LXVI. On pluquaternions, and homoid products of sums of n squares

Rev. Thomas P. Kirkman A.B.

To cite this article: Rev. Thomas P. Kirkman A.B. (1848) LXVI. On pluquaternions, and homoid products of sums of n squares , Philosophical Magazine Series 3, 33:224, 447-459, DOI: [10.1080/14786444808646141](https://doi.org/10.1080/14786444808646141)

To link to this article: <http://dx.doi.org/10.1080/14786444808646141>

Published online: 30 Apr 2009.

Submit your article to this journal [↗](#)

Article views: 2

View related articles [↗](#)

Citing articles: 2 View citing articles [↗](#)

LXVI. *On Pluquaternions, and Homoid Products of Sums of n Squares.* By the Rev. THOMAS P. KIRKMAN, A.B., Rector of Croft with Southworth, Lancashire*.

THE following analysis is the fruit of my meditations on Professor Sir W. R. Hamilton's elegant theory of quaternions, and on a pregnant hint kindly communicated to me, without proof, by Arthur Cayley, Esq., Fellow of Trinity College, Cambridge, about the connexion between a system of triplets having no duad in common, and the property that the product of two sums of $2n$ squares shall be a sum of $2n$ squares. In a note with which the latter gentleman recently favoured me, he writes, "The complete test of the possibility of the product of 2^n squares by 2^n squares reducing itself to a sum of 2^n squares, is the following:—Forming the complete system of triplets for (2^n-1) things, if $eab\ ecd\ fac\ fdb$ be any four of them, we must have, paying attention to the signs only,

$$(\pm eab)(\pm ecd) = (\pm fac)(\pm fdb);$$

that is, if the first two are of the same sign, the last two must be so also, and *vice versa*. I believe that, for a system of seven, two conditions of this kind, being satisfied, would imply the satisfaction of all the others. It remains to be shown, that the complete system of conditions cannot be satisfied for a system of fifteen things."

In the following investigation, the truth and completeness of Mr. Cayley's test of the property in question for sums of 2^n squares are established, and the contradiction anticipated by him for a system of fifteen is deduced; while ulterior results about homoid products of sums of n squares are obtained.

Let $a_o b_o c_o \dots r_o$ be $2n-1$ imaginary units, among which no linear relation exists, and let

$$a_o^2 = b_o^2 = \dots = r_o^2 = -1.$$

Let

$$Q_a = w + a_o a + b_o b + \dots + r_o r,$$

$$Q_{a_i} = w_i + a_o a_i + b_o b_i + \dots + r_o r_i,$$

$ww, aa, \dots rr$, being any real quantities: it is proposed to consider under what conditions it is possible that the product of two such functions shall be of the same form. Suppose, then, that, if possible,

$$Q_a Q_{a_i} = Q_{a_{ii}} = w_{ii} + a_o a_{ii} + b_o b_{ii} + \dots + r_o r_{ii},$$

* Communicated by the Author.

where $w, a, \dots, r$ are certain functions of the arbitrary numbers $ww, aa, \dots, rr$.

In this last equation, it is plainly implied, that every duad, $a_o b_o$, that can be made with two imaginaries, is a linear function of those units; *i. e.* we must have

$$a_o b_o = Aa_o + Bb_o + Cc_o + \dots + Rr_o;$$

whence, since $a_o^2 = -1 = b_o^2$,

$$-b_o = -A + Ba_o b_o + Ca_o c_o + \dots + Ra_o r_o,$$

$$-a_o = Aa_o b_o - B + Cc_o b_o + \dots + Rr_o b_o.$$

Therefore $A=0=B$; and

$$-b_o^2 = 1 = Cb_o a_o c_o + Db_o a_o d_o + Eb_o a_o e_o + \dots + Rb_o a_o r_o.$$

As b_o, a_o, c_o , &c. are pure imaginaries, we cannot suppose that any term of the last expression, as $Cb_o a_o c_o$, is of the form $V+W$, where V is real and W imaginary. Some one or more, then, of these terms must be real; say the first three of them. We have then

$$Cb_o a_o c_o = C',$$

$$Db_o a_o d_o = D',$$

$$Eb_o a_o e_o = 1 - C' - D'.$$

It follows that

$$\frac{C}{C'} c_o = \frac{D}{D'} d_o,$$

which is impossible by hypothesis. There can then be only one of the terms real in the value of $-b_o^2$, and the equation before us is of the form

$$-b_o^2 = 1 = Cb_o a_o c_o,$$

or assuming $C = \pm 1$,

$$1 = \pm b_o a_o c_o;$$

which gives

$$b_o = \mp a_o c_o, \quad a_o b_o = \pm c_o,$$

as the form of the linear function in question, equivalent to any duad $a_o b_o$ or $a_o c_o$.

One condition, therefore, of the possibility of the equation

$$Q_a Q_{a_i} = Q_{a_{ii}}$$

is, that all the imaginaries can be combined into triplets, of which no two shall have a duad in common; since if $b_o a_o c_o$ and $d_o a_o c_o$ were both real, there would exist a linear relation between b_o and d_o , contrary to hypothesis. The number $(2n-1)$ of the imaginaries is hereby restricted to the two forms $6m+1$ and $6m+3$, in the case of either of which the triplets can be formed

so as to have no duad in common, and that generally in various ways (vide Camb. and Dub. Math. Journ., N.S. vol. ii. p. 191), upon the comparison of which it is not necessary here to enter, although the restriction might thus be further narrowed.

When $n=2$, or when there are only three imaginaries, Q_a is the quaternion invented by Sir W. R. Hamilton, and profoundly discussed by him in the twenty-fifth volume of the Philosophical Magazine. When $n=4$, it is the biquaternion or octad, of which Professor John T. Graves has made mention at the end of a paper On the Connexion between the General Theory of Normal Couples, and the Theory of Quadratic Functions of two Variables, *ibid.* vol. xxvi. As it will be useful to have a name for Q_a in cases where $n > 4$, I venture to propose the appellation Pluquaternion, as generally applicable to Q_a when there are more than three imaginaries; not dreading here the pluperfect criticism of grammarians, since the convenient barbarism is their own.

Further, if c_o, a_o, b_o be any three imaginaries forming one of the triplets, $c_o a_o b_o$ cannot differ in sign from $a_o b_o c_o$; for if it does differ,

$$c_o a_o b_o = -a_o b_o c_o = +b_o c_o a_o = -c_o a_o b_o; \text{ Q.E.A.} \\ \therefore c_o a_o b_o = a_o b_o c_o = b_o c_o a_o = \mp 1;$$

which is the property of every triplet of the system.

From these we deduce

$$a_o b_o = \pm c_o, \\ c_o a_o = \pm b_o, \\ b_o c_o = \pm a_o;$$

whence also

$$a_o b_o c_o a_o = c_o b_o = a_o a_o b_o c_o = -b_o c_o, \\ c_o a_o b_o c_o = b_o a_o = c_o c_o a_o b_o = -a_o b_o, \\ b_o c_o a_o b_o = a_o c_o = b_o b_o c_o a_o = -c_o a_o;$$

for, because $b_o c_o a_o = a_o b_o c_o$ is real,

$$a_o b_o \cdot c_o a_o = a_o \cdot b_o \cdot c_o a_o = a_o \cdot b_o c_o a_o = a_o \cdot a_o b_o c_o = a_o \cdot a_o \cdot b_o c_o :$$

and so the other two of these three assertions may be established likewise. Wherefore

$$m_o n_o + n_o m_o = 0$$

is true of every duad $m_o n_o$ that can be made with the $(2n-1)$ imaginaries in the equation $Q_a Q_{a_i} = Q_{a_{ii}}$, by virtue of our definitions.

Let $Q_a Q_{a_i}$ and $Q_{a_{ii}}$ be biquaternions, that is, let there be seven imaginaries $a_o b_o c_o d_o e_o f_o g_o$. The number of triplets

that can be formed with seven things, so as to have no duad in common, is seven. Of these, there will be three in which a_o is found; let the three be made all of one sign, by the inversion of a duad if necessary, a change which can alter no value, since $-a_o b_o c_o = +a_o c_o b_o$. Neglecting subindices, let the three triplets be abc, ade, afg . We shall consider the order $abcdefg$ to be alphabetical, and shall inquire what the remaining four triplets will be, when written out in the same order. Because

$$abc = ade = \mp 1;$$

$$bc = de,$$

and

$$-c = b.de.$$

Now $b.de$, whether real or not, if b, d, e be any three imaginaries, cannot differ in sign from $e.db$; for if it does,

$$b.de = -e.bd = d.eb = -b.de, \text{ Q.E.A.}$$

Therefore

$$-c = b.de = e.bd,$$

$$-ec = -bd,$$

or

$$bd = ec;$$

and for the same reasons,

$$be = cd.$$

Hence, in each of the following lines, any two equations are implied in the third:—

$$\left. \begin{array}{lll} bc = de, & be = cd, & bd = -ce; \\ bc = fg, & bg = cf, & bf = -cg; \\ de = fg, & dg = ef, & df = -eg. \end{array} \right\} . . . \text{ (A.)}$$

It is necessary that three similar lines deduced from any three triplets of the system, beginning with another imaginary than a , should be consistent with these. Now it is readily seen, that the system of seven triplets, of which three are abc, ade, afg , must perforce contain either bdf or ceg ; and the reader will convince himself with less pains than the perusal of a demonstration would cost him, that the above conditions can, and thus only can, be fulfilled, by giving to that triplet of these two, which is found in the system of seven, a sign contrary to that of the remaining three. Hence all possible arrangements, if abc, ade, afg are in the system, are comprised in the two following types, each of four systems:—

$$\pm \left\{ \begin{array}{l} +abc \\ +ade \\ +afg \end{array} \right\} \pm \left\{ \begin{array}{l} -bdf + cdg \\ +beg + cef \end{array} \right\}; \quad \pm \left\{ \begin{array}{l} +abc \\ +ade \\ +afg \end{array} \right\} \pm \left\{ \begin{array}{l} +bdg + cdf \\ +bef - ceg \end{array} \right\};$$

of which eight systems any one will satisfy the test invented by Mr. Cayley, of which mention has been made, and which is proved by the preceding considerations to be both necessary and complete for a system of seven triplets.

The congruity of any of these, *e. g.* of those in the first type, may be tested by taking some other order for alphabetical; for instance,

$$\left\{ \begin{array}{l} +fga \\ +fce \\ +fdb \end{array} \right\} \pm \left\{ \begin{array}{l} -gcd + acb \\ +geb + aed \end{array} \right\},$$

which is still of the same type as before.

One of these eight systems, being adopted, will furnish the non-linear conditions among the seven imaginaries, by the aid of which the product of two biquaternions is reduced to a biquaternion. The system of the first type, having the two upper signs, for example, expresses that

$$a_o = b_o c_o = d_o e_o = f_o g_o,$$

and

$$d_o = e_o a_o = -f_o b_o = g_o c_o.$$

Hence in the biquaternion $Q_{a_{ii}}$, formed by this system,

$$a_{ii} = aw_i + wa_i + (bc_i - cb_i) + (de_i - ed_i) + (fg_i - gf_i),$$

and

$$d_{ii} = dw_i + wd_i + (ea_i - ae_i) - (fb_i - bf_i) + (gc_i - cg_i);$$

for these are exactly the terms affected by a_o and d_o in the product $Q_a Q_{a_i}$. But if we adopt the system of the second type which has the two lower signs, we should form the same functions thus,

$$a_{ii} = aw_i + wa_i - (bc_i - cb_i) - (de_i - ed_i) - (fg_i - gf_i)$$

$$d_{ii} = dw_i + wd_i - (ea_i - ae_i) - (gb_i - bg_i) - (fc_i - cf_i);$$

where the given numbers $ww_i, aa_i, \dots, gg_i$ are in both cases the same.

It thus appears, that a biquaternion $Q_{a_{ii}}$ may be formed, that shall be equal to the product of any two Q_a and Q_{a_i} , in at least eight different ways; and the eight different values of the product shall be equally congruous and consistent with our definitions; the real quantities in Q_a and Q_{a_i} being any numbers, and remaining unchanged in value, sign, or order.

Let Q_a and Q_{a_i} now be pluquaternions of more than seven imaginaries. If $Q_a Q_{a_i} = Q_{a_{ii}}$, there must be some complete system of triplets into which the $(6m+1)$ or $(6m+3)$ imaginaries are thrown. Let $a_o d_o e_o$ and $a_o f_o g_o$, both of one sign,

be any two of these; then, neglecting subindices, $\pm dg$ will be equal to some sixth, and $\pm eg$ to some seventh imaginary, say

$$\pm dg = c, \quad \pm eg = b.$$

It follows that

$$cd = \pm be$$

$$cb = \pm ed = \mp a.$$

If then cdg and beg are of the same sign, abc is a triplet in the pluquaternion system, of the same sign with ade and afg ; and since by equations (A.), which here apply,

$$dg = ef \text{ and } eg = fd,$$

it is plain that the imaginaries $abcdefg$ form some one of eight complete systems of seven triplets. If cdg is of a different sign from beg , acb is a triplet in the system, of the same sign with ade and afg ; so that, if we put b' for c , and c' for b , we know that $ab'c'defg$ form one of the same eight systems.

Let it be either of these two,

$$\begin{aligned} &+abc \\ &+ade \pm \left\{ \begin{array}{l} -bdf + cdg \\ +beg + cef \end{array} \right\}; \\ &+afg \end{aligned}$$

and let $+ahi$ be any fourth triplet in the system beginning with a . Then the following are proved to coexist, like the equations (A.), being deduced from the equal values of a :

$$\left. \begin{array}{lll} bc = hi, & bi = ch, & bh = -ci; \\ de = hi, & di = eh, & dh = -ei, \\ fg = hi, & fi = gh, & fh = -gi. \end{array} \right\} . . . \quad (B.)$$

Let now k be the imaginary equivalent to bi in the system. We have then, since $bi = ch$,

$$\mp bdf = bik,$$

$$\mp df = ik,$$

$$\mp dk = fi = gh;$$

and

$$\pm cdg = chk,$$

$$\pm dg = hk,$$

$$\pm dk = gh;$$

or

$$dk = -dk; \text{ Q.E.A.}$$

If the system of seven triplets be of the other type, or one of these,

$$\begin{aligned} &+abc \\ &+ade \pm \left\{ \begin{array}{l} +bdg + cdf \\ +bef - ceg \end{array} \right\}, \\ &+afg \end{aligned}$$

we have

$$\begin{aligned} \pm bef &= bik, \\ \pm ek &= fi = gh; \end{aligned}$$

and

$$\begin{aligned} \mp ceg &= chh, \\ \mp ek &= gh; \end{aligned}$$

or

$$ek = -ek; \text{ Q.E.A.}$$

It is therefore impossible that $b_o i_o$ or $c_o h_o$ should be equal to k_o , or to any other ninth imaginary; and the same contradiction is involved in the supposition that $c_o i_o$, $d_o i_o$, $e_o i_o$, $f_o i_o$ or $g_o i_o$, or their equivalent duads in (B.), are any one of them equal to a single imaginary. Each of these must therefore of necessity appear, by itself or by its equivalent duad, in the product of the pluquaternions Q_a and Q_{a_i} . Let these, *e. g.* be pluquaternions of nine imaginaries; then their product must be of the form

$$\begin{aligned} & (w + a_o a + b_o b + c_o c + f_o f + g_o g + h_o h + i_o i) \\ & \times (w_i + a_o a_i + b_o b_i + c_o c_i + d_o d_i + e_o e_i + f_o f_i + g_o g_i + h_o h_i + i_o i_i) \\ & = w_{ii} + a_o a_{ii} + b_o b_{ii} + c_o c_{ii} + d_o d_{ii} + e_o e_{ii} + f_o f_{ii} + g_o g_{ii} + h_o h_{ii} + i_o i_{ii} \\ & \quad + b_o i_o B_i + c_o i_o C_i + d_o i_o D_i + e_o i_o E_i + f_o i_o F_i + g_o i_o G_i; \end{aligned}$$

i. e. $Q_a Q_{a_i} = Q_{a_{ii}} + R_1$, or the product is a pluquaternion of $(2 \cdot 1 + 7)$ imaginaries, and an additional function of imaginaries of a different form, which, for distinction's sake, we may call the condition-function R_1 , the number of its terms being the number of conditions to be satisfied among the real quantities of Q_a and Q_{a_i} , in order that the product $Q_a Q_{a_i}$ may be a pluquaternion. In the case of nine imaginaries, there are six of these conditions, viz. $R_1 = 0$, imposed upon the sixteen numbers $bb_i cc_i \dots ii_i$. The terms h_{ii} and i_{ii} of the pluquaternion $Q_{a_{ii}}$, are formed with the aid of the triplet $a_o h_o i_o$; and B_i , C_i , &c. are given by the equations (B.), thus:

$$\begin{aligned} B_i &= bi_i - ib_i + ch_i - hc_i, & C_i &= ci_i - ic_i + hb_i - bh_i, \\ h_{ii} &= hw_i + wh_i + (ia_i - ai_i), & i_{ii} &= iw_i + wi_i - (ah_i - ha_i). \end{aligned}$$

Since $+a_o b_o = c_o$, and $+a_o h_o = i_o$, $c_o h_o = -i_o b_o$ gives

$$\left. \begin{aligned} a_o b_o h_o &= -a_o h_o b_o, \\ b_o h_o &= -h_o b_o; \\ c_o i_o &= -i_o c_o; \end{aligned} \right\} \dots \dots (C.)$$

and the like property can be shown to belong to all the duads

containing h_o or i_o , and one of the seven imaginaries $a_o b_o c_o d_o e_o f_o g_o$, none of which duads are in the system of seven triplets.

From what has preceded, we conclude that it is impossible for pluquaternions of more than seven imaginaries to have a product of the same form, if there are no limitations upon the values of the real quantities, $ww_1, aa_1 \dots rr_1$.

If we now suppose that there are eleven imaginaries, we can form a fifth triplet beginning with a_o , viz. $a_o k_o l_o$; and it can be proved, as in the case of nine imaginaries, that $b_o l_o, c_o l_o, d_o l_o, e_o l_o, f_o l_o, g_o l_o$, have each of them an equivalent duad, but that they cannot any of them be put equal to any single imaginary, without a contradiction. The equal values of $a_o, h_o i_o$ and $k_o l_o$, give also

$$h_o i_o = k_o l_o, \quad h_o l_o = i_o k_o, \quad h_o k_o = -i_o l_o; \quad . \quad (D.)$$

of which neither $h_o l_o$ nor $h_o k_o$ can be equal to any of the seven $a_o b_o c_o d_o e_o f_o g_o$; for this equality is what we have just shown to be impossible; nor can $h_o l_o$ be equal to either i_o or k_o , nor $h_o k_o$ to either i_o or l_o . Wherefore the product of two pluquaternions of eleven ($=2 \cdot 2 + 7$) imaginaries must of necessity be of the form

$$Q_a Q_{a_1} = Q_{a_{11}} + R_2,$$

where the condition-function R_2 contains, besides the six terms of R_1 , the six following,

$$+ b_o l_o B_l + c_o l_o C_l + d_o l_o D_l + e_o l_o E_l + f_o l_o F_l + g_o l_o G_l,$$

with the additional pair of terms $+ h_o l_o H_l + i_o l_o I_l$.

The six B_l , &c. differ from the six B_i , &c. only in this, that for the real quantity h or i in the latter, is substituted k or l in the former; and, in the same way, for h_l or i_l is put k_l or l_l . H_l and I_l are formed from equations (D.), and k_{11} and l_{11} the two additional terms in $Q_{a_{11}}$, with the aid of the triplet $a_o k_o l_o$.

It can also be shown, as in (C.), that the additional duads in R_2 , $b_o l_o$, &c., have the property that $b_o l_o + l_o b_o = 0$.

Suppose next that we have thirteen ($=2 \cdot 3 + 7$) imaginaries. There is now nothing to hinder us from forming another system of seven triplets, with $ahiklmn$, of the same type or not as the former. We may, *e. g.* have the fourteen triplets following:

$$\begin{array}{ll} +abc & +ahi \\ +ade - bdf + cdg & +ahl - hkm + ikn \\ +afg + beg + cef; & +amn + hln + ilm; \end{array}$$

for, although we have proved that no one of the seven $ab \dots g$ can form a tenth triplet with any two of $hikl$, such a triplet *can*

be made with two of these and *any twelfth* imaginary n , or m . In the case of thirteen imaginaries, then, we have the product

$$Q_a Q_{a'} = Q_{a_{ii}} + R_3,$$

where R_3 contains, besides the six $b_o i_o B_i \dots g_o i_o G_i$, and the six $b_o l_o B_l \dots g_o l_o G_l$, another six, $b_o n_o B_n \dots g_o n_o G_n$; but it contains not the pair $h_o l_o H_l$ and $i_o l_o I_l$, because the terms (H_l) are now part of the function n_{ii} , and the terms (I_l) of m_{ii} , by virtue of our new triplets. It is hardly necessary to observe, that the six duad equivalents of $b_o n_o \dots g_o n_o$ are deduced from equating $m_o n_o$ to the values of a_o in the first system of seven, as in equations (B.). The reasoning (C.) applies here as before.

If there are fifteen ($=2 \cdot 4 + 7$) imaginaries $ab \dots nop$, R_4 , in the product

$$Q_a Q_{a'} = Q_{a_{ii}} + R_4,$$

will contain, supposing that we add the triplet aop to the preceding fourteen, the eighteen terms of R_3 , and the two additional sixes, $b_o p_o B_p \dots g_o p_o G_p$, and $h_o p_o H_p$, $i_o p_o I_p \dots n_o p_o N_p$.

In the product of two pluquaternions of seventeen, R_5 has the thirty terms of R_4 , and (aqr being the sixteenth triplet) the two sixes $b_o r_o B_r \dots g_o r_o G_r$, and $h_o r_o H_r \dots n_o r_o N_r$, besides the additional pair $o_o r_o O_r$ and $p_o r_o P_r$. And in the product

$$Q_a Q_{a'} = Q_{a_{ii}} + R_6$$

of two pluquaternions of nineteen, the condition-function R_6 contains, besides the preceding seven sixes, the two additional sixes $b_o t_o B_t \dots g_o t_o G_t$, and $h_o t_o H_t \dots n_o t_o N_t$; but not the pair $o_o r_o O_r$ and $p_o r_o P_r$, as the terms (O_r) are now part of the function t_{ii} , and (P_r) of s_{ii} ; the twenty quantities $w_{ii} a_{ii} \dots s_{ii} t_{ii}$ being formed by the twenty-one triplets,

$$\begin{aligned} &+abc \quad \quad \quad +ahi \quad \quad \quad +aop \\ &+ade - bdf + cdg \quad +akl - hkm + ikn \quad +aqr - oqs + pqt \\ &+afg + beg + cef; \quad +amn + hln + iln; \quad +ast + ort + prs; \end{aligned}$$

if we suppose that the three systems are identical in form. But this is not in anywise necessary, for any one of them, *e.g.* the third, may be any one of eight systems containing the three triplets $a_o o_o p_o$, $a_o q_o r_o$, $a_o s_o t_o$, all of one sign: that is, the function a_{ii} , indicated by the imaginary common to all the twenty-one triplets, may have any one of the eight values,

$$\begin{aligned} a_{ii} = & aw_i + wa_i \pm (bc_i - cb_i) \pm (de_i - ed_i) \pm (fg_i - gf_i) \\ & \pm \{hi_i - ih_i\} \pm \{kl_i - lk_i\} \pm \{mn_i - nm_i\} \\ & \pm [op_i - po_i] \pm [qr_i - rq_i] \pm [st_i - ts_i]; \end{aligned}$$

where the similar brackets have all the same sign. Any other function, as q_{ii} , will have the form, if we take the *first* of our two types,

$$q_{ii} = qw_i + wq_i \pm [ra_i - ar_i] \pm \{\overline{so_i - os_i} - \overline{tp_i - pt_i}\};$$

but *which* of these *four* shall be q_{ii} , can only be determined when the system of seven triplets, to which q_o belongs, shall have been exactly defined. We may thus have any number of complete systems of seven, having *one* common imaginary.

It is obvious that the reasoning (C.) applies to every duad coefficient in R_4, R_5, R_6 ; and it is easily shown to be true that the property

$$m_o n_o + n_o m_o = 0$$

belongs to every pair $m_o n_o$ in the product of any two pluquaternions, whether that product be a pluquaternion or not, provided that that product be formed consistently with a system of congruous triplets. And all this is the inevitable consequence of our first definitions of the imaginaries, and of the supposition that Q_a and Q_{a_i} have, *if possible*, a product of the same form.

The preceding analysis shows that

$$R_1 = 6 \cdot 1$$

$$R_2 = 6 \cdot 2 + 2$$

$$R_3 = 6 \cdot 3$$

$$R_4 = 6 \cdot 4 + 6 \cdot 1$$

$$R_5 = 6 \cdot 5 + 6 \cdot 2 + 2$$

$$R_6 = 6 \cdot 6 + 6 \cdot 3$$

$$R_7 = 6 \cdot 7 + 6 \cdot 4 + 6 \cdot 1$$

$$R_8 = 6 \cdot 8 + 6 \cdot 5 + 6 \cdot 2 + 2,$$

$$\&c. \quad \&c.,$$

the law being obvious.

Generally, if we form the product

$$Q_a Q_{a_i} = Q_{a_{ii}} + R_m$$

of two pluquaternions of $(2m+7)$ imaginaries, the number of terms in the condition-function R_m , if $m = 3k + h$ (k not < 0 ; h not < 0 , not > 2), is always $[(3k)^2 + 3k \cdot (2h+3) + h \cdot (h+5)]$.

Since each of the imaginary duads in R_m must of necessity appear, either by itself or by its equivalent, in the product of Q_a and Q_{a_i} , it is impossible that this product should contain fewer than $[(3k)^2 + 3k \cdot (2h+5) + h(h+7) + 8]$ terms; and if

R'_m be any function exhibiting a less number of imaginaries than R_m , the equation

$$Q_a Q_{a_i} = Q_{a_{ii}} + R'_m$$

is a contradiction, if $ww, aa, \dots rr$, are subjected to no limitations.

We shall next inquire how many possible forms there are of the product of two pluquaternions Q_a and Q_{a_i} , or how many different sets, $w_{ii} a_{ii} b_{ii}$ &c., functions of the $4n$ given positive numbers $ww, aa, \dots rr$, $Q_{a_{ii}}$ and the condition-function R can be made, consistently with our definitions, to exhibit. Here it may be useful to remark, that w_{ii} is invariable, and that the two first terms, which contain w and w_i , of any one of $a_{ii} b_{ii} \dots r_{ii}$, are also invariable, through all the changes of form of the pluquaternion $Q_{a_{ii}}$, so long as the values and signs of the given real numbers are unaltered.

We take first $Q_{a_{ii}}$, the product of two biquaternions Q_a and Q_{a_i} . Considering the set (bc, de, fg) as one with (bc, fg, de) and with (cb, ed, gf) , but distinct from (cb, de, fg) and from (bd, ce, fg) , we can form sixty distinct sets of three duads with the six imaginaries $b_o c_o d_o e_o f_o g_o$. Prefixing a_o to the three duads of any set $(c_o b_o d_o e_o f_o g_o)$, the resulting triplets, $a_o c_o b_o, a_o d_o e_o, a_o f_o g_o$, can be completed in eight different ways into a complete system of seven. For example, the three just written, considered as all of one sign, give for a_{ii} the two following values, taking all the upper or all the lower signs together:

$$a_{ii} = aw_i + wa_i \pm (cb_i - bc_i) \pm (de_i - ed_i) \pm (fg_i - gf_i);$$

to either of which values can be appended four distinct sets, $b_{ii} c_{ii} d_{ii} e_{ii} f_{ii} g_{ii}$, two of either type.

Therefore the biquaternion $Q_{a_{ii}}$, the product of two given ones, Q_a and Q_{a_i} , has 480 distinct forms, each exhibiting a different set of eight numbers, $w_{ii} a_{ii} b_{ii} c_{ii} d_{ii} e_{ii} f_{ii} g_{ii}$.

Let it now be supposed that two pluquaternions of nine, Q_a and Q_{a_i} , are to be multiplied together. The product $Q_{a_{ii}} + R_1$ is determined completely, as soon as an eighth triplet is added to a given complete system of seven.

Thus, if we add to any one of the eight systems which give either of the last written values of a_{ii} , the triplet $a_o h_o i_o$ of the same sign with $a_o c_o b_o$, we obtain for $a_{ii} h_{ii} i_{ii}$ the two sets of values,

$$a_{ii} = aw_i + wa_i \pm (cb_i - bc_i) \pm (de_i - ed_i) \pm (fg_i - gh_i) \pm (hi_i - ih_i),$$

$$h_{ii} = hw_i + wh_i \pm (ia_i - ai_i),$$

$$i_{ii} = iw_i + wi_i \pm (ah_i - ha_i);$$

and the function R_1 , the same for either set, is deduced from the equality of the four values of a_o . If the eighth triplet is $a_o i_o h_o$, we obtain the two sets

$$a_{ii} = aw_i + wa_i \pm (cb_i - bc_i) \pm (de_i - ed_i) \pm (fg_i - gf_i) \pm (ih_i - hi_i),$$

$$h_{ii} = hw_i + wh_i \pm (ai_i - ia_i),$$

$$i_{ii} = iw_i + wi_i \pm (ha_i - ah_i);$$

and a different function, R_1 , is given by the equal values of a_o , and is the same for either set.

To each of these four sets of $a_{ii} h_{ii} i_{ii}$ can be appended four distinct sets of $b_{ii} c_{ii} d_{ii} e_{ii} f_{ii} g_{ii}$, two from either of our two types.

But it is not necessary that either $a_o h_o i_o$ or $a_o i_o h_o$ should be the eighth triplet which is added to the given complete system of seven. It may be either $f_o h_o i_o$ or $f_o i_o h_o$, for example. Either of these will add to the value of f_{ii} , which is exhibited by the biquaternion $Q_{a_{ii}}$ (made by the given system of seven), a binomial function of $hi h_i i_i$, while no such addition will appear now in a_{ii} ; and the values of h_{ii} and i_{ii} will now contain f and f_i , but not a or a_i . The function R_1 will in either of these two cases be formed by the comparison of the four equal values of f_o .

It is clear then, that, if $h_o i_o$ be the two imaginaries which are excluded from the given complete system of seven, we can add to any one of the seven terms ($a_{ii} \dots g_{ii}$) of any one of the 480 biquaternions that can be made with the given complete system of seven, either of the quantities $\pm (hi_i - ih_i)$, and append to the eight quantities $w_{ii} a_{ii} \dots g_{ii}$, thus increased, the corresponding values of $h_{ii} i_{ii}$ and R_1 ; that is, we can construct the sixteen functions $w_{ii} a_{ii} b_{ii} \&c.$ in the product $Q_{a_{ii}} + R_1$, in $7 \cdot 2 \cdot 480$ ways. If $a_o b_o$ were the excluded duad, we should have another series of as many different sets of $w_{ii} a_{ii} b_{ii} \&c.$; and, in fact, the total number of ways of forming the product of two pluquaternions of nine is $36 \cdot 7 \cdot 2 \cdot 480 = 241,920$.

Since the case of fifteen or $(2^4 - 1)$ imaginaries possesses some interest, appearing, as it does, to follow next in order after those of $(2^2 - 1)$ and $(2^3 - 1)$, it may not be considered mere learned trifling, to assign the number of ways in which $(Q_{a_{ii}} + R_4)$, the product of two pluquaternions of fifteen, can be constructed. This product is completely determined, when to two given complete systems of seven triplets is added a fifteenth triplet, containing an imaginary which is common to both the systems of seven. Let a_o be that imaginary; and of the fourteen, $b_o c_o \dots m_o n_o$, take any six, $b_o c_o d_o e_o f_o g_o$, and any other six, $h_o i_o k_o l_o m_o n_o$, $o_p o$ being the duad excluded from both the sixes. With a_o and the first six, form a complete system of seven triplets, of which the three containing a_o shall be any

three, $a_o b_o c_o$, $a_o d_o e_o$, $a_o f_o g_o$; and with a_o and the other six, form also such a system, containing e. g. $a_o h_o i_o$, $a_o k_o l_o$, $a_o m_o n_o$. Then a_{ii} will be one of the eight,

$$a_{ii} = aw_i + wa_i \pm (bc_i - cb_i) \pm (de_i - ed_i) \pm (fg_i - gf_i) \\ \pm \{hi_i - ik_i\} \pm \{kl_i - lk_i\} \pm \{mn_i - nm_i\} \pm [op_i - po_i],$$

where the *like* brackets have all the *same* sign. If $(Q_{a_{ii}} + R_4)$ should reduce itself to a biquaternion, by the vanishing of all the real quantities in Q_a and Q_{a_i} except the twice eight $(ww_{ii}aa_{ii}bb_{ii}cc_{ii}dd_{ii}ee_{ii}ff_{ii}gg_{ii})$, we should have before us in the product only the eight functions $(w_{ii}a_{ii}b_{ii}c_{ii}d_{ii}e_{ii}f_{ii}g_{ii})$ belonging to that one of the 480 forms of such biquaternion, which is determined by the system of seven triplets made with $a_o b_o c_o d_o e_o f_o g_o$; and if all the real quantities were to vanish from Q_a and Q_{a_i} except the twice eight $(ww_{ii}aa_{ii}hh_{ii}ii_{ii}kk_{ii}ll_{ii}mm_{ii}nn_{ii})$, we should have before us one of the 480 forms of another biquaternion, determined by the triplets made with $a_o h_o i_o k_o l_o m_o n_o$. And, since any one of the latter 480 may be combined with any one of the former 480, and since in every case the resulting value of a_{ii} may receive either of the increments $\pm [op_i - po_i]$, according as the triplet $a_o o_o p_o$ is supposed of one sign or the other; it is plain, that we can form our product $(Q_{a_{ii}} + R_4)$, so that $o_o p_o$ shall be the duad excluded from both the systems of seven, and that the two sixes shall be any pair of sixes, from which $a_o o_o$ and p_o are excluded, and that a_o shall be the imaginary common to all the fifteen triplets, in $2 \cdot (480)^2$ distinct ways; although, be it remarked, the resulting values of a_{ii} above-mentioned will not amount to a number so great as this. But *any one* of the thirteen, from which o_o and p_o are excluded, may be the imaginary common to all the triplets, and determine, in a different way, to which of $a_{ii} b_{ii} c_{ii}$, &c. the increment $\pm [op_i - po_i]$ shall be added; and we may combine any one of $15 \cdot 7$ excluded pairs $(o_o p_o)$ with any one of $11 \cdot 42$ pairs of sixes which exclude $(o_o p_o)$. Consequently, the total number of ways in which the forty-six functions $w_{ii}a_{ii}b_{ii} \dots p_{ii}$ may be formed, in the product of two given pluquaternions of fifteen imaginaries, is $2 \cdot (480)^2 \cdot 13 \cdot 15 \cdot 7 \cdot 11 \cdot 42$, or 290,580,480,000.

In all these cases, the condition-function R_m is easily formed, being given by the comparison of the fifteen equal values of that imaginary which is common to all the fifteen triplets; and the values o_{ii} and p_{ii} , indicated by the pair of imaginaries excluded from the two complete systems of seven, are given by the single triplet, which is added to those systems.

[To be continued.]