the $3m$ parameters of the inflexions are the zeroes of the determinant whose constituents are the three coordinates and their first and second differential coefficients with respect to u ; the sum of the parameters of the $3m$ inflexions therefore vanishes, and a curve of order m other than the given elliptic curve will pass through all the double points and the $3m$ inflexions. Thus, when $p = 1$, it is possible to replace the Hessian by a curve of order m which, like the Hessian, has a simple intersection with the given curve at each inflexion; but, whereas the Hessian has a double point at each double point of the given curve, counting as a six-fold intersection, the new curve only passes through the double point

The Syzygetic Theory of Orthogonal Binariants. By E. B. ELLIOTT.

Read and received December 13th, 1900.

1. Attention has been redirected to systems of orthogonal invariants and covariants of binary forms by Major MacMahon in a paper read at the 1900 meeting of the British Association. His method is symbolical. The following pages are, it is believed, quite independent of his work. Their aim is to show that a complete syzygetic theory of the concomitants in question can be based on very elementary and simple considerations.

In a binary p -ic, in fact, the complete irreducible system of non-absolute orthogonal concomitants consists of a pair $x \pm y$ of linear universal covariants and $p+1$ linear invariants; and for a system of a p_1 -ic, a p_2 -ic, &c., it consists of $x \pm y$ and p_1+1 , p_2+1 , &c., linear invariants of the p_1 -ic, the p_2 -ic, &c., separately. The theory of the complete irreducible system of *absolute* orthogonal concomitants for a given p -ic or system is, in effect, the theory of irreducible products, obeying a certain law of isobarism, of the appropriate linear system.

In the relation expressive of the invariancy for all linear transformations of a concomitant of a quantic or quantics, the factor is well known to be always a power of the modulus. All proofs of this depend for their validity on the irreducibility of the general modulus. No such fact holds as to concomitants which have the property of invariancy only for a restricted class of linear transformations, in case

the modulus for that class be resolvable into factors. For instance, in expressions of invariancy for the class of transformations

$$x = lX - mY, \quad y = mX + lY,$$

the factor may be a product of different powers of $l + im$ and $l - im$. In particular, for the direct orthogonal transformation

$$x = X \cos \theta - Y \sin \theta, \quad y = X \sin \theta + Y \cos \theta,$$

the factor in an expression of invariancy will be a positive zero or negative power of $e^{i\theta}$.

Orthogonal invariants, &c., which are non-absolute for direct turning are left out of sight in chap. xv. of my *Algebra of Quantics*. They are not left out of sight in Andoyer's *Théorie des Formes*; but M. Andoyer's direction to find them (p. 143) by first investigating all absolute concomitants, and then combining $x + iy$ with the system found, reverses the order which will be seen to be the one readily applicable to the construction of syzygetic theory.

$$2. \text{ From } x = X \cos \theta - Y \sin \theta, \quad y = X \sin \theta + Y \cos \theta, \quad (1)$$

it at once follows that

$$\left. \begin{aligned} X + iY &= e^{-i\theta} (x + iy) \\ X - iY &= e^{i\theta} (x - iy) \end{aligned} \right\}. \quad (2)$$

Thus $x + iy$ and $x - iy$, which call ξ and η , are fundamental—universal—concomitants, of factors $e^{-i\theta}$, $e^{i\theta}$ respectively. $\xi^m \eta^n$ is one of factor $e^{-i(m-n)\theta}$. Any rational integral function of x and y , with numerical coefficients, may be given the form

$$\sum \lambda_{mn} \xi^m \eta^n,$$

by putting $\frac{1}{2}(\xi + \eta)$ and $\frac{1}{2i}(\xi - \eta)$ for x and y . It will be a universal concomitant if and only if for each term, when it is thus written, $m - n$ is the same.

Moreover, $\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$ are fundamental invariant operators, of factors $e^{-i\theta}$, $e^{i\theta}$ respectively. If, in fact,

$$F(X, Y) \equiv f(x, y)$$

be any equivalence, we have

$$\begin{aligned} \left(\frac{\partial}{\partial X} + i \frac{\partial}{\partial Y} \right)^r \left(\frac{\partial}{\partial X} - i \frac{\partial}{\partial Y} \right)^s F(X, Y) \\ \equiv e^{-i(r-s)\theta} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)^r \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)^s f(x, y). \end{aligned} \quad (3)$$

For shortness, write F_{rs} and $e^{-i(r-s)\theta} f_{rs}$ for the two sides here. A consequence is that

$$F_{r_1 s_1} F_{r_2 s_2} F_{r_3 s_3} \dots = e^{-i(\Sigma r - \Sigma s)\theta} f_{r_1 s_1} f_{r_2 s_2} f_{r_3 s_3} \dots \quad (4)$$

This affords us the means of writing down all rational integral differential equations which are unaltered in form by any transformation such as (1) applied to the independent variables, and of deciding whether a given equation has the property. Replace, in an equation

$$\Sigma A x^a y^b z^c \frac{\partial^{m_1+n_1} z}{\partial x^{m_1} \partial y^{n_1}} \frac{\partial^{m_2+n_2} z}{\partial x^{m_2} \partial y^{n_2}} \dots = 0,$$

x, y by $\frac{1}{2} \{ (x+iy) + (x-iy) \}$, $\frac{1}{2i} \{ (x+iy) - (x-iy) \}$ respectively, and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ by the like linear functions of $\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}, \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$, thus giving it the form

$$\Sigma B (x+iy)^{a'} (x-iy)^{b'} z^{c'} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)^{r_1} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)^{s_1} z \times \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)^{r_2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)^{s_2} z \dots = 0,$$

from which the transformation (1) produces the like form in X, Y with each B replaced by the corresponding $B e^{i(\alpha' - \beta' + \Sigma r - \Sigma s)\theta}$. The necessary and sufficient condition for permanence of form is then that for all terms

$$\alpha' - \beta' + \Sigma r - \Sigma s = \text{const.}$$

The property common to equations with this permanence is that their general solutions are unaltered in functional form when in them $x+iy$ is multiplied, and $x-iy$ divided, by any the same constant e^θ .

3. Take now the infinitesimal transformation

$$x = X - Y\theta', \quad y = X\theta' + Y \quad (4)$$

included in (1). To express properties of invariancy for the one-parameter group (1) it, we know, suffices to express the facts for (4). For a form K associated with

$$u \equiv (a_0, a_1, a_2, \dots, a_p)(x, y)^p \quad (5)$$

to have the property of invariancy for (1) it is necessary and suffices.

that its infinitesimal increment, when X, Y from (4) are substituted in it for x, y , and the consequential substitutions for $a_0, a_1, a_2, \dots, a_p$ are made, be a multiple of $\theta'K$. Accordingly the condition is that, for some value of λ ,

$$\left(\Omega - O - y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) K = \lambda K, \quad (6)$$

where
$$\Omega \equiv a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + \dots + pa_{p-1} \frac{\partial}{\partial a_p}, \quad (7)$$

$$O \equiv pa_1 \frac{\partial}{\partial a_0} + (p-1)a_2 \frac{\partial}{\partial a_1} + \dots + a_p \frac{\partial}{\partial a_{p-1}}. \quad (8)$$

We have already found the two solutions $x \pm iy$, of equations such as (6), which are linear in the variables and free from the coefficients. Let us now consider solutions linear in the coefficients in (5) and free from the variables, *i.e.*, linear orthogonal *invariants* of u .

For
$$k_0 a_0 + k_1 a_1 + k_2 a_2 + \dots + k_p a_p, \quad (9)$$

to be such an invariant the equations to be satisfied by λ and the k 's are

$$\left. \begin{aligned} k_1 &= \lambda k_0 \\ 2k_2 - pk_0 &= \lambda k_1 \\ 3k_3 - (p-1)k_1 &= \lambda k_2 \\ \dots &\dots \dots \\ pk_p - 2k_{p-2} &= \lambda k_{p-1} \\ -k_{p-1} &= \lambda k_p \end{aligned} \right\}, \quad (10)$$

and there is consequently one set of values of ratios of the k 's corresponding to every value of λ which satisfies the equation

$$\begin{vmatrix} \lambda & -1 & & & & \\ p & \lambda & -2 & & & \\ p-1 & & \lambda & -3 & & \\ & & & \dots & & \\ & & & & \dots & \\ & & & & & 3 & \lambda & -(p-1) \\ & & & & & & 2 & \lambda & -p \\ & & & & & & & 1 & \lambda \end{vmatrix} = 0. \quad (11)$$

The direct algebraical solution of this equation presents difficulties. But it will be seen indirectly that the equation must, in fact, be

$$(\lambda^2 + 1^2)(\lambda^2 + 3^2) \dots (\lambda^2 + p^2) = 0$$

or
$$\lambda(\lambda^2 + 2^2)(\lambda^2 + 4^2) \dots (\lambda^2 + p^2) = 0,$$

according as p is odd or even. In fact,

$$\Omega - O - y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

annihilates $u \equiv (a_0, a_1, a_2, \dots, a_p)(x, y)^p$, (5)

to which apply the transformation $x = \frac{1}{2}(\xi + \eta)$, $y = \frac{1}{2}(\xi - \eta)$ of § 2, and write it

$$u \equiv (a'_0, a'_1, a'_2, \dots, a'_p)(\xi, \eta)^p. \quad (11)$$

Now, from the term $\binom{p}{r} a'_r \xi^{p-r} \eta^r$ in this, $-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ produces

$$-(p-2r) \binom{p}{r} a'_r \xi^{p-r} \eta^r.$$

Consequently $\Omega - O$ must produce from it

$$-(p-2r) \binom{p}{r} a'_r \xi^{p-r} \eta^r.$$

In other words, $(\Omega - O) a'_r = -(p-2r) a'_r$ ($r = 0, 1, 2, \dots, p$). (12)

There are then $p+1$ distinct values of λ which make the equations (10) consistent, namely, the values $-p, -(p-2), \dots, (p-2), p$; and $p+1$ distinct corresponding linear orthogonal invariants (9), namely, the coefficients $a'_0, a'_1, a'_2, \dots, a'_p$ in (11).

These $p+1$ linear orthogonal invariants are linearly independent. For, were the coefficients in (11) linearly connected, so would be the $p+1$ linear functions of them $a_0, a_1, a_2, \dots, a_p$; but these are general.

4. To find the expressions for these linear invariants, we notice from (11) that

$$\frac{\partial^p}{\partial \xi^{p-r} \partial \eta^r} u = p! a'_r,$$

and we also observe that

$$\frac{\partial}{\partial \xi} = \frac{1}{2} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \eta} = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right). \quad (13)$$

Thus $a'_r = \frac{1}{p! 2^p} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)^{p-r} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^r u$ ($r = 0, 1, 2, \dots, p$). (14)

Let us now change the notation, and write I_{p-2r} for $2^p a'_r$, thus taking for our $p+1$ independent linear invariants

$$I_p, I_{p-2}, I_{p-4}, \dots, I_1, I_{-1}$$

if p is odd, and $I_p, I_{p-2}, I_{p-4}, \dots, I_0$

if p is even, where, for every s from 0 to the greatest integer in $\frac{1}{2}p$,

$$\begin{aligned} I_{\pm(p-2s)} &= \frac{1}{p!} \left(\frac{\partial}{\partial x} \mp \frac{\partial}{\partial y} \right)^{p-s} \left(\frac{\partial}{\partial x} \pm \frac{\partial}{\partial y} \right)^s u \\ &= \frac{1}{p!} \left(\frac{\partial}{\partial x} \mp \frac{\partial}{\partial y} \right)^{p-2s} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^s u. \end{aligned} \quad (15)$$

Now, to adopt a notation* in which all the "symbolic" theory of forms can be advantageously expressed without the use of any symbol which has not actual meaning either as a quantity or as a differential operator,

$$\begin{aligned} u &\equiv (a_0, a_1, a_2, \dots, a_p)(x, y)^p \\ &\equiv (x+yd)^p a_0, \end{aligned} \quad (16)$$

$$\text{where} \quad d \equiv a_1 \frac{\partial}{\partial a_0} + a_2 \frac{\partial}{\partial a_1} + \dots + a_p \frac{\partial}{\partial a_{p-1}}. \quad (17)$$

Hence (15) gives

$$\begin{aligned} I_{\pm(p-2s)} &= \frac{1}{(p-2s)!} \left(\frac{\partial}{\partial x} \mp \frac{\partial}{\partial y} \right)^{p-2s} (x+yd)^{p-2s} (1+d^2)^s a_0 \\ &= (1 \mp id)^{p-2s} (1+d^2)^s a_0 \\ &= (1 \mp id)^{p-2s} (a_0, a_2, a_4, \dots, a_{2s})(1, 1)^s \\ &= \{ (a_0, a_2, \dots, a_{2s})(1, 1)^s, (a_1, a_3, \dots, a_{2s+1})(1, 1)^s, \dots \\ &\quad \dots, (a_{p-2s}, a_{p-2s+1}, \dots, a_p)(1, 1)^s \} (1, \mp i)^{p-2s}, \end{aligned} \quad (18)$$

a pair of expressions of conjugate imaginary form. When p is even the single I_0 given by $s = \frac{1}{2}p$ is of real form, namely,

$$(a_0, a_2, a_4, \dots, a_p)(1, 1)^{1/2p}. \quad (19)$$

The factor in the expression of invariancy of any of these linear

* Called attention to by Mr. Kempe, *Proc. Lond. Math. Soc.*, Vol. xxiv., p. 102.

[The kindness of Major MacMahon supplies me with a note on the convenience of the pure German symbolism, which he has adopted and extended in his own work on orthogonal systems.

Let the quantic be $(a_1 x_1 + a_2 x_2)^p \equiv a_i^p$, where a_1, a_2 are umbræ. Invariant umbral expressions are $a_1 - ia_2 \equiv a_1$ and $a_1 + ia_2 \equiv a_2$; and $x_1 + ix_2 \equiv \xi_1$, $x_1 - ix_2 \equiv \xi_2$ are invariant. Now $2a_x \equiv a_1 \xi_1 + a_2 \xi_2 \equiv a_1$. Also all products of integral powers of a_1 and a_2 are invariants in symbolic form for some value of p , so that in $\frac{1}{2^p} a_i^p$ all the coefficients are linear invariants, and the variables are covariants. The form $\frac{1}{2^p} a_i^p$, i.e. (22A), is canonical.

In place of my I notation he would recommend one of double suffixes. Thus, in the case of the p -ic a_i^p , he would regard $I_{p0}, I_{p-11}, I_{p-22}, \dots, I_{0p}$ as the most expressive notation for $a_1^p, a_1^{p-1} a_2, a_1^{p-2} a_2^2, \dots, a_2^p$, or for the linear invariants (14).]

invariants I_m is the power of $e^{\pm\theta}$ whose index is the suffix m . For, in (15), $\frac{\partial}{\partial x} \mp i \frac{\partial}{\partial y}$ are by § 2 invariant operators of respective factors $e^{\pm i\theta}$. The doctrine of the infinitesimal substitution of § 3 gives the same information.

Every product $I_m I_{-m}$ of a conjugate pair is a quadratic absolute orthogonal invariant of real form.

Examples.—For low values of p the linear systems are as follows:—

(i.) For the linear form $ax + by$,

$$I_1, I_{-1} \equiv a \mp ib; \text{ factors } e^{\pm i\theta}.$$

(ii.) For the quadratic $ax^2 + 2bxy + cy^2$,

$$I_2, I_{-2} \equiv a \mp 2ib - c; \text{ factors } e^{\pm 2i\theta},$$

$$I_0 \equiv a + c; \text{ factor } 1.$$

(iii.) For the cubic $(a, b, c, d)(x, y)^3$,

$$I_3, I_{-3} \equiv a \mp 3ib - 3c \pm id; \text{ factors } e^{\pm 3i\theta},$$

$$I_1, I_{-1} \equiv a + c \mp i(b + d); \text{ factors } e^{\pm i\theta}.$$

(iv.) For the quartic $(a, b, c, d, e)(x, y)^4$,

$$I_4, I_{-4} \equiv a \mp 4ib - 6c \pm 4id + e; \text{ factors } e^{\pm 4i\theta},$$

$$I_2, I_{-2} \equiv a + c \mp 2i(b + d) - (c + e); \text{ factors } e^{\pm 2i\theta},$$

$$I_0 \equiv a + 2c + e; \text{ factor } 1.$$

(v.) For the quintic $(a, b, c, d, e, f)(x, y)^5$,

$$I_5, I_{-5} \equiv a \mp 5ib - 10c \pm 10id + 5e \mp if; \text{ factors } e^{\pm 5i\theta},$$

$$I_3, I_{-3} \equiv a + c \mp 3i(b + d) - 3(c + e) \pm i(d + f); \text{ factors } e^{\pm 3i\theta},$$

$$I_1, I_{-1} \equiv a + 2c + e \mp i(b + 2d + f); \text{ factors } e^{\pm i\theta}.$$

5. It must be remarked that these systems of linear orthogonal invariants are obtained as invariant only for *direct* orthogonal transformations

$$x = X \cos \theta - Y \sin \theta, \quad y = X \sin \theta + Y \cos \theta. \quad (1)$$

A *skew* orthogonal transformation

$$x = X \cos \theta + Y \sin \theta, \quad y = X \sin \theta - Y \cos \theta, \quad (20)$$

i.e., the result of the sequence of a direct orthogonal transformation and a change of sign of Y and second, fourth, &c., coefficients in the

transformed p -ic, produces as equal to $e^{m\theta} I_m$, not the function I_m of final coefficients, but the conjugate I_{-m} . The real quadratic product $I_m I_{-m}$ is absolutely invariant for skew as well as direct transformations.

In like manner $\xi \equiv x + iy$ of factor $e^{-i\theta}$ and $\eta \equiv x - iy$ of factor $e^{i\theta}$ are invariant only for direct orthogonal transformations. For skew transformations they are interchanged as well as affected by factors. The product $\xi\eta \equiv x^2 + y^2$, the absolute, is, of course, absolutely invariant for skew as well as direct transformations.

A product $I_m'' I_n'' I_q'' \dots \xi'' \eta''$ involves y and the alternate coefficients $a_1, a_3, a_5, \dots$ only to even dimensions in its real part and only to odd dimensions in its part affected by i . It is an orthogonal invariant or covariant for direct transformations, which is absolute if, and only if,

$$m\mu + n\nu + q\kappa + \dots - \alpha + \beta = 0; \quad (21)$$

and, if this be the case, its conjugate $I_{-m}'' I_{-n}'' I_{-q}'' \dots \eta'' \xi''$ will also be absolute. For skew transformations neither product will be absolute, or indeed invariant, unless the products are identical, *i.e.*, unless their form be the real one $(I_m I_{-m})'' (I_n I_{-n})'' \dots (\xi\eta)''$. Half the sum of the two, however, and $\frac{1}{2i}$ times their difference, will, if (21) be satisfied, be real and invariant also for skew transformations. The former will be absolute and the latter will have the factor -1 for the skew case, though absolute for the direct.

The systems presently investigated as systems of absolute orthogonal invariants will be absolute for direct transformations. To avoid repetition it may be stated, once for all, that when for any quantic or quantics a system of products of the I 's and ξ, η , say the products $A, B, \dots P, P', Q, Q', \dots$, in which the pairs P, P' , &c., are conjugate has been seen to form a complete irreducible system of absolute orthogonal invariants, or invariants and covariants, for direct transformations, the completed facts will be that a direct absolute invariant or covariant when expressed, as it can be, rationally and integrally in terms of

$$A, B, \dots, \frac{1}{2}(P+P'), \frac{1}{2i}(P-P'), \frac{1}{2}(Q+Q'), \frac{1}{2i}(Q-Q'), \dots,$$

will be also invariant for skew transformations if, and only if, in every one of the products of which it consists the number of difference factors $P-P', Q-Q', \dots$ is either odd or even. Only in the case of evenness is the invariancy for skew transformations absolute. In the case of oddness the factor is -1 .

6. The system of $p+1$ independent linear orthogonal invariants

$$\left. \begin{array}{l} I_p, I_{p-2}, \dots, I_{-(p-2)}, I_{-p} \\ a'_0, a'_1, \dots, a'_{p-1}, a'_p \end{array} \right\} \quad (22)$$

or

constitutes the complete irreducible system of non-absolute and absolute direct orthogonal invariants for one p -ic. In fact, remembering how (11) was formed from (5), we recognize that every rational integral function of $a_0, a_1, a_2, \dots, a_p$ can be rationally and integrally expressed in terms of $a'_0, a'_1, a'_2, \dots, a'_p$. In particular, then, any invariant can. The one requirement and sufficiency that a rational integral function thus expressed may possess invariancy is that the sum of the suffixes—in either the a' or the I notation—be constant throughout, because I_m is of factor $e^{m\theta}$.

The same system (22) with the addition of ξ and η is the complete irreducible system of direct orthogonal invariants and covariants, in like manner. The sum of the suffixes of the I 's, diminished by the exponent of ξ and increased by that of η , is what has to be constant throughout a rational integral function of the system which is a covariant.

Notice that the p -ic is not one of its own irreducible system. Its expression by means of that system is

$$\frac{1}{2^p} (I_p, I_{p-2}, \dots, I_{-p+2}, I_{-p})(\xi, \eta)^p. \quad (22A)$$

Products, of one degree, of members of the system (22) cannot be connected by any syzygy. Indeed, no relation connects $a'_0, a'_1, \dots, a'_p$, which are quite independent, because the equally numerous linear functions $a_0, a_1, \dots, a_p$ of them are. Neither can products of one degree and order of (22) and ξ, η be connected, for a like reason.

This syzygetic independence of products, and the constancy of sums of suffixes in the terms of such invariants as are sums of products, give us means of applying partition theory to estimate the number of linearly independent orthogonal invariants which have the same factor.

Write ρ for $e^{i\theta}$. The factor which occurs in the expression of invariancy of a product of powers of $I_p, I_{p-2}, \dots, I_{-p+2}, I_{-p}$ for which the sum of suffixes is κ , or of a linear function of such products with the same κ , is then ρ^κ . If we adopt the notation $a'_0, a'_1, \dots, a'_p$, the sum of suffixes, we say, is Σr , where $\kappa = \Sigma (p-2r)$, so that, if i be the degree of the product, or of each product,

$$\kappa = ip - 2w,$$

i.e.,

$$w = \frac{1}{2}(ip - \kappa).$$

There is, then, a one to one correspondence between orthogonal invariants of degree i and factor ρ^* and isobaric functions of degree i and weight $\frac{1}{2}(ip - \kappa)$. The number of the asyzygetic invariants is then the number of partitions denoted by

$$\left(\frac{ip - \kappa}{2}; i, p\right) = \left(\frac{ip + \kappa}{2}; i, p\right). \quad (23)$$

The number of factor ρ^* is the same. There is also the law of reciprocity that the number of orthogonal invariants of degree i and factor ρ^* of a p -ic is equal to the number of degree p and factor ρ^* of an i -ic.

The facts may also be exhibited by generating functions. Thus the *real* generating function for orthogonal invariants of a p -ic, each affected with the appropriate factor of its expression of invariancy, is

$$\{(1 - I_p \rho^p)(1 - I_{p-2} \rho^{p-2}) \dots (1 - I_{-p} \rho^{-p})\}^{-1}. \quad (24)$$

The expansion of this in powers of ρ has for the coefficient of ρ^* in it the sum of all the products of the I 's which are the asyzygetic invariants of factor ρ^* . The *numerical* generating function

$$\{(1 - a\rho^p)(1 - a\rho^{p-2}) \dots (1 - a\rho^{-p})\}^{-1} \quad (25)$$

has for the coefficient of $a^i \rho^*$ in it the number of the asyzygetic invariants of factor ρ^* and of degree i .

For orthogonal covariants the real and numerical generating functions are in like manner

$$\{(1 - I_p \rho^p) \dots (1 - I_{-p} \rho^{-p})(1 - \xi \rho^{-1})(1 - \eta \rho)\}^{-1}, \quad (26)$$

$$\text{and} \quad \{(1 - a\rho^p) \dots (1 - a\rho^{-p})(1 - x\rho^{-1})(1 - x\rho)\}^{-1}. \quad (27)$$

A familiar method enables us to expand (25) in powers of a . Noticing that multiplying (25) by $1 - a\rho^{-p}$ is equivalent to putting $a\rho^2$ for a in it and then multiplying by $1 - a\rho^{p+2}$, we obtain, by equating coefficients of a^i in the two products,

$$C_i = \frac{1}{\rho^p} \frac{1 - \rho^{2i+2p}}{1 - \rho^{2i}} C_{i-1},$$

where C_i means the coefficient of a^i in the expansion of (25). Repeating the application of this to C_{i-1} , C_{i-2} , ..., and observing that $C_0 = 1$, we have

$$C_i = \frac{1}{\rho^p} \frac{(1 - \rho^{2i+2p})(1 - \rho^{2i+2p-2}) \dots (1 - \rho^{2p+2})}{(1 - \rho^{2i})(1 - \rho^{2i-2}) \dots (1 - \rho^2)}. \quad (28)$$

Thus the number of orthogonal invariants of degree i and factor ρ^* is the coefficient of ρ^* in the expansion of (28).

7. Systems of Absolute Orthogonal Invariants, &c.

To obtain the real generating function for absolute orthogonal invariants (for direct transformations) of a p -ic, we have to extract from (24) the part free from ρ in its expansion. For absolute orthogonal covariants and invariants the like extraction from (26) has to be effected. More generally for absolute orthogonal covariants of a number of quantics in a number of cogredient pairs of variables $x_1, y_1; x_2, y_2; \&c.$, we have to extract the part free from ρ from

$$\{ \Pi [(1 - I_p \rho^p) \dots (1 - I_{-p} \rho^{-p})] \Pi [(1 - \xi \rho^{-1})(1 - \eta \rho)] \}^{-1}. \quad (29)$$

As I do not know of a work or memoir to which to refer for the extractions in question, I have considered in the articles which follow a few early cases. I first consider only invariants.

8. For $p = 1$, *i.e.*, for the case of $ax + by$, (24) is

$$\frac{1}{(1 - I_1 \rho)(1 - I_{-1} \rho^{-1})};$$

and the terms free from ρ are

$$1 + I_1 I_{-1} + (I_1 I_{-1})^2 + \dots,$$

so that the real generating function for direct absolute orthogonal invariants is

$$\frac{1}{1 - I_1 I_{-1}}; \quad (30)$$

and

$$I_1 I_{-1} \equiv a^2 + b^2 \quad (31)$$

is the only irreducible absolute orthogonal invariant. It is (*cf.* § 5, *end*) invariant also for skew transformations.

9. For $p = 2$, *i.e.*, for the quadratic $ax^2 + 2bxy + cy^2$, (24) is

$$\frac{1}{(1 - I_2 \rho^2)(1 - I_0)(1 - I_{-2} \rho^{-2})};$$

and the real generating function is at once

$$\frac{1}{(1 - I_0)(1 - I_2 I_{-2})}, \quad (32)$$

so that
$$\left. \begin{aligned} I_0 &\equiv a+c \\ \text{and } I_3 I_{-3} &\equiv (a-c)^2 + 4b^2 \end{aligned} \right\} \quad (33)$$

form the irreducible system. No syzygy connects them. They are invariant also for skew transformations.

10. For $p = 3$, i.e., for the cubic $(a, b, c, d)(x, y)^3$, (24) is

$$\frac{1}{(1-I_3\rho^3)(1-I_1\rho)(1-I_{-1}\rho^{-1})(1-I_{-3}\rho^{-3})}.$$

Now
$$\frac{1}{(1-I_3\rho^3)(1-I_{-3}\rho^{-3})} = \left\{ \frac{1}{1-I_3\rho^3} + \frac{1}{1-I_{-3}\rho^{-3}} - 1 \right\} \frac{1}{1-I_3 I_{-3}}$$

$$= \frac{1}{1-I_3 I_{-3}} \left\{ 1 + \sum_{m=1}^{\infty} (I_3^m \rho^{3m} + I_{-3}^m \rho^{-3m}) \right\}$$

and
$$\frac{1}{(1-I_1\rho)(1-I_{-1}\rho^{-1})} = \frac{1}{1-I_1 I_{-1}} \left\{ 1 + \sum_{n=1}^{\infty} (I_1^n \rho^n + I_{-1}^n \rho^{-n}) \right\}.$$

Hence the part free from ρ in the product is

$$\begin{aligned} &\frac{1}{(1-I_1 I_{-1})(1-I_3 I_{-3})} \left\{ 1 + \sum_{m=1}^{\infty} (I_1^{3m} I_{-3}^m + I_{-1}^m I_3^m) \right\} \\ &= \frac{1}{(1-I_1 I_{-1})(1-I_3 I_{-3})} \left\{ \frac{1}{1-I_1^3 I_{-3}} + \frac{1}{1-I_{-1}^3 I_3} - 1 \right\} \\ &= \frac{1-I_1^3 I_{-1}^3 I_3 I_{-3}}{(1-I_1 I_{-1})(1-I_3 I_{-3})(1-I_1^3 I_{-3})(1-I_{-1}^3 I_3)}. \end{aligned} \quad (34)$$

There are then four irreducible direct absolute orthogonal invariants of the cubic, which are best taken as the four real ones

$$\left. \begin{aligned} I_3 I_{-3} &\equiv (a-3c)^2 + (3b-d)^2, \\ I_1 I_{-1} &\equiv (a+c)^2 + (b+d)^2, \\ \frac{1}{2} \{ I_3 I_{-1}^3 + I_{-3} I_1^3 \} &\equiv (a+c)(a-3c) \{ (a+c)^2 - 3(b+d)^2 \} \\ &\quad + (b+d)(3b-d) \{ 3(a+c)^2 - (b+d)^2 \}, \\ -\frac{1}{2i} \{ I_3 I_{-1}^3 - I_{-3} I_1^3 \} &\equiv (a+c)(3b-d) \{ (a+c)^2 - 3(b+d)^2 \} \\ &\quad - (a-3c)(b+d) \{ 3(a+c)^2 - (b+d)^2 \}. \end{aligned} \right\} \quad (35)$$

They are, by the numerator of (34), connected by one syzygy of degree 8, which expresses the sum of the squares of the last two as the product of the first and the cube of the second. For skew transformations they are also invariant by § 5, the first three being absolute, and the last having the factor -1 .

The numerical generating function is

$$\frac{1-a^3}{(1-a^2)^2(1-a^4)^2} = \frac{1+a^4}{(1-a^2)^2(1-a^4)}. \quad (36)$$

11. For $p = 4$, i.e., for the quartic $(a, b, c, d, e)(x, y)^4$, (24) is

$$\frac{1}{(1-I_4\rho^4)(1-I_2\rho^2)(1-I_0)(1-I_{-2}\rho^{-2})(1-I_{-4}\rho^{-4})},$$

in which put z for ρ^2 .

Now, just as before,

$$\frac{1}{(1-I_4z^2)(1-I_{-4}z^{-2})} = \frac{1}{1-I_4I_{-4}} \left\{ 1 + \sum_{m=1}^{\infty} (I_4^m z^{2m} + I_{-4}^m z^{-2m}) \right\}$$

$$\text{and } \frac{1}{(1-I_2z)(1-I_{-2}z^{-1})} = \frac{1}{1-I_2I_{-2}} \left\{ 1 + \sum_{n=1}^{\infty} (I_2^n z^n + I_{-2}^n z^{-n}) \right\}.$$

Hence the real generating function required is

$$\begin{aligned} & \frac{1}{(1-I_4I_{-4})(1-I_2I_{-2})(1-I_0)} \left\{ 1 + \sum_{m=1}^{\infty} (I_4^m I_{-4}^{2m} + I_{-4}^m I_4^{2m}) \right\} \\ &= \frac{1-I_4I_{-4}I_2^2I_{-2}^2}{(1-I_4I_{-4})(1-I_2I_{-2})(1-I_0)(1-I_4I_{-4}I_2^2)(1-I_{-4}I_{-2}^2)}. \quad (37) \end{aligned}$$

The irreducible system for the quartic accordingly consists of five invariants, best written

$$\left. \begin{aligned} I_4I_{-4} &\equiv (a-6c+e)^2+16(b-d)^2, \\ I_2I_{-2} &\equiv (a-e)^2+4(b+d)^2, \\ I_0 &\equiv a+2c+e, \\ \frac{1}{2}\{I_4I_{-2}^2+I_{-4}I_2^2\} &\equiv (a-6c+e)\{(a-e)^2-4(b+d)^2\} \\ &\quad +16(a-e)(b^2-d^2), \\ \frac{1}{8}\{I_4I_{-2}^2-I_{-4}I_2^2\} &\equiv (a-6c+e)(a-e)(b+d) \\ &\quad -(b-d)\{(a-e)^2-4(b+d)^2\}. \end{aligned} \right\} \quad (38)$$

We are also told by (37) that these are connected by one syzygy of degree 6, which expresses the sum of the squares of the fourth and four times the fifth as the product of the first and the square of the second.

All but the last are absolute also for skew transformations. For such the last has the factor -1 .

The numerical generating function is

$$\frac{1-a^6}{(1-a)(1-a^2)^2(1-a^3)^2} = \frac{1+a^3}{(1-a)(1-a^2)^2(1-a^3)}. \quad (39)$$

12. For $p = 5$, i.e., for the quintic, (24) is

$$\frac{1}{(1-A\rho^5)(1-B\rho^3)(1-C\rho)(1-C'\rho^{-1})(1-B'\rho^{-3})(1-A'\rho^{-5})},$$

where, for temporary convenience, we have altered the notation.

Now

$$\begin{aligned} \frac{1}{(1-A\rho^5)(1-A'\rho^{-5})} &= \frac{1}{1-AA'} \left\{ 1 + \sum_{i=1}^{\infty} A^i \rho^{5i} + \sum_{i'=1}^{\infty} A'^{i'} \rho^{-5i'} \right\}, \\ \frac{1}{(1-B\rho^3)(1-B'\rho^{-3})} &= \frac{1}{1-BB'} \left\{ 1 + \sum_{m=1}^{\infty} B^m \rho^{3m} + \sum_{m'=1}^{\infty} B'^{m'} \rho^{-3m'} \right\}, \\ \frac{1}{(1-C\rho)(1-C'\rho^{-1})} &= \frac{1}{1-CC'} \left\{ 1 + \sum_{n=1}^{\infty} C^n \rho^n + \sum_{n'=1}^{\infty} C'^{n'} \rho^{-n'} \right\}. \end{aligned}$$

In the product of the three expansions no term can contain both A and A' , or both B and B' , or both C and C' . The real generating function for absolute invariants which is required, i.e., the expression whose expansion is the sum of the terms free from ρ in the product of the three right-hand members, will then be the expression for

$$\begin{aligned} \frac{1}{(1-AA')(1-BB')(1-CC')}, & \left\{ 1 + \sum_{5i+3m+n'} A^i B^m C'^{n'} + \sum_{5i'+3m'+n} A'^{i'} B'^{m'} C^n \right. \\ & + \sum_{3m+n=5i'} B^m C^n A'^{i'} + \sum_{3m'+n'=5i} B'^{m'} C'^{n'} A^i \\ & + \sum_{n+5i=3m'} C^n A^i B'^{m'} + \sum_{n'+5i'=3m} C'^{n'} A'^{i'} B^m \\ & - \sum_{m=1}^{\infty} B^m C'^{3m} - \sum_{n=1}^{\infty} C'^{5n} A'^n - \sum_{r=1}^{\infty} A'^{3r} B'^{5r} \\ & \left. - \sum_{m=1}^{\infty} B'^{m'} C^{3m'} - \sum_{n=1}^{\infty} C^{5n} A^n - \sum_{r=1}^{\infty} A^{3r} B^{5r} \right\}, \quad (40) \end{aligned}$$

in which the last six sums are subtracted because each is reckoned in two previous sums. Simultaneous zero indices in any term of a sum are excluded.

Now the simple sets of positive integral solutions of

$$5l + 3m = n'$$

are

$$1 \quad 0 \quad 5 \quad .$$

and

$$0 \quad 1 \quad 3.$$

All other sets of solutions are sums of multiples of these sets. Hence

$$\left. \begin{aligned} \sum_{5l+3m=n'} A^l B^m C^{n'} &= \sum_{r=0, s=0}^{r=\infty, s=\infty} (AC^5)^r (BC^3)^s \\ &= \frac{1}{(1-AC^5)(1-BC^3)} - 1, \\ \text{and similarly} \\ \sum_{5l'+3m'=n} A^{l'} B^{m'} C^n &= \frac{1}{(1-A'C^5)(1-B'C^3)} - 1. \end{aligned} \right\} \quad (41)$$

Again, the simple sets of solutions of

$$3m + n = 5l'$$

are

$$(a) \quad 0 \quad 5 \quad 1,$$

$$(b) \quad 5 \quad 0 \quad 3,$$

$$(c) \quad 1 \quad 2 \quad 1,$$

$$(d) \quad 3 \quad 1 \quad 2,$$

which are connected by the two irreducible syzygies

$$(a) + (d) \equiv 3 (c),$$

$$(b) + (c) \equiv 2 (d),$$

all other syzygies being, in fact, consequences of these, for instance,

$$(a) + (b) \equiv 2 (c) + (d).$$

The justification of these statements is involved in what follows.

We proceed to see that every set of solutions of $3m + n = 5l'$ is included once in the systems of sets

$$s(c) + t(d), \quad q(a) + s(c), \quad r(b) + t(d),$$

for positive integral (not both zero) values of the multipliers, and that all sets are included once only, except sets $s(c)$, $t(d)$, which occur twice.

That every set

$$q(a) + r(b) + s(c) + t(d)$$

can be expressed in one of the three forms is clear, because by use of $(a) + (b) \equiv 2(c) + (d)$ it can be given one of the forms

$$q(a) + s(c) + t(d), \quad r(b) + s(c) + t(d),$$

and by use of $(a) + (d) \equiv 3(c)$ the first of these can be given one of the forms $q(a) + s(c)$, $s(c) + t(d)$, and by use of $(b) + (c) \equiv 2(d)$ the second can be given one of the forms $r(b) + t(d)$, $s(c) + t(d)$.

Can, however, any set of solutions be more than once included in the three systems? It cannot be twice included in either one of the three; if, for instance, $s(c) + t(d)$ and $s'(c) + t'(d)$ were the same, we should have $s + 3t = s' + 3t'$ and $2s + t = 2s' + t'$, i.e., $s = s'$ and $t = t'$. Neither can it be included in two different systems, unless it be of one of the forms $s(c)$, $t(d)$, which occur in two. If, for instance, we could have $s(c) + t(d)$ identical with $q'(a) + s'(c)$, we should have satisfied

$$s + 3t = s', \quad 2s + t = 5q' + 2s';$$

whence

$$5q' = -5t,$$

which would have to be negative unless $t = 0$. In like manner the identity of $s'(c) + t(d)$ and $r'(b) + t'(d)$ would necessitate

$$s + 3t = 5r' + 3t', \quad 2s + t = t';$$

whence

$$5r' = -5s,$$

negative unless $s = 0$. And finally the identity of $q(a) + s(c)$ and $r'(b) + t'(d)$ would necessitate

$$s = 5r' + 3t', \quad 5q + 2s = t';$$

whence

$$5r' = -15q - 5s,$$

which is negative.

Accordingly we have

$\sum_{3m+n=5l'} B^m C^n A^{l'}$ (in which terms with one of m, n zero are included)

$$= \sum (B C^2 A')^s (B^3 O A'^3)^t + \sum (C^5 A')^s (B C^2 A')^t + \sum (B^5 A'^3)^s (B^3 C A'^2)^t \\ - \sum (B C^3 A')^s - \sum (B^3 C A'^2)^t$$

$$= \frac{1}{(1 - B C^2 A')(1 - B^3 C A'^2)} + \frac{1}{(1 - C^5 A')(1 - B C^2 A')} \\ + \frac{1}{(1 - B^5 A'^3)(1 - B^3 O A'^2)} \\ - \frac{1}{1 - B C^3 A'} - \frac{1}{1 - B^3 O A'^2} - 1$$

$$= \frac{1}{(1-BO^3A')(1-B^3CA'^3)} + \frac{O^5A'}{(1-C^5A')(1-BO^3A')} \\ + \frac{B^5A'^3}{(1-B^5A'^3)(1-B^3CA'^3)} - 1. \quad (42)$$

Similarly for the fourth summation in (40).

Once more, the simple solutions of

$$n + 5l = 3m'$$

are

$$(\alpha) \quad 3 \quad 0 \quad 1,$$

$$(\beta) \quad 0 \quad 3 \quad 5,$$

$$(\gamma) \quad 1 \quad 1 \quad 2,$$

connected by the one syzygy

$$(\alpha) + (\beta) \equiv 3(\gamma),$$

so that all solutions are once included in

$$q(\alpha) + r(\beta) + s(\gamma),$$

with s equal to 0 or 1 or 2. Thus

$$\sum_{n+5l=3m'} C^n A^l B'^{m'} = \{1 + \sum (C^3 B')^q (A^3 B'^3)^r\} \{1 + CAB'^2 + (CAB'^2)^2\} - 1 \\ = \frac{1 - (CAB'^2)^3}{(1 - C^3 B')(1 - A^3 B'^3)(1 - CAB'^2)} - 1; \quad (43)$$

and similarly for the sixth summation in (40).

The subtracted sums in (40) amount to

$$-6 + \frac{1}{1-BO'^3} + \frac{1}{1-B'O^3} + \frac{1}{1-C^5A'} + \frac{1}{1-C^5A} + \frac{1}{1-A^3B'^3} + \frac{1}{1-A^3B^3} \\ (44)$$

Consequently the desired real generating function is given by (40) in the form

$$\frac{1}{(1-AA')(1-BB')(1-CC')} \\ \left\{ 1 + \frac{1}{(1-AO'^3)(1-BO'^3)} + \frac{1}{(1-A'O^3)(1-B'O^3)} \right. \\ + \frac{1}{(1-BO'^2A')(1-B^3CA'^2)} + \frac{1}{(1-B'O'^2A)(1-B'^3CA'^2)} \\ \left. + \frac{B^5A'^3}{(1-B^5A'^3)(1-B^3CA'^2)} + \frac{B'^5A^3}{(1-B'^5A^3)(1-B^3CA'^2)} \right\}$$

$$\begin{aligned}
& + \frac{O^3 A}{(1-O^3 A)(1-B'O^2 A)} + \frac{O^3 A'}{(1-O^3 A')(1-BO^2 A')} \\
& + \frac{1-(OAB'^2)^3}{(1-O^3 B')(1-A^3 B'^3)(1-OAB'^2)} \\
& + \frac{1-(O'A'B^3)^3}{(1-O'^3 B)(1-A'^3 B^3)(1-O'A'B^3)} - \frac{1}{1-BO^3} - \frac{1}{1-B'O^3} \\
& - \frac{1}{1-O^3 A'} - \frac{1}{1-O'^3 A} - \frac{1}{1-A^3 B'^3} - \frac{1}{1-A'^3 B^3} \}. \quad (45)
\end{aligned}$$

This real generating function is readily written as one fraction, not in its lowest terms, with denominator

$$\begin{aligned}
& (1-AA')(1-BB')(1-CC')(1-AC^3)(1-A'O^3)(1-BC^3)(1-B'O^3) \\
& \times (1-A^3 B'^3)(1-A'^3 B^3)(1-AB'C^3)(1-A'BC^3)(1-A^3 B^3 O') \\
& \times (1-A^2 B^3 O)(1-AB^2 O)(1-A'B^2 O'); \quad (46)
\end{aligned}$$

and, from the examination above of the critical diophantine equations for their simple sets of solutions, it follows that there are no irreducible products of the absolute class which are still unrepresented in this denominator. Thus, reverting to our ordinary notation, the irreducible absolute system for the quintic consists of the fifteen orthogonal invariants

$$\left. \begin{aligned}
& I_5 I_{-5}, \quad I_3 I_{-3}, \quad I_1 I_{-1}; \\
& I_5 I_{-1}^3, \quad I_3 I_{-1}^3, \quad I_5^3 I_{-3}^3, \quad I_5 I_{-3} I_{-1}^3, \quad I_5 I_1 I_{-3}^3, \quad I_5^3 I_{-3}^3 I_{-1}; \\
& I_{-5} I_1^3, \quad I_{-3} I_1^3, \quad I_{-5}^3 I_5^3, \quad I_{-5} I_3 I_1^3, \quad I_{-5} I_{-1} I_3^3, \quad I_{-5}^3 I_3^3 I_1.
\end{aligned} \right\} \quad (47)$$

After the first three they go in conjugate pairs. Each pair is better replaced by half the sum and $\frac{1}{2i}$ times the difference of the pair, the whole system being thus given reality of form and applicability to skew transformations.

Three of the fifteen are of degree 2, six of degree 4, four of degree 6, and two of degree 8.

The syzygies among the fifteen would be revealed by a study of the numerator when (45) is written as one fraction with (46) for denominator.

The *numerical* generating function, obtained by putting a for each

letter, accented as well as unaccented, in (45), is

$$\frac{1}{(1-a^2)^8} \left\{ 1 + \frac{4+2a^6}{(1-a^4)(1-a^6)} + \frac{2a^8}{(1-a^6)(1-a^8)} + \frac{2-2a^{12}}{(1-a^8)^2(1-a^8)} \right. \\ \left. - \frac{2}{1-a^4} - \frac{2}{1-a^6} - \frac{2}{1-a^8} \right\} \\ = \frac{1+a^2+6a^4+9a^6+12a^8+9a^{10}+6a^{12}+a^{14}+a^{16}}{(1-a^2)^2(1-a^4)(1-a^6)(1-a^8)}, \quad (48)$$

which is easily expanded; and the number of distinct absolute orthogonal invariants of any degree i is given as the coefficient of a^i in the expansion.

If (48) be written with denominator

$$(1-a^2)^2(1-a^4)^2(1-a^6)^2(1-a^8)^2,$$

in which all the irreducible products (47) are represented, we find that the numerator is

$$1-13a^8-18a^{10}+16a^{12}+76a^{14}+66a^{16}-76a^{18}-208a^{20}-130a^{22}+157a^{24} \\ +356a^{26}+192a^{28}-208a^{30}-422a^{32}-208a^{34}+192a^{36}+356a^{38}+157a^{40} \\ -130a^{42}-208a^{44}-76a^{46}+66a^{48}+76a^{50}+16a^{52}-18a^{54}-13a^{56}+a^{58}. \quad (49)$$

Thus the lowest syzygies are of degree 8, of which degree there is also a "ground-form," and are thirteen in number. It will be found that they express in two ways, in terms of the irreducible products of (46) or (47), the thirteen products which consist of

$$BB'O^8C^8, AA'B^3B^3CO', AA'BB'O^3C^3, ABB^3CC^3, A'B^3C^3C^3, \\ ABB'O^8, AA'BB^3C^3, A^3B^3CO^3, \quad (50)$$

and the results of interchanging accented and unaccented letters in the last five. Of degree 10, besides the products of these syzygies of degree 8 and AA' , BB' , CO' , there are eighteen further syzygies. These are found to express in two ways the nine products

$$AB'O^8C^8, A^3B^3CO^8, AA'B^3C^8, AA'BO^3C^8, A^2B^3CO^8, A^2B^3BO^4, \\ AA^2B^3B^3O^3, A^2B^3CO^3, AA^2B^3BO^3, \quad (51)$$

and the results of interchanging accented and unaccented letters in them. As to products of the next degree 12, the number of distinct ones is sixteen more than the result of subtracting from the number of products of degree 12, formed out of (47), thirteen times the number of degree 4 and eighteen times the number of degree 2,

because many will have been subtracted more than once because of syzygies (50) and (51); and so on.

13. For $p = 6$, i.e., for the sextic, the analysis is somewhat simpler than for the quintic. Putting z for ρ^3 , and changing the notation, (24) is now

$$\frac{1}{(1-Iz^3)(1-Jz^3)(1-Kz)(1-L)(1-K'z^{-1})(1-J'z^{-2})(1-I'z^{-3})}.$$

Now

$$\frac{1}{(1-Iz^3)(1-I'z^{-3})} = \frac{1}{1-II'} \left\{ 1 + \sum_{l=1}^{\infty} I^l z^{3l} + \sum_{l'=1}^{\infty} I'^{l'} z^{-3l'} \right\},$$

$$\frac{1}{(1-Jz^3)(1-J'z^{-2})} = \frac{1}{1-JJ'} \left\{ 1 + \sum_{m=1}^{\infty} J^m z^{2m} + \sum_{m'=1}^{\infty} J'^{m'} z^{-2m'} \right\},$$

$$\frac{1}{(1-Kz)(1-K'z^{-1})} = \frac{1}{1-KK'} \left\{ 1 + \sum_{n=1}^{\infty} K^n z^n + \sum_{n'=1}^{\infty} K'^{n'} z^{-n'} \right\}.$$

The real generating function required is then the expression for

$$\frac{1}{(1-II')(1-JJ')(1-KK')(1-L)}$$

$$\left\{ 1 + \sum_{3l+2m=n'} I^l J^m K'^{n'} + \sum_{3l'+2m'=n} I'^{l'} J'^{m'} K^n \right.$$

$$+ \sum_{2m+n=3l'} J^m K^n I'^{l'} + \sum_{2m'+n'=3l} J'^{m'} K'^{n'} I^l$$

$$+ \sum_{n+3l=2m'} K^n I^l J^{m'} + \sum_{n'+3l'=2m} K'^{n'} I'^{l'} J^m$$

$$- \sum_{m=1}^{\infty} J^m K'^{2m} - \sum_{l=1}^{\infty} K^{3l} I'^l - \sum_{n=1}^{\infty} I^{2n} J'^{3n}$$

$$\left. - \sum_{m=1}^{\infty} J'^m K^{2m} - \sum_{l=1}^{\infty} K'^{3l} I^l - \sum_{n=1}^{\infty} I'^{2n} J^{3n} \right\}, \quad (52)$$

the last six sums being subtracted because they have been reckoned twice in previous sums.

The simple sets of solutions of

$$3l+2m=n'$$

are

$$1 \quad 0 \quad 3$$

and

$$0 \quad 1 \quad 2.$$

Hence

$$\sum_{3l+2m=n'} I^l J^m K'^{n'} = \sum_{l+m=1}^{l=\infty, m=\infty} (IK'^3)^l (JK'^2)^m$$

$$= \frac{1}{(1-IK'^3)(1-JK'^2)} - 1 \quad (53)$$

and similarly for the second summation in (52).

Again, the simple sets of solutions of

$$2m+n=3l'$$

are

$$(a) \quad 3 \quad 0 \quad 2,$$

$$(b) \quad 0 \quad 3 \quad 1,$$

$$(c) \quad 1 \quad 1 \quad 1,$$

which are connected by the syzygy

$$(a) + (b) \equiv 3(c).$$

Every set of solutions is then included once and once only in the set

$$q(a) + r(b) + s(c),$$

with $q+r+s \geq 1$ and $s = 0$ or 1 or 2 . Hence

$$\begin{aligned} \sum_{2m+n=3l'} J^m K^n I'^n &= \left\{ 1 + \sum_{n=1}^{n=\infty} (J^3 I'^3)^n (K^3 I')^n \right\} \{1 + JKI' + (JKI')^2\} - 1 \\ &= \frac{1 - (JKI')^3}{(1 - J^3 I'^3)(1 - K^3 I')(1 - JKI')} - 1; \end{aligned} \quad (54)$$

and similarly for the fourth summation in (52).

Once more the simple sets of solutions of

$$n+3l=2m'$$

are

$$(\alpha) \quad 2 \quad 0 \quad 1,$$

$$(\beta) \quad 0 \quad 2 \quad 3,$$

$$(\gamma) \quad 1 \quad 1 \quad 2,$$

with the one syzygy $(\alpha) + (\beta) \equiv 2(\gamma)$.

The form which includes every set of solutions once is then

$$q(\alpha) + r(\beta) + s(\gamma),$$

with $q+r+s \geq 1$ and $s = 0$ or 1 . Consequently

$$\begin{aligned} \sum_{n+3l=2m'} K^n I' J'^n &= \left\{ 1 + \sum_{n=1}^{n=\infty} (K^3 J')^n (I' J'^3)^n \right\} (1 + KIJ'^2) - 1 \\ &= \frac{1 - (KIJ'^2)^3}{(1 - K^3 J')(1 - I' J'^3)(1 - KIJ'^2)} - 1; \end{aligned} \quad (55)$$

and similarly for the sixth summation.

The seventh to twelfth summations in (52) are at once performed;

and we thus arrive at our desired real generating function, the equivalent of (52), in the form

$$\frac{1}{(1-II')(1-JJ')(1-KK')(1-L)} \\ \left\{ 1 + \frac{1}{(1-IK'^2)(1-JK'^2)} + \frac{1}{(1-I'K^2)(1-J'K^2)} \right. \\ + \frac{1-(I'JK)^2}{(1-I'^2J^2)(1-I'K^2)(1-I'JK)} + \frac{1-(IJ'K')^2}{(1-I^2J'^2)(1-IK'^2)(1-IJ'K')} \\ + \frac{1-(IJ^2K)^2}{(1-J'^2K^2)(1-I^2J'^2)(1-IJ^2K)} + \frac{1-(I'J^2K')^2}{(1-JK'^2)(1-I'^2J'^2)(1-I'J^2K')} \\ \left. - \frac{1}{1-JK'^2} - \frac{1}{1-J'K^2} - \frac{1}{1-I'K^2} - \frac{1}{1-IK'^2} - \frac{1}{1-I^2J'^2} - \frac{1}{1-I'^2J^2} \right\}. \quad (56)$$

It may be written as one fraction, not in its lowest terms, with the denominator

$$(1-L)(1-II')(1-JJ')(1-KK')(1-JK'^2)(1-J'K^2)(1-IJ'K') \\ \times (1-I'JK)(1-IK'^2)(1-I'K^2)(1-IJ^2K)(1-I'J^2K') \\ \times (1-I^2J^2)(1-I'^2J'^2);$$

and in this denominator all the irreducible products which are absolute orthogonal invariants occur. There are altogether fourteen in the irreducible system, namely, in our old notation,

one linear	$I_0,$	}	(57)
three quadratic	$I_6I_{-6}, I_4I_{-4}, I_2I_{-2},$		
four cubic	$I_4I_{-2}^2, I_{-4}I_2^2; I_6I_{-4}I_{-2}, I_{-6}I_4I_2,$		
four quartic	$I_6I_{-2}^3, I_{-6}I_2^3; I_6I_{-4}^2I_2, I_{-6}I_4^2I_{-2},$		
and two quintic	$I_6^2I_{-4}^2, I_{-6}^2I_4^2.$		

After the first four they go in conjugate pairs, and each pair is better replaced by the equivalent pair of real forms obtained by taking half the sum and $\frac{1}{2i}$ times the difference of the two.

The numerical generating function is

$$(1-a)(1-a^2)^2 \left\{ 1 + \frac{2}{(1-a^2)(1-a^4)} + \frac{4-2a^2-2a^6}{(1-a^2)(1-a^4)(1-a^6)} \right. \\ \left. - \frac{2}{1-a^2} - \frac{2}{1-a^4} - \frac{2}{1-a^6} \right\}$$

$$\begin{aligned}
&\equiv \frac{1+a^2+3a^3+4a^4+4a^5+4a^6+3a^7+a^8+a^{10}}{(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)} \\
&\equiv \{1-6a^6-10a^7-12a^8+2a^9+29a^{10}+48a^{11}+41a^{12}-6a^{13}-66a^{14} \\
&\quad -102a^{15}-77a^{16}+4a^{17}+90a^{18}+128a^{19}+90a^{20}+4a^{21}-77a^{22} \\
&\quad -102a^{23}-66a^{24}-6a^{25}+41a^{26}+48a^{27}+29a^{28}+2a^{29}-12a^{30}-10a^{31} \\
&\quad -6a^{32}+a^{33}\} \\
&\div (1-a)(1-a^2)^2(1-a^3)^2(1-a^4)^2(1-a^5)^2. \tag{58}
\end{aligned}$$

In the last form of this all of the irreducible systems are represented in the denominator. The numerator gives information as to the syzygies. Thus the lowest syzygies are of degree 6, and are six in number. They are at once seen to be

$$\left. \begin{aligned}
I_4 I_{-2}^2 \cdot I_{-4} I_2^2 &\equiv I_4 I_{-4} \cdot (I_2 I_{-2})^2, \\
I_6 I_{-4} I_{-2} \cdot I_{-6} I_4 I_2 &\equiv I_6 I_{-6} \cdot I_4 I_{-4} \cdot I_2 I_{-2}, \\
I_4 I_{-2}^2 \cdot I_6 I_{-4} I_{-2} &\equiv I_4 I_{-4} \cdot I_6 I_{-2}^3, \\
I_4 I_{-2}^2 \cdot I_{-6} I_4 I_2 &\equiv I_2 I_{-2} \cdot I_{-6} I_4^2 I_{-2},
\end{aligned} \right\} \tag{59}$$

and the results of changing the signs of the suffixes in the last two. Again, the term $-10a^7$ in the numerator tells us that besides I_6 times (59) there are ten new syzygies of degree 7. These give other expressions for the five products

$$\begin{aligned}
I_4 I_{-2}^2 \cdot I_{-6} I_2^3, \quad I_4 I_{-2}^2 \cdot I_6 I_{-4}^2 I_2, \quad I_6 I_{-4} I_{-2} \cdot I_{-6} I_4^3 I_{-2}, \\
I_6 I_{-4} I_{-2} \cdot I_{-6} I_2^3, \quad I_6 I_{-4} I_{-2} \cdot I_6 I_{-4}^2 I_2,
\end{aligned} \tag{60}$$

and their five conjugates; and so on.

14. Absolute Orthogonal Invariants for Systems of Quantics.

It will, I have no doubt, suffice to write down the other results at which I have arrived without exhibiting the tedious, but not difficult, calculations. No complication of a kind not already sufficiently exemplified arises.

I use R.G.F. and N.G.F. to denote respectively real and numerical generating functions, and A.O.I. and A.O.C. for absolute orthogonal invariants and covariants (including invariants) respectively.

(A) *Two linear forms* $ax+by$, $a'x+b'y$. — For this system the R.G.F. A.O.I. is

$$\frac{1-I_1I_{-1}J_1J_{-1}}{(1-I_1I_{-1})(1-J_1J_{-1})(1-I_1J_{-1})(1-I_{-1}J_1)},$$

where I_1 , I_{-1} and J_1 , J_{-1} are the fundamental linear orthogonal invariants of the two forms respectively. The irreducible system of A.O.I.'s consists then of the four invariants

$$I_1I_{-1} \equiv a^2+b^2,$$

$$J_1J_{-1} \equiv a'^2+b'^2,$$

$$\frac{1}{2} \{I_1J_{-1}+I_{-1}J_1\} \equiv aa'+bb',$$

$$\frac{1}{2i} \{I_1J_{-1}-I_{-1}J_1\} \equiv ab'-a'b,$$

connected by the one syzygy

$$(a^2+b^2)(a'^2+b'^2) = (aa'+bb')^2 + (ab'-a'b)^2.$$

(B) *Three linear forms.*—If I_1 , I_{-1} ; J_1 , J_{-1} ; K_1 , K_{-1} refer to the three forms, the R.G.F. A.O.I. is

$$\frac{1}{(1-I_1I_{-1})(1-J_1J_{-1})(1-K_1K_{-1})} \left\{ 1 + \frac{J_1K_{-1}}{(1-I_1K_{-1})(1-J_1K_{-1})} + \frac{J_{-1}K_1}{(1-I_{-1}K_1)(1-J_{-1}K_1)} \right. \\ + \frac{I_1J_{-1}}{(1-J_{-1}K_1)(1-I_1J_{-1})} + \frac{I_{-1}J_1}{(1-J_1K_{-1})(1-I_{-1}J_1)} \\ \left. + \frac{I_1K_{-1}}{(1-I_1J_{-1})(1-I_1K_{-1})} + \frac{I_{-1}K_1}{(1-I_{-1}J_1)(1-I_{-1}K_1)} \right\}.$$

The N.G.F. is, if a , a' , a'' refer to the three forms,

$$\frac{1 + \Sigma aa' - aa'a'' \Sigma a - a^2a'^2a''^2}{(1-a^2)(1-a'^2)(1-a''^2)(1-aa')(1-aa'')(1-a'a'')},$$

or, if we do not distinguish between degrees in the different pairs of coefficients,

$$\frac{1+4a^2+a^4}{(1-a^2)^3} \equiv \frac{1-9a^4+16a^6-9a^8+a^{12}}{(1-a^2)^6}.$$

There are nine invariants in the irreducible system, three of each of the types

$$a^2 + b^2, \quad aa' + bb', \quad ab' - a'b,$$

no new type occurring which did not present itself in the case of two linear forms. (We readily convince ourselves that no new type occurs for any greater number of linear forms.)

The lowest syzygies are of degree 4 and are nine in number. There are three of each of the types

$$I_1 I_{-1} \cdot J_1 J_{-1} \equiv I_1 J_{-1} \cdot I_{-1} J_1, \quad I_1 I_{-1} \cdot J_1 K_{-1} \equiv I_1 K_{-1} \cdot I_{-1} J_1,$$

$$I_1 I_{-1} \cdot J_{-1} K_1 \equiv I_1 J_{-1} \cdot I_{-1} K_1.$$

The next coefficient 16 in the numerator of the last written form of the N.G.F. is positive. The new syzygies (if there be any) of degree 6 are then less in number by sixteen than the number of times we subtract products of degree 6 from the whole number of products of degree 6 which are obtained as products of three of the nine irreducible products of degree 2, when we take away the number of products of syzygies of degree 4 and the products of degree 2. For instance,

$$I_1 I_{-1} J_1 J_{-1} K_1 K_{-1},$$

which must have coefficient 1 in the expansion of the R.G.F., is reckoned six times as a product of three of $I_1 I_{-1}$, $I_1 J_{-1}$, ..., is taken away nine times, once because of each of the syzygies of degree 4, and has accordingly to be reckoned four times more.

(C) *Quadratic and linear form.*—If I_2 , I_0 , I_{-2} and J_1 , J_{-1} refer to the two forms, the R.G.F. A.O.I. is

$$\frac{1 - I_2 I_{-2} J_1^2 J_{-1}^2}{(1 - I_0)(1 - I_2 I_{-2})(1 - J_1 J_{-1})(1 - I_2 J_{-1}^2)(1 - I_{-2} J_1^2)}.$$

There is then one syzygy of partial degrees 2, 4 among a system of five irreducible invariants. The five are, if we take half the sum and $\frac{1}{4}$ times the difference of the conjugate pair, the two of the quadratic

$$a + c, \quad (a - c)^2 + 4b^2,$$

the one of the linear form $a^2 + b^2$,

and the two

$$(a - c)(a^2 - b^2) + 4ba'b', \quad (a - c)a'b' - b(a^2 - b^2).$$

(D) *Quadratic and two linear forms.*—If $I_2, I_0, I_{-2}; J_1, J_{-1}; K_1, K_{-1}$ refer to the forms respectively, the R.G.F. A.O.I. is

$$\frac{1}{(1-I_0)(1-I_2I_{-2})(1-J_1J_{-1})(1-K_1K_{-1})} \left\{ 1 + \frac{J_1K_{-1}}{(1-I_2K_{-1}^2)(1-J_1K_{-1})} + \frac{J_{-1}K_1}{(1-I_2K_1^2)(1-J_{-1}K_1)} \right. \\ + \frac{J_{-1}K_1}{(1-I_2J_{-1}^2)(1-J_{-1}K_1)} + \frac{J_1K_{-1}}{(1-I_2J_1^2)(1-J_1K_{-1})} \\ + \frac{1-(I_{-2}J_1K_1)^2}{(1-I_{-2}J_1^2)(1-I_{-2}K_1^2)(1-I_{-2}J_1K_1)} \\ + \frac{1-(I_2J_{-1}K_{-1})^2}{(1-I_2J_{-1}^2)(1-I_2K_{-1}^2)(1-I_2J_{-1}K_{-1})} \\ \left. - \frac{1}{1-J_1K_{-1}} - \frac{1}{1-J_{-1}K_1} \right\}.$$

The irreducible system consists of the twelve forms

$$I_0, I_2I_{-2}, J_1J_{-1}, K_1K_{-1}, J_1K_{-1} \pm J_{-1}K_1, I_2J_{-1}^2 \pm I_{-2}J^2, \\ I_2K_{-1}^2 \pm I_{-2}K_1^2, I_2J_{-1}K_{-1} \pm I_{-2}J_1K_1.$$

Eight of them are of types such as occur in (C), and the remaining four are

$$a'a'' + b'b'', \quad a'b'' - a''b', \\ (a-c)(a'a'' - b'b'') + 2b(a'b'' + a''b'), \\ (a-c)(a'b'' + a''b') - 2b(a'a'' - b'b'').$$

(E) *Two quadratics.*—The R.G.F. A.O.I. is

$$\frac{1-I_2I_{-2}J_2J_{-2}}{(1-I_0)(1-J_0)(1-I_2I_{-2})(1-J_2J_{-2})(1-I_2J_2)(1-I_{-2}J_{-2})};$$

and the irreducible system consists of six members

$$a+c, \quad a'+c', \\ (a-c)^2 + 4b^2, \quad (a'-c')^2 + 4b'^2, \\ (a-c)(a'-c') + 4bb', \quad (a-c)b' - b(a'-c'),$$

which are connected by the one syzygy of partial degrees 2, 2

$$\{(a-c)(a'-c') + 4bb'\}^2 + 4\{(a-c)b' - b(a'-c')\}^2 \\ \equiv I_2I_{-2}J_2J_{-2} \\ \equiv \{(a-c)^2 + 4b^2\} \{(a'-c')^2 + 4b'^2\}.$$

(F) *Two quadratics and linear form.*—The R.G.F. A.O.I. is

$$\frac{1}{(1-I_0)(1-J_0)(1-I_2I_{-2})(1-J_2J_{-2})(1-K_1K_{-1})} \\ \times \left\{ 1 - \frac{J_2K_{-1}^2}{(1-I_2K_{-1}^2)(1-J_2K_{-1}^2)} + \frac{J_{-2}K_1^2}{(1-I_{-2}K_1^2)(1-J_{-2}K_1^2)} \right. \\ + \frac{I_2J_{-2}}{(1-J_{-2}K_1^2)(1-I_2J_{-2})} + \frac{I_{-2}J_2}{(1-J_2K_{-1}^2)(1-I_{-2}J_2)} \\ \left. + \frac{I_2K_{-1}^2}{(1-I_2J_{-2})(1-I_2K_{-1}^2)} + \frac{I_{-2}K_1^2}{(1-I_{-2}J_2)(1-I_{-2}K_1^2)} \right\};$$

and there are eleven members of the irreducible system, of seven types, viz.,

$$I_0, \quad J_0, \quad I_2I_{-2}, \quad J_2J_{-2}, \quad K_1K_{-1}, \quad I_2J_{-2} \pm I_{-2}J_2, \quad I_2K_{-1}^2 \pm I_{-2}K_1^2, \\ J_2K_{-1}^2 \pm J_{-2}K_1^2.$$

(G) *Three quadratics.*—The R.G.F. A.O.I. is

$$\frac{1}{(1-I_0)(1-J_0)(1-K_0)(1-I_2I_{-2})(1-J_2J_{-2})(1-K_2K_{-2})} \\ \times \left\{ 1 + \frac{J_2K_{-2}}{(1-I_2K_{-2})(1-J_2K_{-2})} + \frac{J_{-2}K_2}{(1-I_{-2}K_2)(1-J_{-2}K_2)} \right. \\ + \frac{I_2J_{-2}}{(1-J_{-2}K_2)(1-I_2J_{-2})} + \frac{I_{-2}J_2}{(1-J_2K_{-2})(1-I_{-2}J_2)} \\ \left. + \frac{I_2K_{-2}}{(1-I_2J_{-2})(1-I_2K_{-2})} + \frac{I_{-2}K_2}{(1-I_{-2}J_2)(1-I_{-2}K_2)} \right\}.$$

Accordingly the irreducible system consists of twelve invariants, all of types which have occurred in the system for two quadratics. It will be found that no new types occur for n quadratics.

The N.G.F. for three quadratics is that of (B) multiplied by

$$\frac{1}{(1-a)(1-a')(1-a'')}.$$

(H) *Cubic and linear form.*—The R.G.F. A.O.I. is

$$\frac{1}{(1-I_3I_{-3})(1-I_1I_{-1})(1-J_1J_{-1})} \\ \times \left\{ 1 + \frac{I_1J_{-1}}{(1-I_3J_{-1}^3)(1-I_1J_{-1})} + \frac{I_{-1}J_1}{(1-I_{-3}J_1^3)(1-I_{-1}J_1)} \right\}.$$

$$\begin{aligned}
& + \frac{I_{-1}J_1}{(1-I_{-1}J_1)(1-I_8I_{-1}^3)} + \frac{I_1J_{-1}}{(1-I_1J_{-1})(1-I_{-8}I_1^3)} \\
& + \frac{1}{(1-I_{-8}I_1^2J_1)(1-I_{-8}I_1J_1^2)} + \frac{1}{(1-I_8I_{-1}^2J_{-1})(1-I_8I_{-1}J_{-1}^2)} \\
& + \frac{I_{-1}J_1^3}{(1-I_{-8}J_1^3)(1-I_{-8}I_1J_1^2)} + \frac{I_8J_{-1}^3}{(1-I_8J_{-1}^3)(1-I_8I_{-1}J_{-1}^2)} \\
& + \frac{I_8I_{-1}^3}{(1-I_8I_{-1}^3)(1-I_8I_{-1}^2J_{-1})} + \frac{I_{-8}I_1^3}{(1-I_{-8}I_1^3)(1-I_{-8}I_1^2J_1)} \\
& - \frac{1}{1-I_1J_{-1}} - \frac{1}{1-I_{-1}J_1} \Big\}.
\end{aligned}$$

There are then thirteen members of the irreducible system, viz., the four (35) of the cubic, the one (31) of the linear form, and the four pairs

$$I_1J_{-1} \pm I_{-1}J_1, \quad I_8J_{-1}^3 \pm I_{-8}J_1^3, \quad I_8I_{-1}J_{-1}^2 \pm I_{-8}I_1J_1^2, \quad I_8I_{-1}^2J_{-1} \pm I_{-8}I_1^2J_1.$$

(K) *Cubic and quadratic.*—If I_8, I_1, I_{-1}, I_{-8} refer to the cubic, and J_3, J_0, J_{-3} to the quadratic, the R.G.F. A.O.I. and the irreducible system are what those, (56) and (57), for one sextic become when in them for

$$\begin{array}{l}
I, I', J, J', K, K', L, \\
\text{or} \quad I_6, I_6, I_4, I_{-4}, I_2, I_{-2}, I_0, \\
\text{we put} \quad I_8, I_{-8}, J_3, J_{-3}, I_1, I_{-1}, J_0.
\end{array}$$

(L) *Quartic and linear form.*—The R.G.F. A.O.I. is

$$\begin{aligned}
& \frac{1}{(1-I_0)(1-I_2I_{-2})(1-I_4I_{-4})(1-I_1J_{-1})} \\
& \times \left\{ 1 + \frac{I_2J_{-1}^2}{(1-I_4J_{-1}^4)(1-I_2J_{-1}^2)} + \frac{I_{-2}J_1^2}{(1-I_{-4}J_1^4)(1-I_{-2}J_1^2)} \right. \\
& + \frac{I_{-2}J_1^2}{(1-I_4I_{-2}^2)(1-I_{-2}J_1^2)} + \frac{I_2J_{-1}^2}{(1-I_{-4}I_2^2)(1-I_2J_{-1}^2)} \\
& + \frac{1-(I_{-4}I_2J_1^2)^2}{(1-I_{-4}J_1^2)(1-I_{-4}J_1^4)(1-I_{-4}I_2J_1^2)} \\
& + \frac{1-(I_4I_{-2}J_{-1}^2)^2}{(1-I_4I_{-2}^2)(1-I_4J_{-1}^4)(1-I_4I_{-2}J_{-1}^2)} \\
& \left. - \frac{1}{1-I_2J_{-1}^2} - \frac{1}{1-I_{-2}J_1^2} \right\}.
\end{aligned}$$

The irreducible system consists of twelve forms, viz., the five (38) of the quartic, the one (31) of the linear form, and the three pairs

$$I_2 J_{-1}^2 \pm I_{-2} J_1^2, \quad I_4 I_{-2} J_{-1}^2 \pm I_{-4} I_2 J_1^2, \quad I_4 J_{-1}^4 \pm I_{-4} J_1^4.$$

(M) *Quartic and quadratic*.—If $I_4, I_2, I_0, I_{-2}, I_{-4}$ refer to the quartic, and J_2, J_0, J_{-2} to the quadratic, the R.G.F. A.O.I. is obtained from that of (D) for a quadratic and two linear forms by dividing by $1 - J_0$, and replacing

$$I_2, I_0, I_{-2}, J_1, J_{-1}, K_1, K_{-1}$$

by

$$I_4, I_0, I_{-4}, I_2, I_{-2}, J_2, J_{-2},$$

respectively. There are thirteen invariants in the irreducible system, viz., the five (38) of the quartic, the two (33) of the quadratic, and three pairs

$$I_2 J_{-2} \pm I_{-2} J_2, \quad I_4 I_{-2} J_{-2} \pm I_{-4} I_2 J_2, \quad I_4 J_{-2}^2 \pm I_{-4} J_2^2.$$

15. It will be well to explain and generalise some coincidences in the above list. The R.G.F. A.O.I. for a quartic is of exactly the same form as that for a quadratic and a linear form. Again, that for a sextic is of the same form as that for a cubic and a quadratic. The general fact which accounts for these identities of form is that there is a complete one to one correspondence of orthogonal invariants—non-absolute as well as absolute—of a $2p$ -ic with those of a p -ic and a $(p-1)$ -ic jointly. In fact the one generating function is

$$\{(1 - I_{2p} \rho^{2p})(1 - I_{2p-2} \rho^{2p-2}) \dots (1 - I_{-2p+2} \rho^{-2p+2})(1 - I_{-2p} \rho^{-2p})\}^{-1},$$

and the other is

$$\{(1 - J_p \rho^p)(1 - J_{p-2} \rho^{p-2}) \dots (1 - J_{-p} \rho^{-p})(1 - K_{p-1} \rho^{p-1})(1 - K_{p-3} \rho^{p-3}) \dots \\ \dots (1 - K_{-p+1} \rho^{-p+1})\}^{-1}.$$

and, replacing ρ by ρ^2 in the second, it becomes the first with

$$J_p, K_{p-1}, J_{p-2}, \dots, J_{-p+2}, K_{-p+1}, J_{-p},$$

respectively, written for

$$I_{2p}, I_{2p-2}, I_{2p-4}, \dots, I_{-2p+4}, I_{-2p+2}, I_{-2p}.$$

The reason is not hard to discover. If x, y are orthogonally transformed—turned through θ —so also are $x^2 - y^2, 2xy$ —turned through 2θ . Now our $2p$ -ic is (11) a numerical multiple of

$$(I_{2p}, I_{2p-2}, \dots, I_{-2p+2}, I_{-2p})(\xi, \eta)^{2p},$$

i.e., of $(J'_p, J'_{p-2}, \dots, J'_{-p})(\xi^2, \eta^2)^p + \xi\eta (K'_{p-1}, K'_{p-3}, \dots, K'_{-p+1})(\xi^2, \eta^2)^{p-1}$,

where $J'_p, K'_{p-1}, J'_{p-2}, \dots, K'_{-p+1}, J'_{-p}$

are numerical multiples of

$$I_{2p}, I_{2p-2}, I_{2p-4}, \dots, I_{-2p+2}, I_{-2p};$$

also $\xi^2 \equiv x^2 - y^2 + 2ixy$, $\eta^2 \equiv x^2 - y^2 - 2ixy$, and $\xi\eta \equiv x^2 + y^2$ the absolute. The J 's and K 's are then the fundamental linear orthogonal invariants of a p -ic and a $(p-1)$ -ic in $x^2 - y^2, 2xy$.

16. *Systems of Absolute Orthogonal Covariants (including Invariants).*

By (26) and (29) the generating function for covariants of a given binary form, or system of forms, is the same as that for invariants of the system of forms which consists of the given form or system and a linear form whose fundamental orthogonal invariants I'_1, I'_{-1} are η, ξ respectively. Accordingly the results of § 14 also give us the following:—

(N) *System for one linear form $ax + by$.*—The R.G.F. A.O.C. is obtained by putting ξ, η for J_{-1}, J_1 in that of § 14 (A). The system consists of

$$a^2 + b^2, \quad x^2 + y^2, \quad ax + by, \quad ay - bx,$$

with the one syzygy

$$(a^2 + b^2)(x^2 + y^2) \equiv (ax + by)^2 + (ay - bx)^2.$$

(P) *Two linear forms $ax + by, a'x + b'y$.*—The R.G.F. A.O.C. is the result of putting ξ, η for K_{-1}, K_1 in that of § 14 (B). The system consists of the four invariants of (A), the absolute $x^2 + y^2$, and the four covariants

$$ax + by, \quad a'x + b'y, \quad ay - bx, \quad a'y - b'x.$$

(Q) *One quadratic $(a, b, c)(x, y)^2$.*—For the R.G.F. A.O.C. we put ξ, η for J_{-1}, J_1 in that of (C). There are in the irreducible system, besides the absolute $x^2 + y^2$ and the two invariants

$$a + c, \quad (a - c)^2 + 4b^2,$$

the two covariants

$$(a - c)(x^2 - y^2) + 4bxy, \quad (a - c)xy - b(x^2 - y^2).$$

The one connecting syzygy is

$$\{(a - c)^2 + 4b^2\}(x^2 + y^2)^2 \equiv \{(a - c)(x^2 - y^2) + 4bxy\}^2 + 4\{(a - c)xy - b(x^2 - y^2)\}^2,$$

and the N.G.F. is
$$\frac{1 + ax^2}{(1 - a)(1 - a^2)(1 - x^2)(1 - ax^2)}.$$

The quadratic itself is a linear combination of the first covariant and $(a+c)(x^2+y^2)$.

(R) *Quadratic and linear form.*—For the R.G.F. A.O.C. we have to put ξ, η for K_{-1}, K_1 in the R.G.F. of (D). The irreducible system consists of the five invariants given in (C), the absolute x^2+y^2 , and the six covariants

$$\begin{aligned} (a-c)(x^2-y^2) + 4bxy, \quad (a-c)xy - b(x^2-y^2), \\ a'x + b'y, \quad b'x - a'y, \\ (a-c)(a'x - b'y) + 2b(a'y + b'x), \quad (a-c)(a'y + b'x) - 2b(a'x - b'y). \end{aligned}$$

(S) *Two quadratics.*—The R.G.F. A.O.C. is the result of putting ξ, η for K_{-1}, K_1 in the R.G.F. of (F). There are eleven members of the irreducible system, viz., x^2+y^2 , the six invariants given in (E), and the four covariants

$$\begin{aligned} (a-c)(x^2-y^2) + 4bxy, \quad (a'-c')(x^2-y^2) + 4b'xy, \\ (a-c)xy - b(x^2-y^2), \quad (a'-c')xy - b'(x^2-y^2). \end{aligned}$$

The N.G.F. for the system is

$$\frac{1 + aa' + (a+a')(1-aa')x^2 - aa'x^4 - a^2a'^2x^4}{(1-a)(1-a')(1-a^2)(1-a'^2)(1-aa')(1-a^2x^2)(1-a'^2x^2)(1-x^2)}.$$

(T) *One cubic* $(a, b, c, d)(x, y)^3$.—For the R.G.F. A.O.C. put ξ, η for J_{-1}, J_1 in the R.G.F. of (H).

The irreducible system consists of the absolute x^2+y^2 , the four invariants of § 10, and eight covariants given by the last line of (H). Rejecting factors 2 and 2i, these eight covariants are

$$\begin{aligned} (a+c)x + (b+d)y, \\ (b+d)x - (a+c)y, \\ (a-3c)x(x^2-3y^2) + (3b-d)y(3x^2-y^2), \\ (3b-d)x(x^2-3y^2) - (a-3c)y(3x^2-y^2), \\ \{(a+c)^2 - (b+d)^2\} \{ (a-3c)x + (3b-d)y \\ \quad + 2(a+c)(b+d) \{ (3b-d)x - (a-3c)y \}, \\ \{(a+c)^2 - (b+d)^2\} \{ (3b-d)x - (a-3c)y \\ \quad - 2(a+c)(b+d) \{ (a-3c)x + (3b-d)y \}, \\ \{(a-3c)(a+c) + (3b-d)(b+d)\} (x^2-y^2) \\ \quad + 2 \{ (a+c)(3b-d) - (a-3c)(b+d) \} xy, \\ \{(a+c)(3b-d) - (a-3c)(b+d)\} (x^2-y^2) \\ \quad - 2 \{ (a-3c)(a+c) + (3b-d)(b+d) \} xy. \end{aligned}$$

Note that the full covariant $(a^2d-3abc+2b^3)x^3+\dots$, regarded as an absolute orthogonal covariant, is reducible in terms of the others.* With this exception the irreducible system of full covariants and invariants of a cubic and quadratic yield the equivalents of the above system upon taking the absolute x^2+y^2 for the quadratic.

(U) *One quartic* $(a, b, c, d, e)(x, y)^4$.—The R.G.F. A.O.C. is got by putting ξ, η for J_{-1}, J_1 in the R.G.F. of (L). The irreducible system consists then of x^2+y^2 , the five invariants of § 11, and six covariants

$$I_2\xi^2\pm I_{-2}\eta^2, \quad I_4I_{-2}\xi^2\pm I_{-4}I_2\eta^2, \quad I_4\xi^4\pm I_{-4}\eta^4.$$

These prove to be, on rejection of powers of 2 and i as factors,

$$\begin{aligned} & (a-e)(x^2-y^2)+4(b+d)xy, \\ & (b+d)(x^2-y^2)-(a-e)xy, \\ & \{(a-6c+e)(a-e)+8(b^2-d^2)\}(x^2-y^2) \\ & \quad -4\{(a-6c+e)(b+d)-2(a-e)(b-d)\}xy, \\ & \{(a-6c+e)(b+d)-2(a-e)(b-d)\}(x^2-y^2) \\ & \quad +\{(a-6c+e)(a-e)+8(b^2-d^2)\}xy, \\ & (a-6c+e)(x^4-6x^2y^2+y^4)+16(b-d)xy(x^2-y^2), \\ & (b-d)(x^4-6x^2y^2+y^4)-(a-6c+e)xy(x^2-y^2). \end{aligned}$$

On some Properties of Groups of Odd Order. (Second Paper.)

By W. BURNSIDE. Communicated and received December 13th, 1900.

In the present paper I have extended the method used in the paper "On Groups of Degree n and Class $n-1$," communicated to the Society at the June meeting, so as to make it applicable to any transitive substitution group. As the title of the paper indicates, the method is considered mainly in its application to groups of odd

* [A remark by a referee leads me to emphasize the above. Absolute orthogonal invariants of a p -ic are invariants of the p -ic and x^2+y^2 , and *vice versa*. It has therefore been hastily supposed elsewhere that the complete irreducible invariant system for a p -ic and quadratic produces exactly the complete (absolute) orthogonal system for the p -ic, when x^2+y^2 is taken for the quadratic. A first case of the redundancy of the former system for the latter purpose is exhibited above. The search for complete absolute orthogonal systems is not identical with the search for invariant systems of forms one of which is a quadratic.]