes the anharmonic ratio of the four planes $L1$, $L2$, $L3$, $L4$.

§ I.

6. *Lemma A.*—The intersection of two quadrics is a quartic curve.

A quadric is a surface which is met by any straight line in two points; and, consequently, its intersection with a plane is a conic. Draw any plane intersecting the two given quadrics in two conics. These two conics intersect in four points (Chasles, *Traité des Sections Coniques*, Chap. xiii., Nos. 328, 335); these four points are common to the two quadrics, and, therefore, lie on their curve of intersection. Therefore every plane meets the curve of intersection of two quadrics in four points, *i.e.*, the curve of intersection is a quartic.

7. The intersection of two quadrics is not in all cases a proper curve of the fourth degree. It may break up into four lines. Thus two hyperboloids may pass through a *gauche* quadrilateral. Again, if the two quadrics are ruled surfaces, and have a common generator, their intersection consists of this generator and a *gauche* cubic.

It is worth noticing here that every *gauche* cubic may be considered as the partial intersection of two quadrics having a common generator. For the cone containing a *gauche* cubic and having its vertex on the curve is easily seen to be a quadric, and the intersection of two such cones is the cubic itself and the common generator joining their vertices.*

* Hence we infer that only one *gauche* cubic can be described through six given points; or, six points determine a *gauche* cubic (Salmon, *Geometry of Three Dimensions*, No. 302, first edition; No. 365, third edition).

Hence, with the help of the anharmonic properties of cones and hyperboloids, it is easy to show that a hyperboloid can be constructed which will pass through the gauche cubic and have any two given chords of the curve as generators of the same system; from which it follows that a gauche cubic may be considered as the partial intersection of two hyperboloids, or of a hyperboloid and a cone.

8. *Lemma B.*—Through seven points in space it is, in general, possible to describe an infinite number of hyperboloids.

If the seven given points are 1, 2, 3, 4, 5, 6, 7, and if we take any two lines of the system 12, 34 as fixed axes, and draw two variable planes π , π' passing, respectively, through 12 and 34, in such a manner that the anharmonic ratio of the four planes 125, 126, 127, and π is equal to the anharmonic ratio of the four planes 345, 346, 347, and π' , the locus of the intersection of π and π' will be a hyperboloid passing through the seven given points (No. 4).

Next, let L be any line drawn through one of the points 1, and let us examine if it be possible to draw a line M through 2, so that the system of planes $L3$, $L4$, $L5$, $L6$, $L7$ may be homographic with the system $M3$, $M4$, $M5$, $M6$, $M7$. Using the notation of No. 5, we are to have

$$L(3456) = M(3456),$$

and also

$$L(3457) = M(3457).$$

The first equality shows, since $L(3456)$ is given, that $M(3456)$ is given, and, therefore, M is a generator of a known cone of the second degree, having also 23, 24, 25, 26 for generators (No. 4). The second equality shows that M is a generator of a known cone having 23, 24, 25, 27 for generators. Now two cones having a common vertex have four generators common, because their sections by any plane have four points common; and these two cones have three generators common; therefore M is their fourth common generator. And it is easy to determine (by a construction with the ruler alone) the fourth point of intersection of the two conics which are the sections of these two cones by the plane 345. We can thus determine M without difficulty; and it is evident (by No. 4) that the hyperboloid having L and M as generators, and passing through 3, 4, 5, will also pass through 6 and 7. In what follows we denote by $(ab, a'b')(c, c', x)$ the hyperboloid having the lines ab , $a'b'$ for generators of one system, and passing through the points c , c' , x .

9. *Lemma C.*—If a transversal intersect a system of quadrics

having a common curve of intersection, the pairs of points in which it meets those surfaces are in involution.

For, draw any plane through the transversal. This plane will meet the common curve of intersection in four points (Lemma A), and it will intersect the quadrics in a system of conics passing through those four points. Therefore, by Sturm's extension of Desargues' theorem (Chasles, *Traité des Sections Coniques*, Chap. XII., No. 302), the pairs of points in which the transversal meets the conics are in involution. But those points are the points in which the transversal meets the quadrics. Therefore the lemma is proved.

Cor. 1.—If a transversal intersect a hyperboloid, the three pairs of points, in which it cuts the surface and the opposite faces of any quadrilateral formed by four generators, are in involution.

For we may consider each pair of opposite faces as a quadric; and these two quadrics and the hyperboloid have a common intersection, namely, the four sides of the quadrilateral.

We may also prove this directly, as we proved the lemma, by drawing a plane π through the transversal. Thus, if L, M, L', M' be the four generators forming the quadrilateral, and if π meet those generators in a, b, c, d respectively, and the hyperboloid in a conic C which passes through a, b, c, d ; by Desargues' theorem, the three

pairs of points, in which the transversal meets the conic C and the opposite sides of the plane quadrilateral $abcd$, are in involution. Now the points in which the transversal meets C are the points in which it meets the hyperboloid; and the points in which it meets the opposite sides of the plane quadrilateral $abcd$ are the points in which it meets the opposite faces of the *gauche* quadrilateral $LMM'L'$; which proves the theorem.

We may also deduce this corollary directly from the anharmonic property of a hyperboloid (No. 4), just as Chasles deduces Desargues' theorem from the anharmonic property of conics (*Traité des Sections Coniques*, Chap. II., No. 20). If we conceive two planes turning round L and L' , in such a manner as to always intersect in a second generator of the surface, they will form on the transversal two homographic systems of points, whose double points e, f (real or

imaginary) will be the points of the hyperboloid situated on the transversal. Now, if the transversal cut the two pairs of opposite faces of the quadrilateral in l, l' ; m, m' respectively, l, m and m', l' are two pairs of corresponding points of the two homographic systems on the transversal. Therefore, the three pairs l, l' ; m, m' , and c, f are in involution (*Géométrie Supérieure*, 259, 1^{re} édition, 267, 2^{me} édition).

Cor. 2.—Conversely, if a transversal meet a hyperboloid in a point c , and the opposite faces of a quadrilateral formed by four generators in l, l' and m, m' ; and if we take f the conjugate of c in the involution determined by ll' and mm' , then f also lies on the hyperboloid.

Cor. 3.—Given a gauche hexagon $abca'b'c'$, and a transversal xy (meeting its faces, not its sides, as in a plane polygon), if $ab(cc'xy) = a'b'(cc'xy)$, then the points x, y and the two pairs of points in which the transversal meets the opposite faces $c'ab, ca'b'$ and $abc, a'b'c'$, are in involution.

Let xy meet the planes $abc, a'b'c', abc', a'b'c$ in $\beta, \beta', \alpha, \alpha'$, respectively. Then

$$(\beta\alpha xy) = (\alpha'\beta'xy).$$

Now an anharmonic ratio is not altered by interchanging two of the four points, lines, or planes, which determine it, provided the two other points, lines, or planes are also interchanged; and therefore

$$(\beta\alpha xy) = (\alpha\beta yx).$$

Consequently

$$(\alpha\beta yx) = (\alpha'\beta'xy),$$

and therefore the three pairs of points $x, y; \alpha, \alpha'; \beta, \beta'$ are in involution.

Otherwise, thus:—Draw M, M' through c, c' respectively, to meet

ab and $a'b'$. Then (Lemma B) a hyperboloid can be described, having $ab, a'b'$ as generators, and passing through c, c' , and x ; and, since

$$ab(cc'xy) = a'b'(cc'xy),$$

it will also pass through y . Therefore, by Cor. 1, the pair of points x, y , and the two pairs of points in which xy meets the opposite faces of the quadrilateral formed by $ab, M, a'b', M'$, are in involution. But the two last pairs of points are the points in which xy meets the opposite faces $c'ab, ca'b'$ and $abc, a'b'c'$ of the hexagon. The corollary is, therefore, proved.

Cor. 4.—Conversely, if xy and the two pairs of points, in which xy is cut by the opposite faces $c'ab, ca'b'$ and $abc, a'b'c'$ of the hexagon $abca'b'c'$, are in involution, then

$$ab(cc'xy) = a'b'(cc'xy).$$

This is seen to be true by reversing the steps of the first proof of Cor. 3; or it may be deduced from Cor. 2, as Cor. 3 was deduced from Cor. 1.

10. Lemma D.—Given a gauche hexagon $abca'b'c'$, and a transversal xy meeting its opposite faces in $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$; if

$$ab(cc'xy) = a'b'(cc'xy), \quad \text{and} \quad bc(aa'xy) = b'c'(aa'xy);$$

then

$$ca'(bb'xy) = c'a'(bb'xy).$$

By Lemma C, Cor. 3, $\alpha\alpha', \beta\beta', xy$ are three segments in involution; and $\beta\beta', \gamma\gamma', xy$ are three segments in involution; therefore $\gamma\gamma', \alpha\alpha', xy$ are three segments in involution; and therefore, by Lemma C, Cor. 4,

$$ca'(bb'xy) = c'a'(bb'xy).$$

Cor. 1.—Conversely, being given two of the three involutions just mentioned, the three equalities of anharmonic ratios hold good. For, from two of these involutions the third follows; and, consequently, by Lemma C, Cor. 4, the three equalities of anharmonic ratios.

Cor. 2.—Using the notation indicated at the end of No. 8, the three hyperboloids $(ab, a'b')(c, c', x)$, $(bc, b'c')(\alpha, \alpha', x)$, $(ca', c'a')(b, b', x)$ all pass through the seven points $a, b, c, \alpha, \alpha', b', c', x$; and, if y be any point on the intersection of two of them

$$(ab, a'b')(c, c', x) \quad \text{and} \quad (bc, b'c')(\alpha, \alpha', x),$$

we have the two equalities

$$ab(cc'xy) = a'b'(cc'xy) \quad \text{and} \quad bc(aa'xy) = b'c'(aa'xy);$$

therefore, by the lemma, we have also

$$ca'(bb'xy) = c'a(bb'xy),$$

and, consequently, y lies also on the third hyperboloid $(ca', c'a)(b, b', x)$. The three hyperboloids have, then, a common intersection, which (Lemma A) is a quartic curve.

11. *Lemma B.*—If two quadrics have a common generator, their intersection consists of this generator and a cubic curve, and the common generator is a chord of the cubic (No. 7).

It is a known theorem that a cubic lying on a hyperboloid meets the generators of one system once, and those of the other system twice (Salmon, *Geometry of Three Dimensions*, No. 303, pp. 242, 243, first edition; No. 366, p. 305, second edition). The present lemma follows easily, as a limiting case, from that theorem. As, however, the lemma is important, a proof is here given in full.

Let L be a generator common to two hyperboloids H_1, H_2 ; L_1, L_2 generators of the same system as L in H_1, H_2 respectively. Draw any plane π through L , and let π_1, π_2 be the planes through L_1 and L_2 corresponding to π in the homographic systems which determine H_1, H_2 respectively; so that M_1 , the intersection of π and π_1 , is a second generator of H_1 ; and M_2 , the intersection of π and π_2 , is a second generator of H_2 . Let M_1, M_2 meet L in l_1, l_2 , respectively.

Then M_1 and M_2 , being both in one plane π , intersect each other in a point p . Now the section of H_1 by π consists of the two lines L and M_1 , and the section of H_2 by π consists of the two lines L and M_2 . The point p and the points of the line L are, therefore, the only points common to H_1 and H_2 in that plane. And p does not, in general, lie on L ; therefore p lies on the cubic which forms part of the intersection of H_1 and H_2 . But π must meet the cubic in three points (one of those points being p), and those three points must be all common to H_1 and H_2 . Hence the two other points in which π meets the cubic must lie on L . The positions of those points can be found in this way: The planes π and π_1 correspond homographically, and the same is true of the planes π and π_2 (No. 4); therefore the planes π_1 and π_2 correspond homographically, and the points l_1 and l_2 form two homographic divisions on the line L . The points l_1 and l_2 coincide at the double points of these two divisions; and when they coincide p also coincides with them. Thus the

cubic cuts L at the double points of the two homographic divisions l_1 and l_2 .

12. *Lemma F*.—Given seven points in space, 1, 2, 3, 4, 5, 6, 7, the three hyperboloids $(12, 34)(5, 6, 7)$, $(34, 56)(1, 2, 7)$, $(56, 12)(3, 4, 7)$ have one, and only one, other point common (see end of No. 8 for explanation of the notation).

The enunciation assumes that the three hyperboloids can be constructed, that they have not a common curve of intersection, and that they do not all coincide. In general, the given points being taken arbitrarily, these conditions are fulfilled. The conditions would not be satisfied if five or more of the points lay in a plane. If five of the points are co-planar, every quadric through the seven points passes through the conic determined by the five points; and if six or seven points lie in a plane it may not be possible to describe a quadric through them. These particular cases are excluded, and also those cases in which three of the points lie in a straight line. If four of the points are co-planar, the hyperboloids may resolve into cones, or into pairs of planes. Thus, if 1, 2, 3, 4 are in a plane, the hyperboloid $(12, 34)(5, 6, 7)$ becomes a cone, while the two other hyperboloids $(34, 56)(1, 2, 7)$ and $(56, 12)(3, 4, 7)$ both coincide with the two planes 1234 and 567. It does not follow, however, that three hyperboloids cannot be described through the seven points, which will also pass through one eighth homologous point (Def. 1). Thus, if 1, 3, 5, 7 are in one plane, the construction given below can be performed, and the eighth homologous point exists. We now proceed to prove the lemma.

Let $7A$ be the generator through 7 of the first system in the hyperboloid $(34, 56)(1, 2, 7)$; and $7B, 7C$ be generators of the first system in the hyperboloids $(56, 12)(3, 4, 7)$ and $(12, 34)(5, 6, 7)$, respectively.

If 8 be a point common to the three hyperboloids, and if we draw through 8 the generators of the second system in the three surfaces, these second generators will meet the first generators already drawn through 7. Let A, B, C be the points of meeting. Then $8B, 8C$, being second generators in $(12, 56)(3, 4, 7)$ and $(12, 34)(5, 6, 7)$ respectively, must meet 12, which is a first generator in both hyperboloids, and thus 12 lies in the plane $(8B, 8C)$, and, consequently, BC intersects 12 in a point a . Similarly, CA intersects 34 in a point b , and AB intersects 56 in a point c . Hence a, b, c are the points where the known planes $(7B, 7C), (7C, 7A), (7A, 7B)$ are cut by the given lines

12, 34, 56; and a, b, c are points on the sides of the triangle ABC ;

therefore A, B, C are the points where the plane of the known triangle abc cuts the known lines $7A, 7B, 7C$. It follows that 8 is the intersection of three known planes (12, BC), (34, CA), and (56, AB). This is a perfectly definite construction, and the above reasoning shows that the three hyperboloids have only one other point common besides the seven given points.

This construction was given by Th. Reye, in 1887, in the *Journal für die Mathematik*, Vol. c., p. 489; see also *Jahrbuch über die Fortschritte der Mathematik*, Band xix., p. 637 (1887).

13. The point 8 is, by Def. 1, the eighth homologous point with respect to the seven given points and the three hyperboloids (12, 34)(5, 6, 7), (34, 56)(1, 2, 7), (56, 12)(3, 4, 7). It has not been proved that the other quadrics, which pass through the seven given points, pass also through 8. The existence of the eighth homologous point has only been proved for the three hyperboloids just mentioned.

§ II.

14. *Theorem I.*—The ten pairs of points in which a chord of a cubic meets the opposite planes of a system of six points lying on the curve, and the pair of points in which the chord meets the curve, all belong to one involution.

Let mn' be a chord of the cubic which passes through 1, 2, 3, 4, 5, 6. Complete the hexahedron $123456pp'$ as in the figure of No. 3, *supra*. Consider the hyperboloid through the lines pp' , 23, 56. The lines $p1$, $4p'$ are generators of the second system, and the four lines 23, $1p$, 56, $4p'$ are four generators which form a *gauche* quadrilateral. Therefore, by Lemma C, Cor. 1, the points where mn' meets the hyperboloid, the points where it meets the planes 123, 456, and the points where it meets the planes 234, 561, are three pairs of points in involution. Similarly, the points where mn' meets the hyperboloid through the lines pp' , 34, 61, the points where it meets the planes 234, 561, and the points where it meets the planes 345, 612, are three pairs of points in involution.

Now, the two hyperboloids have a common generator pp' ; therefore the rest of their intersection is a *gauche* cubic (No. 7). This cubic passes through the six points 1, 2, 3, 4, 5, 6, because they are common to both hyperboloids, and thus coincides with the given cubic, since only one cubic can be drawn through six points (foot-note, No. 7). Consequently, the points n , n' , which lie on the cubic (123456), also lie on both hyperboloids, and the two involutions just mentioned have a common segment mn' . They have, besides, another common segment, namely, the segment between the points where mn' cuts the opposite faces 234, 561. Having, then, two common segments, the two involutions are one and the same.

Therefore, the points n , n' , where the chord meets the cubic, and the three pairs of points in which it meets the opposite faces of the hexahedron $123456pp'$, are in involution. Or, we may state the result thus: 123456 is a hexagon inscribed in a cubic; if mn' be a chord of the cubic, then the points n , n' , and the three pairs of points in which mn' meets the opposite faces of the hexagon, belong to one involution.

Every hexagon that can be formed by taking 1, 2, 3, 4, 5, 6 in different orders, gives rise to an involution. Now, the nine hexagons, distinguished by Roman numerals in the table given below, have one pair of opposite faces common, namely, the pair 123, 456; therefore the involutions determined by those hexagons have one segment common. The segment mn' is also common to all those involutions. The nine involutions have, then, two common segments; and, consequently, are one and the same. Now there are ten different pairs of opposite planes in the table, and only ten pairs of opposite planes can be formed from six points. The table thus includes all the pairs of opposite planes that can be formed from the six points; and, there-

fore, it is proved that the points n, n' , and the ten pairs of points in which the chord nn' meets the opposite planes of the system 1, 2, 3, 4, 5, 6, belong to one and the same involution.

This is the theorem, analogous to that of Desargues, which we undertook to prove. The converse of this is given below (Theorem IV., No. 18).

Hexagon.	Pairs of opposite faces, or planes, distinguished from each other by numbers in brackets.
I. 123456	(1) 123, 456; (2) 234, 561; (3) 345, 612
II. 231456	(1) 231, 456; (4) 314, 562; (5) 145, 623
III. 312456	(1) 312, 456; (6) 124, 563; (7) 245, 631
IV. 123564	(1) 123, 564; (8) 235, 641; (6) 356, 412
V. 231564	(1) 231, 564; (9) 315, 642; (2) 156, 423
VI. 312564	(1) 312, 564; (10) 125, 643; (4) 256, 431
VII. 123645	(1) 123, 645; (5) 236, 451; (10) 364, 512
VIII. 231645	(1) 231, 645; (7) 316, 452; (8) 164, 523
IX. 312645	(1) 312, 645; (3) 126, 453; (9) 264, 531

15. If we refer to the construction of an octahedron in No. 3, we see that the above theorem can be stated thus:

If the corners of an octahedron rest on a cubic, the four pairs of opposite faces of the octahedron, and its six pairs of opposite diagonal planes, determine on any chord of the curve ten pairs of points belonging to the same involution; and to this involution also belong the points in which the chord meets the curve.

16. *Theorem II.*—If the points 7, 8 belong to the involution determined on a chord of the cubic through the points 1, 2, 3, 4, 5, 6, every line of the system of eight points (Def. 3) is a chord of the cubic determined by the six remaining points; and the points of the system which lie on every such line belong to the involution determined on it by the opposite planes with respect to those six remaining points. And if we take two lines of the system of eight points, the anharmonic ratio of the four planes passing through one of the lines and the four remaining points is equal to the anharmonic ratio of the four planes passing through the other line and those four remaining

points. The first part of the theorem may be stated more simply by means of symbols, thus :

If we express the fact, that 7, 8 belong to the involution determined on a chord of the cubic through 1, 2, 3, 4, 5, 6 by the opposite planes with respect to those points, by the symbolic equation

$$(123456) = (78),$$

then the numbers may be transposed in all possible ways; so that we may write, for example,

$$(712456) = (38).$$

First, any side 12 of any one of the hexagons determined by 1, 2, 3, 4, 5, 6, is a chord of the cubic (345678), drawn through 7, 8 and the remaining points of the hexagon. For 7, 8, the points

where the line 78 cuts 126, 345, and the points where 78 cuts 312, 456, are three pairs of points in involution. Therefore, by Lemma C, Cor. 4,

$$12 (3678) = 45 (3678).$$

Similarly,

$$12 (4578) = 36 (4578).$$

We can, consequently, describe a hyperboloid having 12, 45 as generators, and passing through 3, 6, 7, 8; and another hyperboloid having 12, 36 as generators, and passing through 4, 5, 7, 8. These two hyperboloids have a common generator 12, and the remainder of their intersection is the cubic (345678); therefore by Lemma E, 12 is a chord of this cubic.

Secondly, the points 1, 2 belong to the involution determined on the line 12 by the opposite planes with respect to 3, 4, 5, 6, 7, 8.

For 7, 8, the points where the line 78 cuts 342, 156, and the points where 78 cuts 134, 256, are three pairs of points in involution. Therefore, by Lemma C, Cor. 4,

$$34 (1278) = 56 (1278),$$

which may also be written

$$34 (7812) = 56 (7812).$$

Therefore, by Lemma C, Cor. 3, the points 1, 2 belong to the involu-

tion determined on the line 12 by the two pairs of planes 348, 567, and 347, 568, *i.e.*, to the involution determined on 12 by the opposite planes with respect to the six points 3, 4, 5, 6, 7, 8.

Thus, from the symbolic equation

$$(123456) = (78),$$

we deduce

$$(345678) = (12),$$

where the numbers which appear on the right-hand side of the second equation are both taken from among the six numbers which appear on the left-hand side of the first equation. Similarly, from the second equation, we deduce

$$(124568) = (37).$$

Here one number on the right-hand side (7) originally stood on the same side in the first equation; while the other number (3) originally stood on the left-hand side.

Hence it is clear that we may transpose the numbers in all possible ways. This proves the first part of the theorem. To prove the second part: Take any two lines of the system, 34 and 56. Then, by what has been just shown, 1, 8 belong to the involution determined on the line 18 by the two pairs of planes 347, 256 and 234, 567. Therefore, by Lemma C, Cor. 4,

$$34(2718) = 56(2718).$$

Similarly, the theorem can be proved for any other two lines of the system.

17. *Theorem III.*—If 8 be the eighth homologous point (Def. 1) with respect to the three hyperboloids $(12, 34)(5, 6, 7)$, $(34, 56)(1, 2, 7)$, $(56, 12)(3, 4, 7)$ (see Lemma F), then 78 is a chord of the cubic (123456) , and the points 7, 8 belong to the involution determined on the line 78 by the opposite planes with respect to 1, 2, 3, 4, 5, 6.

For, by supposition, 12 is a generator common to two of the hyperboloids. Therefore, by Lemma E, 12 is a chord of the cubic which forms part of the intersection of those two hyperboloids. Now, both hyperboloids pass through 3, 4, 5, 6, 7, 8; therefore the cubic passes through those points; and thus 12 is a chord of the cubic (345678) .

Again, by the anharmonic property of hyperboloids,

$$34(7812) = 56(7812);$$

therefore, by Lemma C, Cor. 3, the points 1, 2 belong to the involution determined on the line 12 by the pairs of planes 348, 756 and 347, 568, *i.e.*, by the opposite planes with respect to the system of points 3, 4, 5, 6, 7, 8 (Theorem I.).

Thus we have the symbolic equation

$$(345678) = (12) \text{ (see No. 16) ;}$$

and therefore

$$(123456) = (78),$$

or the line 78 is a chord of the cubic (123456), and the points 7, 8 belong to the involution determined on the line 78 by the opposite planes with respect to 1, 2, 3, 4, 5, 6.

Cor.—If 7 lie on one of the planes determined by three of the points 1, 2, 3, 4, 5, 6, then 8 lies on the opposite plane.

18. *Theorem IV.*—If a transversal cut the opposite faces of an octahedron in four pairs of points belonging to one involution, then the transversal is a chord of the cubic determined by the six corners of the octahedron.

This is the converse of Theorem I., and is the second property of *gauche cubics* which we undertook to prove (see No. 1).

Let the transversal meet the planes 612, 345; 123, 456; 234, 561; 135, 246 (see figure No. 3), in the points $a, a'; b, b'; c, c'; d, d'$, respectively. Then, because dd', bb', cc' are in involution, we have, by Lemma C, Cor. 4,

$$23(14dd') = 56(14dd').$$

Similarly, because dd', cc', aa' are in involution,

$$34(25dd') = 61(25dd');$$

and, because dd', aa', bb' are in involution, we have

$$45(36dd') = 12(36dd').$$

Hence, using the notation given at the end of No. 8, the three hyperboloids $(23, 56)(1, 4, d)$, $(34, 61)(2, 5, d)$, $(45, 12)(3, 6, d)$, which all pass through the seven points 1, 2, 3, 4, 5, 6, and d , and have a common intersection (Lemma D, Cor. 2), all pass through d' . Now this common intersection is a quartic curve, which is, therefore, met by

the plane 246 in four, and only four, points. The four points are, clearly, 2, 4, 6, and d' , and there cannot be any other point in the plane 246 common to the three hyperboloids.

Again, if 7 be any point in space, and if we describe the three hyperboloids (12, 34)(5, 6, 7), (34, 56)(1, 2, 7), (56, 12)(3, 4, 7) as in Lemma F, and determine 8, the eighth homologous point with respect to these three hyperboloids; then, by Theorem III., 78 is a chord of the cubic (123456), and the points 7, 8 belong to the involution determined on the line 78 by the pairs of opposite planes with respect to 1, 2, 3, 4, 5, 6. Consequently, by Theorem II., we have

$$23(1478) = 56(1478), \quad 34(2578) = 61(2578), \quad 45(3678) = 12(3678).$$

This is still true if 7 lie in the plane 135. It follows, if 7 coincide with d , that the point 8 lies on the three hyperboloids (23, 56)(1, 4, d), (34, 61)(2, 5, d), (45, 12)(3, 6, d). But in that case, by the Corollary to Theorem III., 8 lies in the plane 246, and must, by what was shown above, be one of the four points 2, 4, 6, or d' . Hence, as 8 is, in general, distinct from 2, 4, or 6, it will coincide with d' .

It follows, by Theorem III., that dd' is a chord of the cubic (123456).*

* [6th October, 1893. If we assume the proposition stated at the beginning of this paper (No. 1), we can give another proof of Theorem IV. Using the notation of No. 18, we have the involution aa', bb', cc', dd' . Let 7 be any point on the transversal and 8 its conjugate in this involution. Then from the involution 78, dd', aa' , we deduce

$$35(1478) = 26(1478).$$

We got, similarly, $13(2578) = 64(2578)$,

and $61(3678) = 42(3678)$.

Hence the three hyperboloids (35, 26)(1, 4, 7), (13, 64)(2, 5, 7), and (61, 42)(3, 6, 7), which pass through 1, 2, 3, 4, 5, 6, 7, will all pass through 8. Those surfaces, in general, do not pass through a common curve of intersection; therefore they have no other points common, since three quadrics have, in general, eight points common, and no more. But, by the proposition assumed, a system of quadrics passing through seven points will pass also through an eighth point. Therefore the three hyperboloids (12, 34)(5, 6, 7), (34, 56)(1, 2, 7), (56, 12)(3, 4, 7), which pass through 1, 2, 3, 4, 5, 6, 7, pass also through 8. Therefore, by Def. 1, and Lemma F, 8 is the eighth homologous point with respect to the three last hyperboloids. It follows, by Theorem III., that the line 78 is a chord of the cubic through 1, 2, 3, 4, 5, 6.]